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Advanced Quantum Field Theory

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2 Lecture 2: The quantum effective action

In the last lecture, we introduced the partition function

$$Z[J,K] = \int \mathcal{D}\Phi \, \exp\left[\frac{i}{\hbar}\left(S[\Phi] - \int \mathrm{d}^4x \; J_a(x)\Phi_a(x) - \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y \; K^{ab}(x,y)\Phi_a(x)\Phi_b(y)\right)\right] \,, \quad (1)$$

which we could use to generate n-point correlation functions. However, it would be nice if, instead of having something that was a function of the sources, we had an object that was a function of the correlation functions themselves. It would be even better if this thing were akin to the classical action, whose variation gave us the equations of motion governing those correlation functions. We're in luck: this object exists, or rather a family of such objects exist, and we call them *quantum effective actions* or just *effective actions*.

To construct them, we first need to take a little diversion into differential geometry. Consider a point x on some manifold M. We can define a tangent space at the point x. As an example, suppose the manifold M is a hemisphere. The tangent space at a point x on the hemisphere is just the plane tangential to the surface at that point. Now, we could build the hemisphere by giving the set of points x or from the envelope of the set of all planes tangential to the surface.

We can do the same thing with convex or concave functions: We can build a function $f(x)$ from the coordinates $(x, f(x))$ or we can build it from the set of tangents to the function, specified in terms of their gradients and intercepts. The mapping between these two sets of coordinates is a *Legendre transform*.

Given that the tangents to the Schwinger functional are the correlation functions, this is precisely what we want to do. Thus, we define the quantum effective action

$$\Gamma[\phi,\Delta] = \max_{J,K}\left[-W[J,K] + J_a(x)\phi_a(x) + \frac{1}{2}K_{ab}(x,y)\big(\phi_a(x)\phi_b(y) + \hbar\Delta_{ab}(x,y)\big)\right] \,. \quad (2)$$

In fact, this is the so-called *two-particle-irreducible* or *2PI effective action* — due to Cornwall, Jackiw and Tomboulis [1] — which will prove particularly helpful in understanding the family of effective actions. Note that, having got rather tired of writing so many integrals, we now use a DeWitt-like notation in which repeated arguments are integrated over, e.g.,

$$J_a(x)\phi_a(x) \equiv \sum_a \int \mathrm{d}^4x\, J_a(x)\phi_a(x) \ . \tag{3}$$

The extremisation gives

$$\phi_a(x) = \frac{\delta W[J,K]}{\delta J_a(x)} \ , \tag{4}$$

$$\hbar\Delta_{ab}(x,y) = 2\frac{\delta W[J,K]}{\delta K_{ab}(x,y)} - \phi_a(x)\phi_b(y) \ , \tag{5}$$

and we identify ϕ and Δ with connected one- and two-point functions.

We have, however, committed a slight abuse of notation here to keep things simple. After the extremisation, we have particular sources J and K, which are those that maximise the functional in square brackets in Eq. (2). Thus, the extremisation defines $J = J[\phi,\Delta]$ and $K = K[\phi,\Delta]$ to be functionals of ϕ and Δ, in much the same way as the Legendre transform between Lagrangian and Hamiltonian mechanics makes the momentum p a function of the time derivative of the generalised coordinate q, i.e., $p = p(\dot{q})$.

With this understood, we can drop the extremisation and write the quantum effective action

$$\Gamma[\phi,\Delta] = -W[J,K] + J_a(x)\phi_a(x) + \frac{1}{2}K_{ab}(x,y)\big(\phi_a(x)\phi_b(y) + \hbar\Delta_{ab}(x,y)\big) \ . \tag{6}$$

The gradients of the effective action are

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\phi_a(x)} = J_a(x) + K_{ab}(x,y)\phi_b(y) \ , \tag{7}$$

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta_{ab}(x,y)} = \frac{1}{2}K_{ab}(x,y) \ . \tag{8}$$

Given the presence of the sources on the right-hand side, we infer that these are nothing other than the equations of motion for the one- and two-point functions. They are, however, not the classical equations of motion; they are the quantum-corrected equations of motion, wherein quantum fluctuations have been integrated out to give effective c-number differential equations for the correlation functions. And this brings us to an important facet of the question, "How do we build theories?": *integrating stuff out*.

We are probably all familiar with the maxim, "Sociology is Biology, Biology is Chemistry, Chemistry is Physics, Physics is Maths, and no one knows what Maths is". As crass as this state-

ment is, it is a reflection of what we might call *constructionism*, which is often and incorrectly assumed to be the corollary of reductionism. To be similarly crass: knowing all the physics that there is to know won't help you one bit to explain why certain species of bird are so partial to a mate who can moonwalk. The reason is that the degrees of freedom are all wrong. Trying to understand the mating rituals of the world's fauna in terms of the dynamics of elementary particles is a fool's errand. We might like to think that the rules of social engagement can be derived from fundamental physics, but *complexity* has taught us otherwise.

But we can consider far less extreme hierarchies. For example, we know that the most useful degrees of freedom for describing quantum chromodynamics at high energies are quarks and gluons and, at low energies, they are hadrons. What we learn from this is that the most effective way of describing the physics can depend on the characteristic energy or length scale of the system of interest. We move from shorter length scales to larger ones by averaging over effects on smaller scales. We call this *coarse graining*. In quantum field theory, this process amounts to integrating out quantum fluctuations on particular scales, and we will meet this in more detail next lecture.

To start with though, we are going to integrate out all the quantum fluctuations to get our bearings on the form of the 2PI effective action, but it will already be enough for us to see how the relevant laws of physics can change. This will be the focus of Exercise 4. Let's start with the case $N = 1$ and impose some additional symmetry on the classical action to constrain the number of operators. To this end, we impose $\mathbb{Z}_2$ symmetry, so that the action is symmetric under $\Phi \to -\Phi$. To keep things simpler still, we will do everything in Euclidean space, and we'll stop at operators quartic in the field Φ. Thus, we take the partition function

$$Z[J,K] = \mathcal{N} \int \mathcal{D}\Phi \exp\left[-\frac{1}{\hbar}\left(S[\Phi] - J(x)\Phi(x) - \frac{1}{2}K(x,y)\Phi(x)\Phi(y)\right)\right] , \tag{9}$$

with classical action

$$S[\Phi] = \int \mathrm{d}^4x \left(\frac{1}{2}Z_\Phi\big(\partial\Phi(x)\big)^2 + \frac{1}{2}m^2\Phi^2(x) + \frac{\lambda}{4!}\Phi^4(x)\right) . \tag{10}$$

For the time being, we fix $m^2 > 0$ and $\lambda > 0$, and leave the case $m^2 < 0$ for Exercise 4.

So far, we have allowed for a general wavefunction factor Z_Φ. However, by making the following redefinitions of the fields and sources, we can eliminate Z_Φ from the action:

$$\Phi(x) \longrightarrow Z_\Phi^{-1/2}\Phi(x) , \qquad J(x) \longrightarrow Z_\Phi^{1/2}J(x) , \qquad K(x,y) \longrightarrow Z_\Phi K(x,y) . \tag{11}$$

This is an important observation about our charting of the theory space in terms of parameters (or operators): some parameters (and indeed operators) can be removed by field redefinitions. We call the corresponding operators *inessential* (see [2]), and they are a redundant part of our mathematical description that do not impact physical observables. Making these redefinitions,

the classical action becomes

$$S[\Phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\big(\partial\Phi(x)\big)^2 + \frac{1}{2}m^2\Phi^2(x) + \frac{\lambda}{4!}\Phi^4(x) \right) \ , \tag{12}$$

and thankfully we no longer have to worry about getting our Z's confused.

The functional integral will be dominated by configurations that extremise the exponent. Our aim is to find these configurations and perform a so-called *saddle-point approximation*. We proceed by writing $\Phi(x) = \varphi(x) + \hbar^{1/2}\delta\Phi(x)$, where the $\hbar^{1/2}$ is for bookkeeping, and expand

$$\begin{aligned} S[\Phi] = S[\varphi] + \hbar^{1/2}\frac{\delta S[\Phi]}{\delta\Phi(x)}\bigg|_{\Phi=\varphi}\delta\Phi(x) + \frac{\hbar}{2!}\delta\Phi(x)\frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\bigg|_{\Phi=\varphi}\delta\Phi(y) \\ + \frac{\hbar^{3/2}}{3!}S^{(3)}(\varphi;x)\delta\Phi^3(x) + \frac{\hbar^2}{4!}S^{(4)}(\varphi;x)\delta\Phi^4(x) \ , \end{aligned} \tag{13}$$

where, e.g.,

$$S^{(3)}(\varphi;x) = \int \mathrm{d}^4y\,\mathrm{d}^4z\,\frac{\delta^n S[\Phi]}{\delta\Phi(x)\delta\Phi(y)\delta\Phi(z)}\bigg|_{\Phi=\varphi} = \lambda\varphi(x) \ . \tag{14}$$

The saddle-point configuration φ is that which satisfies

$$\frac{\delta S[\Phi]}{\delta\Phi(x)}\bigg|_{\Phi=\varphi} = J(x) + K(x,y)\varphi(y) \ , \tag{15}$$

and the partition function can be written

$$\begin{aligned} Z[J,K] = \mathcal{N}\exp\left[-\frac{1}{\hbar}\left(S[\varphi] - J(x)\varphi(x) - \frac{1}{2}K(x,y)\varphi(x)\varphi(y)\right)\right] \\ \times \int \mathcal{D}\delta\Phi\ \exp\left[-\frac{1}{2}\delta\Phi(x)\mathcal{G}^{-1}(x,y)\delta\Phi(y) - \frac{\hbar^{1/2}}{3!}S^{(3)}(\varphi;x)\delta\Phi^3(x) - \frac{\hbar}{4!}S^{(4)}(\varphi;x)\delta\Phi^4(x)\right] \ , \end{aligned} \tag{16}$$

where we have introduced the inverse two-point function

$$\mathcal{G}^{-1}(x,y) = G^{-1}(\varphi;x,y) - K(x,y) = \frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)}\bigg|_{\Phi=\varphi} - K(x,y) \ . \tag{17}$$

A little pedantry is needed here: We must keep in mind that the sources J and K are still functionals of the one- and two-point functions ϕ and Δ.

Assuming that $S^{(3)}$ and $S^{(4)}$ are controlled by small parameters, we can compute the functional integral over the fluctuations perturbatively. The leading term is

$$\int \mathcal{D}\delta\Phi\ \exp\left[-\frac{1}{2}\delta\Phi(x)\mathcal{G}^{-1}(x,y)\delta\Phi(y)\right] \propto \det{}^{-1/2}\mathcal{G}^{-1} \ , \tag{18}$$

wherein we have omitted constant factors. In Euclidean space, the Schwinger functional is

$$W[J,K] = \hbar \ln Z[J,K] \tag{19}$$

and the quantum effective action up to order $\hbar^2$ can be written as

$$\begin{aligned}\Gamma[\phi,\Delta] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}] + \hbar^2\Gamma_{1\mathrm{PR}}[\varphi,\mathcal{G}] \\ + \big(J(x) + K(x,y)\phi(y)\big)\big(\phi(x)-\varphi(x)\big) \\ - \frac{1}{2}K(x,y)\big[\big(\phi(x)-\varphi(x)\big)\big(\phi(y)-\varphi(y)\big) - \hbar\big(\Delta(x,y)-\mathcal{G}(x,y)\big)\big]\,,\end{aligned} \tag{20}$$

or, more neatly, as

$$\begin{aligned}\Gamma[\phi,\Delta] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}] + \hbar^2\Gamma_{1\mathrm{PR}}[\varphi,\mathcal{G}] \\ + \frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)}\big(\phi(x)-\varphi(x)\big) \\ - \frac{1}{\hbar}\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\big[\big(\phi(x)-\varphi(x)\big)\big(\phi(y)-\varphi(y)\big) - \hbar\big(\Delta(x,y)-\mathcal{G}(x,y)\big)\big]\,,\end{aligned} \tag{21}$$

where

$$\Gamma_0[\varphi] = S[\varphi]\,, \qquad \Gamma_1[\varphi,\mathcal{G}] = \frac{1}{2}\mathrm{tr}\Big[\ln\big(\mathcal{G}^{-1}*G(0)\big) + G^{-1}(\varphi)*\mathcal{G} - 1\Big]\,. \tag{22}$$

The $*$ indicates that the two-point functions are integrated over the volume, and we have introduced a normalisation $G(0)$ to make the logarithm dimensionless. Γ_2 and $\Gamma_{1\mathrm{PR}}$ contain 2PI and one-particle reducible (1PR) diagrams, respectively. We will calculate these explicitly in Exercise 3.

If we assume that the path integral is dominated by a single saddle-point configuration — This is not actually true for general $J(x)$ and $K(x,y)$. — we can eliminate ϕ and Δ for φ and $\mathcal{G}$ (see [3, 4]). To this end, we expand

$$\begin{aligned}\Gamma[\phi,\Delta] = \Gamma[\varphi,\mathcal{G}] + \frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}}\big(\phi(x)-\varphi(x)\big) + \frac{1}{2}\big(\phi(x)-\varphi(x)\big)\frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}}\big(\phi(y)-\varphi(y)\big) \\ + \frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\bigg|_{\substack{\phi=\varphi\\\Delta=\mathcal{G}}}(\Delta(x,y)-\mathcal{G}(x,y)) + \mathcal{O}(\hbar^3)\,.\end{aligned} \tag{23}$$

Similarly, we expand

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)} = \frac{\delta\Gamma[\phi,\Delta]}{\delta\phi(x)}\bigg|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} + \frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} \big(\phi(y)-\varphi(y)\big) + \mathcal{O}(\hbar^2)\ , \tag{24}$$

$$\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)} = \frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\bigg|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} + \mathcal{O}(\hbar^2)\ . \tag{25}$$

Putting everything together, we are left with (at order $\hbar^2$)

$$\begin{aligned}\Gamma[\phi,\Delta] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}] + \hbar^2\Gamma_{1\mathrm{PR}}[\varphi,\mathcal{G}] \\ + \frac{1}{2}\big(\phi(x)-\varphi(x)\big)\left[\frac{\delta^2\Gamma[\phi,\Delta]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} - \frac{2}{\hbar}\frac{\delta\Gamma[\phi,\Delta]}{\delta\Delta(x,y)}\right]\big(\phi(y)-\varphi(y)\big)\ .\end{aligned} \tag{26}$$

The terms in the second line cancel (see Exercise 3), eliminating the 1PR diagrams, and, after showing an admirable level of patience, we arrive at the final expression

$$\Gamma[\varphi,\mathcal{G}] = \Gamma_0[\varphi] + \hbar\Gamma_1[\varphi,\mathcal{G}] + \hbar^2\Gamma_2[\varphi,\mathcal{G}]\ . \tag{27}$$

Notice that we have two ways of obtaining the equation of motion for φ, one obtained by taking the partial functional derivative of $\Gamma[\varphi,\mathcal{G}]$ with respect to φ, and the other obtained from the stationarity condition (15). For these to be consistent, it is clear that we require that

$$\frac{\Gamma[\phi,\Delta]}{\delta\phi(x)}\bigg|_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} = \Big[J(x) + K(x,y)\phi(y)\Big]_{\substack{\phi=\varphi\\ \Delta=\mathcal{G}}} = 0\ , \tag{28}$$

and we can show it to be the case if we recall that J and K are functionals of ϕ and Δ. This approach is particularly useful when the extremum of the quantum effective action and the extremum of the classical action differ by more than some small amount [3]. Thus, φ satisfies the following quantum-corrected equation of motion at order $\hbar$:

$$\frac{\delta\Gamma[\varphi,\mathcal{G}]}{\delta\varphi(x)} = \frac{\delta S[\varphi]}{\delta\varphi(x)} + \frac{\hbar}{2}\frac{\delta G^{-1}(\varphi)}{\delta\varphi(x)}\,\mathcal{G} = -\partial^2\varphi(x) + m^2\varphi(x) + \frac{\lambda}{3!}\varphi^3(x) + \frac{\hbar\lambda}{2}\varphi(x)\mathcal{G}(x,x)\ , \tag{29}$$

wherein we recognise the one-loop tadpole proportional to $\mathcal{G}(x,x)$.

The 2PI effective action is a useful hammer, and it gives us a powerful organising principle for the resummation of loop corrections (see Exercise 3). Moreover, by making various choices of the source K, we can obtain the 1PI effective action ($K \to 0$) or preserve symmetry properties in truncations of the effective action. We can also use it to understand how theories change with energy scale via the exact renormalization group, and this will be the focus of the next lecture.

2.1 Exercises

Exercise 3: The Schwinger–Dyson equation

In the second lecture, we derived the order-$\hbar$ term in the 2PI effective action, which corresponded to the integral over quadratic fluctuations. In this exercise, we will derive the order-$\hbar^2$ corrections to the 2PI effective action, show that the 1PR diagrams cancel, and obtain the Schwinger–Dyson equation for the two-point function:

- Start by expanding the exponential in the partition function as a power series in the vertices $S^{(3)}$ and $S^{(4)}$, being careful with the $\hbar$ power counting.
- Use Wick's theorem to compute the functional integral in terms of the two-point function $\mathcal{G}$.
- Now consider the logarithm of this result and expand again to order $\hbar^2$ in the expression for the effective action.
- By deriving an expression for the one-point function ϕ, show that the 1PR diagrams cancel in the 2PI effective action.
- By varying the final result for the 2PI quantum effective action with respect to the two-point correlation function obtain the Schwinger–Dyson equation for the two-point function. What level of resummation does this amount to?

Exercise 4: Convexity and the Maxwell construction

Since it is defined via a Legendre transform, the effective action must be a convex functional. In this exercise, we will show, in the case of a non-convex classical potential, that this is ensured by the Maxwell construction (which you might have met in thermodynamics):

- Consider the $\lambda\phi^4$ theory with a tachyonic mass term $m^2 < 0$, such that the classical potential has two degenerate minima separated by a potential barrier. Since both minima will contribute equally to the partition function, we must sum over them.
- We want to calculate the effective potential and so are only interested in the zero mode or the infinite-wavelength contribution. This can be projected out by a neat trick, akin to Faddeev and Popov's. We will skip this step, and start from the result

$$Z[J,K] = \sum_{\pm} \exp\left[-\frac{\Omega}{\hbar}\left(\bar{S}(\varphi_\pm) - \bar{J}\varphi_\pm - \frac{1}{2}\bar{K}\varphi_\pm^2 + \frac{\hbar}{2\Omega}\mathrm{tr}\ln\mathcal{G}_\pm^{-1}G(0) + \mathcal{O}(\hbar^2)\right)\right] , \qquad (30)$$

  where Ω is a volume factor and $\varphi_\pm = \pm\sqrt{6}m/\sqrt{\lambda}$.
- Using the expression above, determine the one-point function in terms of $\varphi_\pm$ and $Z_\pm$.

- Rearrange this to find the ratio Z_+/Z_- and find an expression for $\bar{J}$. What happens when $\phi < \varphi_-$ or $\phi > \varphi_+$?
- Substitute for $\bar{J}$ in the exponents of the path integrals, working to zeroth order in $\hbar$ to keep things simple. These exponents should now be equal.
- Calculate the effective action and take the limit $\bar{K} \to 0$ to show that the effective potential $U = \Gamma/\Omega$ is convex.

References

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