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Advanced Quantum Field Theory

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4 Lecture 4: The exact renormalization group

In the previous lecture, we wrote down the renormalization group flow equation for the 2PI effective action

$$\partial_t \Gamma[\phi, \Delta] = \frac{\delta \Gamma[\phi, \Delta]}{\delta \Delta^\alpha} \, \partial_t \Delta^\alpha = K_\alpha \partial_t \Delta^\alpha \, , \tag{1}$$

where $t \propto \ln k$ is the RG time. To obtain this, we associated the Fourier transform of the two-point source K_α with a regulator, which allows us to control which of the quantum fluctuations of the fields we integrate out.

We will choose this regulator so that it "freezes" infrared modes below some scale k, such that we only integrate out ultraviolet modes in the path integral. We therefore require that the regulator $R_k(p) = -\tilde{K}(p)$ has the following properties:

- $R_k(p) \to 0$ as $k \to 0$, so that when the scale k is zero, all the modes are integrated out.
- $R_k(p) \to 0$ as $p^2/k^2 \to \infty$, and the UV modes are integrated out as usual in the path integral.
- $R_k(p) \to \infty$ as $p^2/k^2 \to 0$, and the IR modes are *not* integrated out.

In the limit $R_k(p) \to 0$, the scale-dependent 2PI effective action reduces to the 1PI effective action. In the other limit, $R_k(p) \to 0$, we instead recover the classical action $S[\phi]$, since there are no quantum fluctuations for us to integrate out in the path integral. At some finite k, we integrate out modes with momenta $|p| \gtrsim |k|$. Note that the cut-off k is, in principle, implemented smoothly, and we may still integrate some modes in the vicinity of the scale k. This will depend on the precise form of the regulator, and the choice is not unique.

At this point, we might be worried that there is something unphysical about the effective action at arbitrary values of the scale k, since it will depend on the choice of regulator. We're right to worry. However, we are not interested in the particular form of the effective action, but

rather what the evolution in the theory space looks like for a given regulator and, in particular, whether there are any special points in that evolution. We call these *fixed points*, and they don't change under RG evolution. At these points, the correlation length will be zero or infinity, and if it is infinity, we have a critical point (where phase boundaries terminate), we see the emergence of scale invariance, and we have universal behaviour.

The 2PI flow equations are relatively new on the market, and much more common are those derived from the so-called 1PI average effective action, which is a shifted version of the functional $\Gamma[\bar{\phi}; \bar{K}]$ that we introduced in the previous lecture. It has the following form:

$$\Gamma_{\rm av}[\phi, K] = -W[J, K] + J_a\phi^a + \frac{1}{2}K_{ab}\phi^a\phi^b \,, \tag{2}$$

where the maximisation in the Legendre transform has only been performed with respect to J_a. We therefore have the same expression for the one-point function

$$\phi^a = \frac{\delta W[J, K]}{\delta J_a} \,, \tag{3}$$

but we do not have the corresponding expression for the two-point function, as we did in the 2PI case. This means that we do not have the advantage of the implicit resummation of 1PI self-energies implicit to the 2PI approach. Nevertheless, it remains the case that

$$\Delta^{ab} = \frac{\delta W[J, K]}{\delta J_a \delta J_b} = 2\frac{\delta W[J, K]}{\delta K_{ab}} - \phi_a\phi_b \,. \tag{4}$$

At this point, we need to be a bit careful. The 1PI average effective action is *not* the Legendre transform of the Schwinger functional; it is shifted by the term $K_{ab}\phi^a\phi^b/2$. The first two terms in $\Gamma_{\rm av}[\phi, K]$ are, however, the terms present in the 1PI effective action

$$\Gamma_{\rm 1PI}[\phi] = -W[J] + J_a\phi^a \,. \tag{5}$$

Compared to the 2PI case, the convexity of the 1PI effective action follows from the much simpler identity

$$\frac{\delta W[J, K]}{\delta J_a \delta J_b}\frac{\delta^2\Gamma_{\rm 1PI}[\phi]}{\delta\phi^a\delta\phi^b} = \mathbb{I} \,, \tag{6}$$

where we are being rather blasé about the meaning of the $\mathbb{I}$ on the right-hand side. Even so, we can hopefully convince ourselves that

$$\Delta^{ab} = \left(\frac{\delta^2\Gamma_{\rm 1PI}[\phi]}{\delta\phi^a\delta\phi^b}\right)^{-1} \,. \tag{7}$$

It then follows that the two-point function is obtained from the 1PI average effective action as

$$\Delta^{ab} = \left(\frac{\delta^2 \Gamma_{\rm av}[\phi, K]}{\delta\phi^a \delta\phi^b} - K_{ab} \right)^{-1} \ . \tag{8}$$

In order to obtain the flow equation, we again associate the Fourier transform of K with the regulator, i.e., $\tilde{K}(p) = -R_k(p)$, as we did in the 2PI case. Varying the 1PI average effective action with the respect to the RG time $t \propto \ln k$ for fixed ϕ, we then obtain the flow equation

$$\partial_t \Gamma_{\rm av}[\phi, K] = \left(-\frac{\delta W[J,K]}{\delta K_{ab}} + \phi^a \phi^b \right) \partial_t K_{ab} = -\frac{1}{2} \Delta^{ab} \partial_t K_{ab} \ , \tag{9}$$

which we notice differs from the 2PI flow equation by a total derivative with respect to the RG time $\partial_t (K_{ab} \Delta^{ab})$. The flow equation (9) is due to Wetterich [1], Morris [2] and Ellwanger [3], and it is more often written in momentum space in the form

$$\partial_k \Gamma_k = \frac{1}{2} {\rm STr}\, \partial_k R_k(p) \left[\Gamma_k^{(2)} + R_k \right] \ , \tag{10}$$

where $\Gamma_k \equiv \Gamma_{\rm av}$, STr is the trace in momentum space and over all fields and quantum numbers.

In the previous lecture, we saw that RG evolution moved us around in the configuration space of correlation functions, which we were able to equip with a geometrical structure stolen from information theory. In this lecture, we will see how the flow equation can be used to describe how the couplings in our theory change with energy scale, and therefore allow us to understand how RG evolution moves us around in theory space, when we instead use the couplings as coordinates. To do this, we will solve the flow equation by an Ansatz, using what is commonly called the local potential approximation. We will also work to the lowest order in the derivative expansion, i.e., we consider terms that contain at most two derivatives of the field. We will also neglect the infamous *anomalous dimension*, but there are all manner of excellent reviews of the exact renormalization group that we can dive into to find out more.

One such review, and the one that we will follow relatively closely in what follows, is the review by Berges, Tetradis and Wetterich [4]. Focusing, as these authors did, on the symmetry breaking case of the ϕ^4 theory that we considered in the exercises to the second lecture, we write the following Ansatz for the Euclidean 1PI average effective action:

$$\Gamma_k = \int {\rm d}^d x \ \left[\frac{1}{2} Z_k(\rho) \partial_\mu \phi \partial_\mu \phi + U_k(\rho) \right] \ , \tag{11}$$

where $\rho = \phi^2/2$ and

$$U_k(\rho) = \frac{1}{2} g_k (\rho - \rho_k)^2 + \Lambda_k \ . \tag{12}$$

The parameters g_k, ρ_k and Λ_k are dimensionful in general dimensions d, and it is convenient to

scale out their dimensionality by defining

$$\kappa_k = \bar{Z}_k k^{2-d} \rho_k \ , \tag{13}$$

$$\lambda_k = \bar{Z}_k^{-2} k^{d-4} g_k \ , \tag{14}$$

where

$$\bar{Z}_k \equiv Z_k(\rho_k) \ . \tag{15}$$

The scale-dependent potential then takes the form

$$U_k(\rho) = \frac{1}{2} k^d \lambda_k \big(\bar{Z}_k k^{2-d} \rho - \kappa_k\big)^2 + \Lambda_k \tag{16}$$

For constant ρ, the flow equation for the average effective action quickly yields the flow equation for the effective potential

$$\partial_t U_k(\rho) = \frac{1}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \partial_t R_k(q) \Delta_k(\rho, q) \ . \tag{17}$$

Using Eq. (8), the momentum-space two-point function is

$$\Delta_k(\rho, q) = \frac{1}{Z_k(\rho) q^2 + R_k(q) + U_k'(\rho) + 2\rho U_k''(\rho)} \ . \tag{18}$$

For simplicity, we will take $Z_k(\rho) = \bar{Z}_k$ and neglect the scale-dependence of $\bar{Z}_k$. (This is what causes us to neglect the anomalous dimension $\eta = -\partial_t \ln \bar{Z}_k$.) We now just need to pick a regulator. A particularly straightforward choice for our purposes is the Litim regulator [5]:

$$R_k(q) = \bar{Z}_k (k^2 - q^2) \theta(k^2 - q^2) \ , \tag{19}$$

where θ is the unit step function. This regulator has the advantage of making the so-called *threshold functions*, which we will obtain when performing the integral over the momentum q, particular easy to evaluate.

The first of these threshold functions arises in the flow of the effective potential, and we have

$$\begin{aligned} \frac{1}{2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \, \partial_t R_k(q) \Delta_k(\rho, q) &= \frac{s_d}{2} \int \mathrm{d}q \, q^{d-1} \frac{2 \bar{Z}_k k^2 \theta(k^2 - q^2)}{\bar{Z}_k(\rho) q^2 + \bar{Z}_k (k^2 - q^2) \theta(k^2 - q^2) + U_k'(\rho) + 2\rho U_k''(\rho)} \\ &= \frac{s_d}{d} k^d \left[1 + \frac{U_k'(\rho) + 2\rho U_k''(\rho)}{\bar{Z}_k k^2} \right]^{-1} \ , \end{aligned} \tag{20}$$

where $s_d = \Omega_d/(2\pi)^d$, with $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ being the solid angle in d dimensions. Thus,

$$\partial_t U_k(\rho) = \frac{s_d}{d} k^d \left[1 + \frac{U_k'(\rho) + 2\rho U_k''(\rho)}{\bar{Z}_k k^2}\right]^{-1} . \tag{21}$$

Evaluating this at the minimum of the potential $\rho = \rho_k$, we obtain the flow equation for the cosmological constant

$$\partial_t \Lambda_k = \frac{s_d}{d} \frac{k^d}{1 + 2\kappa_k \lambda_k} . \tag{22}$$

To obtain the flow equations for κ_k and λ_k, we take derivatives with respect to ρ, and again evaluate at $\rho = \rho_k$. After a little algebra, we find

$$\partial_t \kappa_k = (2-d)\kappa_k + \frac{6 s_d}{d} \frac{1}{(1 + 2\kappa_k \lambda_k)^2} , \tag{23}$$

$$\partial_t \lambda_k = (d-4)\kappa_k + \frac{18 s_d}{d} \frac{\lambda_k^2}{(1 + 2\kappa_k \lambda_k)^3} . \tag{24}$$

If we were to solve for the RG evolution of this system, we would find that the the potential barrier between the two minima of the potential would lower as we integrated out more and more modes (i.e., lowered the scale k towards the infrared). As we take the scale $k \to 0$, and integrate out all of the quantum fluctuations, we would find a convex potential in agreement with our analysis of the effective potential and the emergence of the Maxwell construction in Exercise 4.

The first thing that we notice is that in $d > 4$ the signs of the leading terms in the flow equations (which result from the canonical scaling dimension of the original dimensionful couplings) are opposite for κ_k and λ_k. For κ_k, the right-hand side of the flow equation, otherwise called the *beta function*, is negative. This means that the corresponding operator (the mass term) increases into the infrared for $d > 2$; we say that the operator is *relevant*. On the other hand, for $d > 4$, the leading term in the beta function for λ_k is positive, and we would say that the operator is *irrelevant*, and it decreases into the infrared. In $d = 4$, the leading term in the beta function for λ_k is actually zero. If it were strictly zero, we would say that the corresponding operator was *marginal*. However, we see that the contribution from the loop corrections (via the threshold functions) is positive and there is the anomalous dimension too that we neglected. In fact, the coupling λ_k is said to be *marginally irrelevant*, and it will evolve comparably slowly in RG time.

We see already from this analysis that we can have theories who share common relevant operators and differ only in their irrelevant operators. Such theories from a *universality class*: they give the same infrared physics. If we think in terms of effective field theories, where we allow higher-dimensional, and therefore a menagerie of irrelevant operators, the same infrared physics can be realised from multiple effective field theories.

Going back to our theory space, and spanning it with the couplings, we also see that if we are close to some fixed point, where the beta function vanishes, we have the possibility to move

in irrelevant and relevant directions. If we try to move in an irrelevant direction, the RG flow will drive us back to the fixed point. This is because irrelevant directions do not affect the long-distance physics. On the other hand, we can happily move in a relevant direction, and the RG flow will kindly take us away from the fixed point. This is because relevant operators *do* affect the long-distance physics. (For a much more lucid discussion on these points than I could ever hope to give, I strongly recommend David Tong's lecture notes [6].)

Some concluding remarks

Sadly, this brings us to the end of our summer Smörgåsbord of quantum field theory morsels. And there are many things that we did not get chance to discuss further or, indeed, to meet at all:

- We worried a little about some foundational topics in quantum field theory, but we did not have time to meet the axiomatic field theory or algebraic quantum field theory that were the reaction to the types of concerns raised by Haag and others.

- We didn't get to meet the other kinds of integrating out, in which we trace over some subsystem in order to build an open quantum field theoretic system, and this would have been a lot of fun. We would have introduced the Schwinger–Keldysh closed-time path formalism, which is geared up to calculating equal-time expectation values of local products of operators and readily allows us to treat field-theoretic statistical ensembles. From there, we would have looked at the field-theoretic description of the evolution of reduced density matrices and met the Feynman–Vernon influence functional, where we really would have to worry about the boundary conditions on the path integral. We could have looked at how causality is borne out in the path-integral representation, and how this is naturally framed within the closed-time path formalism, thanks to the presence of both time- and anti-time-ordered prescriptions to ensure convergence of the path integral.

- We only touched the surface of the (exact) renormalization group, which remains one of the most powerful non-perturbative techniques that we have in quantum field theory. And we didn't get to look at the role of non-linear field redefintions in eliminating redundancies in our description, allowing us to distinguish between essential and inessential couplings. For this, I point you to the excellent paper by Baldazzi, Zinati and Falls [7].

- We didn't get to worry about quantum field theory in curved spacetime or how all of the tools that we developed allow us to treat non-trivial field configurations, like instantons and solitions, or to worry about the evolution of metastable field configurations.

- We didn't get to play with defining different state spaces to consider much wider classes of quantum field theories, even finding ones with complex actions that nevertheless have real,

bounded eigenspectra and a (pseudo-)unitary evolution. Such theories are said to be *pseudo-Hermitian*. Maybe we'll get to discuss these pseudo-Hermitian quantum field theories and what they teach us about "standard" quantum field theory another time.

- And we didn't get to see that the ϕ^4 theory suffers from a Landau pole (divergent coupling at a finite scale) and the problem of triviality. Even the seemingly innocent toy theories can turn out to be less than upright in character in the end. As Winne-the-Pooh would say, "One never can tell with theories."

And that was just scalar field theories: no fermions, no gauge fields, no gravity.

So thank you for listening, for all the great questions, and for all the great discussions. I look forward to chatting with you all again and seeing all that you achieve. But please, never let anyone try to convince you that quantum field theory is all figured out — it most certainly is not. All the best for your PhDs and the adventures that will follow, wherever they take you.

References

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