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Advanced Quantum Field Theory

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3 Lecture 3: The geometry of theory space

In the last lecture, we made a sightseeing tour of the 2PI equantum effective action

$$\Gamma[\phi,\Delta] = -W[J,K] + J(x)\phi(x) + \frac{1}{2}K(x,y)\big[\phi(x)\phi(y) + \hbar\Delta(x,y)\big] \ . \tag{1}$$

To kick off this lecture, we are going to write the same effective action in the following more condensed form:

$$\Gamma[\bar{\phi},\bar{\Delta};\emptyset] = -W[\bar{J},\bar{K}] + \bar{J}_a\bar{\phi}^a + \bar{K}_\alpha\bar{\Delta}^\alpha \ , \tag{2}$$

where we have defined $\bar{J} = J$, $\bar{K} = K/2$, $\bar{\phi} = \phi$ and

$$\bar{\Delta}^\alpha = \Delta^{a_1a_2}(x_1,x_2) + \phi^{a_1}(x_1)\phi^{a_2}(x_2) \ , \tag{3}$$

and we are going to set $\hbar = 1$ from now on. (We worried a lot in previous lectures about working with connected n-point functions and keeping track of factors of $\hbar$, but not any more, and we will see why shortly.) Herein, we also use the full power of the DeWitt notation in which indices contain any number of discrete indices and a spacetime coordinate, and repeated indices are summed and integrated over, as appropriate. For example,

$$\bar{J}_a\bar{\phi}^a \equiv \sum_a \int \mathrm{d}^4x \ \bar{J}_a(x)\bar{\phi}^a(x) \ , \tag{4}$$

$$\bar{K}_\alpha\bar{\Delta}^\alpha = \sum_{a_1,a_2} \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \ \bar{K}_{a_1a_2}(x_1,x_2)\bar{\Delta}^{a_1,a_2}(x_1,x_2) \ . \tag{5}$$

All of this notation and the discussion that follows are based on [1].

In the previous lecture, we also remarked that there is actually a whole family of effective actions. In fact, even for our choice of two sources J and K, we have four functionals, which are

pairwise related by double Legendre transforms. Together with the 2PI effective action above, these four functionals are

$$\Gamma[\emptyset;\bar{J},\bar{K}] = -W[\bar{J},\bar{K}] \;, \tag{6}$$

$$\Gamma[\bar{\phi};\bar{K}] = -W[\bar{J},\bar{K}] + \bar{J}_a\bar{\phi}^a \;, \tag{7}$$

$$\Gamma[\bar{\Delta};J] = -W[\bar{J},\bar{K}] + \bar{K}_\alpha\bar{\Delta}^\alpha \;, \tag{8}$$

$$\Gamma[\bar{\phi},\bar{\Delta};\emptyset] = -W[\bar{J},\bar{K}] + \bar{J}_a\bar{\phi}^a + \bar{K}_{ab}\bar{\Delta}^{ab} \;. \tag{9}$$

The arguments of the functionals are such that any dependence on n-point functions is written to the left of the semicolon and any dependence on the sources is written to the right of the semicolon.

Each of these functionals depends on some number m of the sources and some number $2-m$ of the n-point functions. Thus, for each functional, we can define a coordinate Q_A whose upper components are the m sources and whose lower components are the $2-m$ n-point functions. For example, the Schwinger functional depends on the coordinate

$$Q_A = \begin{pmatrix} \bar{J}_a \\ \bar{K}_\alpha \end{pmatrix} \;, \tag{10}$$

on the infinite-dimensional manifold of sources — remember this is all functional — this is our configuration space.

So now we can start making use of all the technology from differential geometry, and our first task is to introduce an affine connection, which is defined by the functional derivatives with respect to the coordinate Q_A. Thus, given some tensor $T^{C_1C_2\cdots}_{B_1B_2\ldots}[Q]$, we can introduce the connection $\bar{D}$ acting as

$$\bar{D}^A T^{C_1C_2\cdots}_{B_1B_2\ldots}[Q] = \frac{\delta}{\delta Q_A} T^{C_1C_2\cdots}_{B_1B_2\ldots}[Q] \;. \tag{11}$$

We can also introduce a metric based on the Hessian of some functional of the coordinates $\bar{F}[Q]$

$$g^{AB} = \bar{D}^A\bar{D}^B\bar{F}[Q] = \frac{\delta^2\bar{F}[Q]}{\delta Q_A\delta Q_B} \;, \tag{12}$$

where the latter equality holds only in the affine coordinate frame. The geometry that results from this form of metric is called a *Hessian geometry*, and our configuration space is a *Hessian manifold*. Such geometries are common in information theory, so much so that we can refer to information geometry. The Fischer metric is one example.

The affine connection $\bar{D}$ may not be metric compatible, e.g., $\bar{D}^A g^{BC} \neq 0$, in general, and we can introduce a tensor

$$Y^A_{BC} = \frac{1}{2}\bar{D}^A g_{BC} \;, \tag{13}$$

named for Amari and Chentsov (see [2] and references therein), which keeps track of this. We can

use the Amari–Chentsov tensor to construct new connections, and two are particularly interesting. The first is the metric-compatible, Levi–Civita connection

$$\nabla = \bar{D} - Y \otimes \mathbb{I} \,, \tag{14}$$

where the tensor product with the identity map $\mathbb{I}$ takes into account that the Amari–Chentsov tensor acts in accordance with $\bar{D}$ on any given tensor [3]. The second is

$$D = \bar{D} - 2Y \otimes \mathbb{I} \,, \tag{15}$$

which is always affine, and the affine coordinates are the functional derivatives of the functional $\bar{\Gamma}[Q]$

$$P^A = \frac{\delta \bar{F}[Q]}{\delta Q_A} \,, \tag{16}$$

such that

$$D_A = \frac{\delta}{\delta P^A} \,. \tag{17}$$

Thus, we see that the coordinates P^A are dual to the coordinates Q_A. Going further, we can define the inverse of the metric g^{AB} in terms of the Hessian of a functional $F[P]$ via

$$g_{AB} = \frac{\delta F[P]}{\delta P^A \delta P^B} \,. \tag{18}$$

This functional

$$F[P] = -\bar{F}[Q] + Q_A P^A \tag{19}$$

is nothing other than the Legendre transform of $\bar{F}[Q]$. So now we can associate $\bar{F}[Q]$ with the Schwinger functional $W[J,K] = -\Gamma[\emptyset; \bar{J}, \bar{K}]$, $F[P]$ with the 2PI effective action $\Gamma[\bar{\phi}, \bar{\Delta}; \emptyset]$, and

$$P^A = \begin{pmatrix} \bar{\phi}^a \\ \bar{\Delta}^\alpha \end{pmatrix} \,, \tag{20}$$

and we can write

$$g_{AB} = \frac{\delta Q_A}{\delta P^B} \,. \tag{21}$$

With this technoloy, our n-point correlation functions are now obtained from action of the $\bar{D}$ connection on the Schwinger functional, i.e.,

$$\langle \bar{\phi}^{a_1} \cdot \bar{\phi}^{a_n} \rangle_{\text{conn.}} = \bar{D}^{a_1} \cdots \bar{D}^{a_n} W[\bar{J}, \bar{K}] \,. \tag{22}$$

And now we should be thankful that this connection is not metric compatible, because otherwise we would have no n-point functions higher than the two-point function! Moving away from the

affine coordinate frame of barred variables, we can express everything entirely in terms of the the connected two-point functon Δ and the field ϕ and write

$$\langle \bar{\phi}^{a_1} \cdots \bar{\phi}^{a_n} \rangle_{\text{conn.}} = \left[\prod_{j=1}^{n-2} g^{a_j B_j} D_{B_j} \right] \Delta^{a_{n-1} a_n} \ , \tag{23}$$

where

$$\begin{aligned} g^{aB} D_B &= \frac{\delta W[J,K]}{\delta \bar{J}_a \delta \bar{J}_b} \frac{\delta}{\delta \bar{\phi}^b} + \frac{\delta^2 W[J,K]}{\delta \bar{J}_a \delta \bar{K}_\beta} \frac{\delta}{\delta \bar{\Delta}^\beta} \\ &= \Delta^{ab} \left[\frac{\delta}{\delta \phi^b} - \frac{\delta^2 \Gamma}{\delta \phi^b \delta \Delta^\gamma} \left(\frac{\delta^2 \Gamma}{\delta \Delta^\gamma \delta \Delta^\delta} \right)^{-1} \frac{\delta}{\delta \Delta^\delta} \right] \ . \end{aligned} \tag{24}$$

This is all very interesting, but what does it mean for the physics. Well, given that the one- and two-point functions $\bar{\phi}$ and $\bar{\Delta}$ are functionals of the sources $\bar{J}$ and $\bar{K}$, changing these sources will move us around in our configuration space or equivalently define trajectories on the associated Hessian manifold. Suppose then that we let the source K depend on some parameter t, such that

$$\frac{\mathrm{d}\bar{K}_\alpha}{\mathrm{d}t} = \text{constant} \ . \tag{25}$$

This will define a particular trajectory along which the source K increases linearly with t. Given that

$$\bar{K}_\alpha = \frac{\delta \Gamma}{\delta \bar{\Delta}^\alpha} \ , \tag{26}$$

and taking an additional derivative with respect to t

$$\begin{aligned} \frac{\mathrm{d}^2 \bar{K}_\alpha}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\delta \Gamma}{\delta \bar{\Delta}^\alpha} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\delta^2 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\phi}^b} \frac{\mathrm{d}\bar{\phi}^b}{\mathrm{d}t} + \frac{\delta^2 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\Delta}^\beta} \frac{\mathrm{d}\bar{\Delta}^\beta}{\mathrm{d}t} \right] \\ &= \frac{\delta^2 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\phi}^b} \frac{\mathrm{d}^2 \bar{\phi}^b}{\mathrm{d}t^2} + \frac{\delta^2 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\Delta}^\beta} \frac{\mathrm{d}^2 \bar{\Delta}^\beta}{\mathrm{d}t^2} + \frac{\delta^3 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\phi}^b \delta \bar{\phi}^c} \frac{\mathrm{d}\bar{\phi}^b}{\mathrm{d}t} \frac{\mathrm{d}\bar{\phi}^c}{\mathrm{d}t} + \frac{\delta^3 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\phi}^b \delta \bar{\Delta}^\gamma} \frac{\mathrm{d}\bar{\phi}^b}{\mathrm{d}t} \frac{\mathrm{d}\bar{\phi}^\gamma}{\mathrm{d}t} \\ &\quad + \frac{\delta^2 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\Delta}^\beta \delta \bar{\phi}^c} \frac{\mathrm{d}\bar{\Delta}^\beta}{\mathrm{d}t} \frac{\mathrm{d}\bar{\phi}^c}{\mathrm{d}t} + \frac{\delta^2 \Gamma}{\delta \bar{\Delta}^\alpha \delta \bar{\Delta}^\beta \delta \bar{\Delta}^\gamma} \frac{\mathrm{d}\bar{\Delta}^\beta}{\mathrm{d}t} \frac{\mathrm{d}\bar{\Delta}^\gamma}{\mathrm{d}t} \ . \end{aligned} \tag{27}$$

Re-expressing this in terms of the geometric objects defined previously, the first term is just

$$g_{\alpha B} \frac{\mathrm{d}^2 P^B}{\mathrm{d}t^2} \tag{28}$$

and the second term can be written as

$$D_\alpha g_{CD} \frac{\mathrm{d}P^C}{\mathrm{d}t} \frac{\mathrm{d}P^D}{\mathrm{d}t} = g_{\alpha B} \bar{D}^B g_{CD} \frac{\mathrm{d}P^C}{\mathrm{d}t} \frac{\mathrm{d}P^D}{\mathrm{d}t} = 2 g_{\alpha B} Y^B_{CD} \frac{\mathrm{d}P^C}{\mathrm{d}t} \frac{\mathrm{d}P^D}{\mathrm{d}t} \ . \tag{29}$$

Putting everything together, we arrive at the following equation:

$$g_{\alpha B} \left[\frac{\mathrm{d}^2 P^B}{\mathrm{d}t^2} + 2 Y^B_{CD} \frac{\mathrm{d}P^C}{\mathrm{d}t} \frac{\mathrm{d}P^D}{\mathrm{d}t} \right] = 0 \ . \tag{30}$$

In cases where the bracket itself vanishes, this is nothing other than the geodesic equation of the $\bar{D}$ connection

$$\frac{\mathrm{d}^2 P^B}{\mathrm{d}t^2} + 2 Y^B_{CD} \frac{\mathrm{d}P^C}{\mathrm{d}t} \frac{\mathrm{d}P^D}{\mathrm{d}t} = 0 \ . \tag{31}$$

These trajectories are just straight lines in the affine coordinate frame of $\bar{D}$, along which the t-derivatives of the sources are, by construction, conserved.

Now, we recall that the source $K(x, y)$ is accompanied by two factors of the field in the path integral. By an appropriate choice K, we can therefore make the field more or less massive. Moreover, since the source K is coordinate dependent, we can use its Fourier transform $\tilde{K}(p)$ to make certain Fourier modes of the field more or less massive. This allows us to re-weight contributions from particular modes, e.g., freezing modes with momentum p less than some scale k, such that we only integrate out modes with momentum $p > k$ when calculating the partition function. In this way, the Fourier transform of the two-point source can be used to implement a smooth cut-off, and we call it a *regulator*. We can then ask how the effective action changes as a function of the scale k — we call this the *renormalization group* or *RG evolution*.

So returning to Eq. (30), if we associated $t \propto \ln k$ with the so-called RG time, then the geodesics of the $\bar{D}$ connection on the configuration space can be used to specify surfaces of constant RG time or to construct RG trajectories. We can make more sense of this by considering the variation of the effective action with respect to t for fixed ϕ. This gives

$$\partial_t \Gamma[\phi, \Delta] = \frac{\delta \Gamma[\phi, \Delta]}{\delta \Delta^\alpha} \, \partial_t \Delta^\alpha = K_\alpha \partial_t \Delta \ , \tag{32}$$

where we have used that Δ is a functional of K. This is the RG flow equation of the 2PI effective action. By acting on it with the $\bar{D}$ connection, we can obtain systems of flow equations for the n-point correlation functions.

Returning then to the geometrical picture, we see that changes in the RG time lead to changes in the two-point correlation function at fixed ϕ. This defines a vector field on the configuration space that connects hypersurfaces of constant RG time. Thus, the renormalization group tells us how we move around in theory space, spanned here by the correlation functions. In the next lecture, we will see how the same renormalization group evolution moves us around in the same

theory space when we span it by the parameters (or couplings) of our theory.

3.1 Exercises

Exercise 5: Obtaining vertex functions

In this exercise, we will prove the convexity of the 2PI effective action, and use the resulting expressions to obtain the operator in Eq. (24). To make our lives easier, we will forget the functional structure and just consider the Schwinger functional and the effective as functions of numbers $\bar{J}$ and $\bar{K}$, and $\bar{\phi}$ and $\bar{\Delta}$, respectively.

- Show that the product of Hessians of the Schwinger "function" with respect to $\bar{J}$ and $\bar{K}$ and the effective action with respect to $\bar{\phi}$ and $\bar{\Delta}$ are inverses of one another. This is encoded in the geometric picture by $g_{AC}g^{CB} = \delta_A^B$. Noting that the Hessian of the Schwinger function is a covariance matrix, argue that the effective action is convex.

- Using the above results, obtain expressions for

$$\frac{\partial W}{\partial \bar{J}\partial \bar{J}} \qquad \text{and} \qquad \frac{\partial^2 W}{\partial \bar{J}\partial \bar{K}} \tag{33}$$

  in terms of the 2PI effective action, $\bar{\phi}$ and $\bar{\Delta}$.

- Using the same results, change coordinates from $(\bar{\phi}, \bar{\Delta})$ to (ϕ, Δ) and write the connected two-point correlation function Δ in terms of variations of the 2PI effective action.

- Hence show that

$$\begin{aligned} g^{aB}D_B &= \frac{\partial W(J,K)}{\partial \bar{J}\partial \bar{J}}\frac{\partial}{\partial \bar{\phi}} + \frac{\partial^2 W(J,K)}{\partial \bar{J}\partial \bar{K}}\frac{\partial}{\partial \bar{\Delta}} \\ &= \Delta\left[\frac{\partial}{\partial \phi} - \frac{\partial^2 \Gamma}{\partial \phi \partial \Delta}\left(\frac{\partial^2 \Gamma}{\partial \Delta \partial \Delta}\right)^{-1}\frac{\partial}{\partial \Delta}\right] . \end{aligned} \tag{34}$$

  This operator is particularly useful for obtaining expressions for the proper n-point vertices from the 2PI effective action by acting on the connected two-point function obtained above.

You might find [4, 5, 6] — all with a Nottingham connection — helpful.

References

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