BUSSTEPP 2025
Advanced Quantum Field Theory

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1 Lecture 1: theory space

The aim of this lecture series is deceptively simple: it is to think about the question, "How do we build theories?". As the whimsical but nevertheless wise Winnie-the-Pooh might say, before one can come to a theory, one might first like to know what a theory is and whether there is any reason to fear such a thing, supposing, as a matter of fact, a theory were ever to jump out from behind a gorse bush and say, "How do you do?". Rest assured, the theories that we will consider in these lectures are all of upright character, but defining what a theory is is not a bad place at all for us to start, so let's start there.

Often, in the spirit of logical positivists like Ernst Mach, we talk of wanting to explain empirical observations. For those of us battling our way through BUSSTEPP, the observed phenomena are likely to be the properties and interactions of elementary particles. But, actually, we might want to do more than explain *observed* phenomena; we might want to predict *new* phenomena for which we (or at least those of us with a flair for experiment or observation) can go off and look. The history of particle physics has seen theories play both these roles.

So, for us, a theory will be a mathematical description of the dynamics of elementary particles that takes some inputs, which we might call model parameters, applies some rules, which we will give some fancy title like *relativistic quantum mechanics* or *quantum field theory*, and makes some prediction. So we have three things to do: (1) set down the mathematical language and rules, (2) decide on the input parameters (i.e., build the model) and (3) derive the relationships between those input parameters and physical observables. We will need to accept that (3) is not independent of (1).

Now, there was a time, not so long ago, when the prevailing attitude was that the only good theories were ones that had a finite number of input parameters and a much larger number of physical observables. This is a win for predictivity, since we can eliminate the input parameters, stick with Mach, and concentrate only on the relationships between physical observables.

Theories like this would be highly coveted within the reductionist philosophy that has shaped much of modern physics, and they would be given the less-than-transparent label of being *perturbatively renormalizable*. (Don't worry, we will come back to what this means later.) But times inevitably change — sometimes for the better and sometimes for the worst — and it turns out that there are many physically relevant theories that do not find themselves lucky enough to be part of the perturbatively renormalizable clique. These theories are called *effective*, and they are in good company, alongside General Relativity (still our best description of classical gravitation), chiral perturbation theory (describing the dynamics of hadrons), the Ginzburg–Landau theory of superconductivity (We have much to learn from our condensed-matter colleagues.) and, indeed, the much-celebrated Standard Model of particle physics if we consider it the low-energy limit of some larger theory (that might, for example, offer an explanation of the origins of dark matter or dark energy). The latter is captured in the Standard Model Effective Field Theory that you will hear about a lot (but not so much in these lectures).

Let's turn then to our little excursion into *advanced quantum field theory*. (In fact, I think all quantum field theory is pretty advanced. And given that we only have four short lectures, we had better rename these lectures *Some Quantum Field Theory*, or better still *Some of the Quantum Field Theory that Pete knows a little about*. Notwithstanding this, it is often the basic foundations that compose the most "advanced" topics.) So what is it that we want our theories to describe? Well, let's restrict ourselves to writing down theories that describe the microscopic origin of the known fundamental forces. We'll leave out gravity — that would need another lecture series — and we'll stick to toy models, too, and just scalar fields, since higher spins are a technical complication to which we do not need to subject ourselves. This will still be more than enough for us to gain some understanding of what we mean by *theory space*.

Now to complete step (1): set down the mathematical language and rules. However, when we said earlier that we would be writing down theories of the dynamics of the elementary particles, we weren't being entirely honest. In fact, we will write down theories of fields — whatever they are — and we will try to interpret the excitations of these fields as elementary particles. It will turn out that defining particles is not the easiest of tasks. We'll worry about this a little more in Exercise 1.

There are a number of ways that we can approach quantum field theory:

- the canonical operator formulation.
- the path integral formulation.
- the operator product expansion.

We will make use of the first two, and leave the third to people who know about conformal field theory.

So, how do we write down a quantum field theory? We would be forgiven for thinking that we just need to write down an action, say

$$S[\Phi] = \int \mathrm{d}^4x \left[\eta^{\mu\nu} Z_{ab} \partial_\mu \Phi_a \partial_\nu \Phi_b - V^{(1)}_a \Phi_a - V^{(2)}_{ab} \Phi_a \Phi_b - V^{(3)}_{abc} \Phi_a \Phi_b \Phi_c - V^{(4)}_{abcd} \Phi_a \Phi_b \Phi_c \Phi_d + \cdots \right] . \tag{1}$$

Here, the square brackets indicate that the action is a *functional* of the field Φ, i.e., it maps functions to a number, $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric, and we consider N real scalar fields Φ_a with $a \in \{1, 2, \ldots, N\}$. We have certainly written down a field theory, but we have not yet written down a *quantum* field theory. The latter is composed of two things:

1. Some generators of the dynamics. (In Minkowski spacetime, these are the generators of the Poincaré group and, in particular, the Hamiltonian, the generator of time translation, for which the action will serve as a proxy.)

2. Some state space around which the generators of dynamics move us. (This will be something like a Hilbert space, but there are some subtleties related to how we deal with continuous spectra, for example, that we won't go into in these lectures.)

We often forget that there is freedom in how we choose *both* of these ingredients of our quantum field theory. However we choose these ingredients, the challenge is to find useful bases to span this state space.

One choice is the basis of *field eigenstates*, which satisfy the eigenequation

$$\hat{\Phi}(t, \mathbf{x}) \, |\Phi(\mathbf{x}), t\rangle = \Phi(\mathbf{x}) \, |\Phi(\mathbf{x}), t\rangle \ . \tag{2}$$

The state $|\Phi(\mathbf{x}), t\rangle$ is an instantaneous eigenstate of the Heisenberg-picture field operator $\hat{\Phi}(t, \mathbf{x})$ at time t with "eigenvalue" $\Phi(\mathbf{x})$. These states are orthonormal and complete in a functional sense with

$$\int [\mathrm{d}\Phi(\mathbf{x})] \, |\Phi(\mathbf{x}), t\rangle \, \langle \Phi(\mathbf{x}), t| = \mathbb{I} \, . \tag{3}$$

This basis is useful for constructing the path integral, which we will come to shortly and derive explicitly in the field eigenbasis in Exercise 2. Note that the field eigenstate is not really a function of the spatial coordinate $\mathbf{x}$. We include the argument to remind us that it contains information about the configuration of the system at all points in space.

Another choice is the Fock basis, composed of multiparticle states and generated from the right-action of creation operators on the Fock vacuum state $|0(t)\rangle$. (The definition of this vacuum state is somewhat problematic, as we will see in Exercise 1.) For a single real scalar field, we have

$$\hat{a}^\dagger(\mathbf{p}, t) \, |0(t)\rangle = |\mathbf{p}(t)\rangle \qquad \text{with} \qquad \hat{a}(\mathbf{p}, t) \, |0(t)\rangle = 0 \, . \tag{4}$$

The single-particle momentum states are orthonormal,

$$\langle \mathbf{p}_1(t)|\mathbf{p}_2(t)\rangle = (2\pi)^3 2E(\mathbf{p})\delta^{(3)}(\mathbf{p}_1 - \mathbf{p}_2) \,, \tag{5}$$

where $E_j(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_j^2}$, and the completeness of the Fock basis takes the unwieldy form

$$\begin{aligned} &|0(t)\rangle \langle 0(t)| + \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2E(\mathbf{p}_1)} \; |\mathbf{p}_1(t)\rangle \langle \mathbf{p}_1(t)| \\ &\quad + \frac{1}{2!}\int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2E(\mathbf{p}_1)}\frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2E(\mathbf{p}_2} \; |\mathbf{p}_1(t), \mathbf{p}_2(t)\rangle \langle \mathbf{p}_1(t), \mathbf{p}_2(t)| + \cdots = \mathbb{I} \,. \end{aligned} \tag{6}$$

Here, we have assumed that the states are orthonormal with respect to the Dirac inner product, i.e., the bra state is just the Hermitian conjugate of the ket state. More generally, we can introduce a metric on the state space, which will play a role similar to a weight function in the orthonormality of the basis. We might be tempted (and, in fact, we should be) to ask why we tend to model build only at the level of the classical action and not also in terms of the way we build the state space.

Either way, we now have our first way of charting our theory space: we can use the parameters of the theory as coordinates, or more precisely the coefficients of the various "operators" appearing in the action. Step (2), building the model, then amounts to deciding which of these operators to include, and how we might be guided to choose them.

However, we are also interested in how the system evolves, and one way to find this out is to kick it and see what happens. The slightly more scientific approach to kicking a quantum field theory is to introduce external sources J_a that perturb the system away from its ground state and measure changes in correlations in the system. A convenient way to generate expressions for these correlation functions is to introduce the particular path integral

$$Z[J] = \mathcal{N}\int \mathcal{D}\Phi \; \exp\left[\frac{i}{\hbar}\left(S[\Phi] - \int \mathrm{d}^4x \; J_a(x)\Phi_a(x)\right)\right] \,, \tag{7}$$

and take functional derivatives with respect to the sources. But let's not go too fast. The functional integral over field configurations Φ is not convergent, because the integrand is oscillatory. We can tame this by taking $V^{(2)}_{ab} \to V^{(2)}_{ab} - \delta_{ab} i\epsilon/2$, where $\epsilon = 0^+$. This is the Feynman prescription, and it will have two effects: first, it will allow us to invert the Klein–Gordon operator

$$\frac{\delta^2 S[\Phi]}{\delta\Phi_{a_1}(x_1)\delta\Phi_{a_2}(x_2)} \tag{8}$$

to obtain the tree-level two-point correlation function; and second, it will give us time ordering of the resulting correlation functions. Moreover, this prescription allows us to construct the Wick contour to rotate to Euclidean space by analytically continuing $t \to -i\tau$. We can then identify the

classical Euclidean action with the classical free energy, and it is for this reason that we often refer to $Z[J]$ as the *partition function*.

With this prescription, $Z[J]$ acts as a generating functional of time-ordered correlation functions. More specifically, we can obtain the n-point time-ordered correlation function via

$$\langle \mathrm{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)]\rangle_J = \frac{1}{Z[J]}\left(\prod_{j=1}^{n} i\hbar\frac{\delta}{\delta J_{a_j}(x_j)}\right) Z[J]\ , \tag{9}$$

where the operator T ensures that the fields are ordered with the latest time to the left. However, the correlation functions obtained from $Z[J]$ will also contain disconnected pieces. To generate only connected correlation functions, we can vary the *Schwinger functional*

$$W[J] = i\hbar \ln Z[J]\ , \tag{10}$$

whose Euclidean counterpart is related to the free energy. For example, the connected two-point function is

$$\begin{aligned}\hbar\Delta_{a_1a_2}(x_1,x_2) &= i\hbar\frac{\delta^2 W[J]}{\delta J_{a_1}(x_1)\delta J_{a_2}(x_2)} = \frac{(i\hbar)^2}{Z[J]}\frac{\delta^2 Z[J]}{\delta J_{a_1}(x_1)\delta J_{a_2}(x_2)} - \frac{(i\hbar)^2}{Z^2[J]}\frac{\delta Z[J]}{\delta J_{a_1}(x_1)}\frac{\delta Z[J]}{\delta J_{a_2}(x_2)} \\ &= \langle \mathrm{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)]\rangle_J - \langle\Phi_{a_1}(x_1)\rangle_J\langle\Phi_{a_2}(x_2)\rangle_J\ .\end{aligned} \tag{11}$$

Of course, we can add any sources that we want, and it will prove useful to introduce a source for the two-point correlation function. To this end, we generalise the partition function to

$$Z[J,K] = \int \mathcal{D}\Phi\ \exp\left[\frac{i}{\hbar}\left(S[\Phi] - \int \mathrm{d}^4x\ J_a(x)\Phi_a(x) - \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\ K^{ab}(x,y)\Phi_a(x)\Phi_b(y)\right)\right]\ , \tag{12}$$

with the Schwinger functional becoming

$$W[J,K] = i\hbar\ln Z[J,K]\ . \tag{13}$$

(We could keep going and add a three-point source and so on. Let's not.) We can now obtain the two-point function from a single variation with respect to the two-point source $K(x,y)$; namely,

$$\langle \mathrm{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)]\rangle_{J,K} = 2\frac{\delta}{\delta K_{a_1a_2}(x_1,x_2)}W[J,K] = \frac{2i\hbar}{Z[J,K]}\frac{\delta}{\delta K_{a_1a_2}(x_1,x_2)}Z[J,K]\ . \tag{14}$$

However we choose to obtain the correlation functions, we now have the second way of charting our theory space: we can use the correlation functions of the theory as coordinates. Moreover, in having written down the generating functionals in the path integral representation, we are now well on our way to completing step (3).

Summary

Taking stock, we have two ways to chart our theory space:

1. We can use the parameters of our theory as coordinates and, more precisely, the coefficients of the various operators appearing in the action.

2. We can use the correlation functions of our theory, and we have two useful objects: the partition function and the Schwinger functional for generating them.

In the next lecture, we will introduce the quantum effective action, which will prove to be invaluable for understanding how to move (or rather how quantum mechanics moves us) around in theory space.

1.1 Exercises

Exercise 1: Haag's theorem

In this exercise, following an argument by Rudolf Haag [1], we will uncover an important subtlety of the Fock representation. In doing so, we'll revisit the canonical operator formalism for the real scalar field, come across Bogoliubov transformations, and think about particle creation.

Consider two of our field operators $\hat{\Phi}_1(x)$ and $\hat{\Phi}_2(x)$, with masses m_1 and m_2, respectively, and with corresponding creation and annihilation operators $\hat{a}^\dagger_{1,2}(\mathbf{k}, 0)$ and $\hat{a}_{1,2}(\mathbf{k}, 0)$:

- Choose the fields and their conjugate momenta $\hat{\Pi}_{1,2}(t, \mathbf{x}) = \partial_t \hat{\Phi}_{1,2}(t, \mathbf{x})$ to have the same initial conditions at $t = 0$, i.e., $\hat{\Phi}_1(0, \mathbf{x}) = \hat{\Phi}_2(0, \mathbf{x})$ and $\hat{\Pi}_1(0, \mathbf{x}) = \hat{\Pi}_2(0, \mathbf{x})$.

- Show that the creation and annihilation operators of the two fields are related via

$$\hat{a}_2(\mathbf{k}, 0) = \cosh\theta\, \hat{a}_1(\mathbf{p}) + \sinh\theta\, \hat{a}^\dagger_1(\mathbf{p}) \,, \tag{15}$$

$$\hat{a}_1(\mathbf{k}, 0) = \cosh\theta\, \hat{a}_2(\mathbf{p}) - \sinh\theta\, \hat{a}^\dagger_2(\mathbf{p}) \,, \tag{16}$$

  and determine θ.

- Consider what this implies for the action of the operator $\hat{a}_2(\mathbf{p}, 0)$ on the vacuum state for the operator $\hat{a}_1(\mathbf{p}, 0)$, i.e., that state for which $\hat{a}_1(\mathbf{p}, 0)\,|0\rangle = 0$.

- Now let $\hat{\Phi}_2$ be the full, interacting (Heisenberg) field and $\hat{\Phi}_1$ be the free, interaction-picture field, such that the masses m_1 and m_2 also coincide at time $t = 0$. The mass m_2 will differ from m_1 for times $t > 0$ due to the interactions — we might call this a fancy name too: *vacuum polarisation*. What do your conclusions above imply for the action of the interaction-picture operators on the interacting vacuum?

- When might this subtlety be relevant?

Exercise 2: The path integral

Often, the path integral is derived for the quantum harmonic oscillator and then extrapolated to quantum fields. In this exercise, we'll derive the path integral for a real scalar field directly, and we'll make sure to keep track of boundary conditions.

- Start by considering discrete time steps. What is the overlap $\langle \Phi_{j+1}(\mathbf{x}), t_{j+1} | \Phi_j(\mathbf{x}), t_j \rangle$ between a field eigenstate at time t_{j+1} and a field eigenstate at time t_j? Rewrite this in terms of the canonical time-evolution operator for the real scalar field, but remember that the subscripts on the field arguments continue to label different integrals over field configurations. We will need to use the eigenequation for the field eigenstates, introduce a basis $|\Pi(\mathbf{x}), t\rangle$ of eigenstates of the conjugate-momentum operator $\hat{\Pi}(t, x)$, and use the corresponding eigenequation, too. The following expression will prove useful:

$$\langle \Phi(\mathbf{x}) | \Pi(\mathbf{x}) \rangle = \exp\left[\frac{i}{\hbar} \int \mathrm{d}^3\mathbf{x}\, \Pi(\mathbf{x}) \Phi(\mathbf{x}) \right] \ . \tag{17}$$

  How can we make the resulting integral over the conjugate-momentum field convergent?

- Now consider the vacuum persistence amplitude for vanishing source

$$Z[0] = \langle 0(+\infty) | 0(-\infty) \rangle \ , \tag{18}$$

  insert N complete sets of field eigenstates at discrete time steps, and make use of your result above for $\langle \Phi_{j+1}(\mathbf{x}), t_{j+1} | \Phi_j(\mathbf{x}), t_j \rangle$.

- Finally, take the limit $N \to \infty$ to obtain the continuum result, and introduce a non-vanishing source to obtain the partition function $Z[J]$.

References

[1] R. Haag, "On quantum field theories," Kong. Dan. Vid. Sel. Mat. Fys. Med. **29N12** (1955), 1-37 [CERN-55-08].

[2] S. B. Giddings and J. Perkins, "Challenges for describing unitary evolution in nontrivial geometries: pictures and representations," [arXiv:2507.13351 [hep-th]].