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Field Theory Techniques for Gravitational Waves

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Abstract: These lectures introduce some modern field theory techniques for the relativistic two-body problem. The primary motivation is to describe the gravitational waveform that a distant observer would detect from a quasi-circular inspiral of two compact objects; predictions of this waveform are directly relevant for the current LIGO/Virgo/KAGRA network of gravitational wave detectors, as well as future missions such as LISA and Einstein Telescope/Cosmic Explorer. The Post-Newtonian expansion is discussed with a focus on how effective field theory can be used to efficiently compute and organise high-order terms in perturbation theory. In order to separate the key conceptual points from the problem of manipulating lengthy tensor expressions, a simple toy model in which the gravitational force is mediated by a scalar field is introduced (and completely solved).

All questions or corrections are welcome and can be sent to: s.melville@qmul.ac.uk

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1 Introduction

Motivation. The advent of gravitational wave astronomy is exciting for many different communities and for lots of different reasons. As a result, "gravitational wave science" (broadly defined) now spans a vast range of different fields; everything from quantum optics and quantum computing to seismology. These lectures will not attempt to cover such a broad scope, and instead will focus on one aspect of gravitational waves that I find particularly interesting:

$$\text{Gravitational waves} \;=\; \text{New window into fundamental physics.} \tag{1.1}$$

The basic idea behind (1.1) is that if gravity in our Universe is described by:

$$G_{\mu\nu} = \frac{2}{M_P^2} T_{\mu\nu} + ... \tag{1.2}$$

where the "+..." indicates possible modifications from Einstein's field equations[1], then gravitational waves are an entirely new way to probe the "+...". The range of mass / length scales that are relevant when an observable gravitational wave is generated (e.g. by the binary merger of two black holes) is unlike any previous solar system or cosmological test of gravity. So gravitational waves (GWs) have the potential to reveal completely new things about the gravitational sector of our Universe.

GW anatomy. The gravitational waveforms detected by LIGO/Virgo/KAGRA are typically treated as three separate problems:

(i) *inspiral*, when the two compact objects are sufficiently far apart that they are non-relativistic and can be treated as a *perturbation around Newtonian orbits*,

(ii) *merger*, when the two compact objects are sufficiently close that there are no large hierarchies in the system: this means there is no small parameter to expand in, but also that *numerical relativity* simulations can be very effective[2]

(iii) *ringdown*, when the objects have merged into a single remnant, which is initially in a highly excited state and settles down into Schwarschild or Kerr by radiating its excess energy as gravitational waves—this is well described by *black hole perturbation theory*.

Old results. Already in 1938, Einstein and others had recognised that General Relativity would correct Newtonian orbits and lead to a fairly complicated pattern of gravitational wave emission. In fact, for two point particles separated by a distance $\mathbf{r}$ and moving with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, the famous Einstein-Infeld -Hoffmann result for their potential energy is

$$V = \frac{\kappa}{r} + \frac{\kappa}{2r}\left[3(\mathbf{v}_1^2 + \mathbf{v}_2^2) - 7(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{(\mathbf{v}_1 \cdot \mathbf{r})(\mathbf{v}_2 \cdot \mathbf{r})}{r^2}\right] - \frac{\kappa^2}{2\mu r^2} + ... \tag{1.3}$$

where $\kappa = G_N m_1 m_2$ and $1/\mu = 1/m_1 + 1/m_2$ depend on the masses of the two bodies. Here the +... represent higher order terms in the small velocity v^2/c^2 or the weak gravitational field κ/r. Even earlier, in 1918, Einstein derived what is now called the "quadrupole formula" for calculating how much energy would be emitted by a system of masses once they trajectories $\mathbf{x}(t)$ are known. These two results already set up the modern "gravitational wave problem", which in a nutshell is:

(a) find the equations of motion that govern the point masses (e.g. the potential),

(b) solve them to find $\mathbf{x}(t)$,

(c) use the quadrupole formula and its higher-order extensions to compute the power emitted in gravitational waves (also its precise waveform, but more on this later).

[1] A common complaint at this stage is: "why modify GR? It's already hard enough." The observational motivation for the +... is that it could provide a microscopically consistent explanation of dark energy (without the fine tuning problems suffered by a cosmological constant). The theoretical motivation is that we know GR is incomplete since it does not include quantum mechanics, so it *must* be modified at some small length scale (which may be significantly lower than the Planck scale). Furthermore, just because GR is already difficult, that does not mean that its successor will be more difficult. In fact, there are plenty of historical examples of theories becoming simpler as they get more fundamental; e.g. Kepler's elliptical orbits are much simpler than iterating large numbers of epicycles.

[2] As a rough rule of thumb, it is more difficult to simulate a system if there are large separations of scale, e.g. if you want to resolve the details of the gravitational field around two small objects that are a very large distance apart, you need a very big box with very fine grid spacing.

	v^0	v^2	v^4	v^6	v^8	v^{10}
G^1	1	1	1	3	3	3
G^2	1	8	7	23	72	
G^3	5	48	212	2702		
G^4	8	317	41676			
G^5	50	116498				
G^6	27582					

(1.4)

Table 1: Number of terms contributing to the conservative potential of two spinless point-like particles in General Relativity up to (next-to)^5-leading order in the velocity v/c and weak field GM/r expansions (in the particular EFT approach of 1104.1122, 1903.05113 and 2010.13672).

What makes this problem tricky is that it is *non-linear*. As the system radiates gravitational waves it loses energy, and this has an effect (called "radiation reaction") on the trajectories $\mathbf{x}(t)$. So in practice one needs to iterate (a-c) repeatedly, each time arriving at a more precise estimate of the radiated power and the orbital motion of the particles. The complexity of this problem grows exponentially with the number of iterations; Table 1 shows the rapid rise in the number of terms appearing at higher orders in perturbation theory (in v^2/c^2 and $\kappa/r \sim G$).

New results. I will now skip over almost 100 years of crucial theoretical development and important insights. This includes the early work on post-Newtonian theory, and developing perturbative schemes in which the Einstein equations can be consistently solved (roughly along the lines (a-c) above). This is a huge disservice to all those that have worked in this field. But since we have only four hours of lectures, I have chosen to highlight how various ideas from quantum field theory have recently been applied to the gravitational wave problem. This includes ideas from effective field theory and renormalisation, as well as modern amplitude techniques. These new ideas (as well as the older classical approaches) have now matured to the point where very high orders in perturbation can be computed and analysed. They provide a systematic way to organise the huge number of different contributions that can arise, and ultimately break the problem down into manageable chunks.

These notes. My goal is to give pedagogical lectures that sketch the big picture ideas and roughly how these modern calculations work. I have tried to include many references to the literature, particularly the EFT-inspired approaches to GWs, for those that want a fuller picture. I will also close this introduction by listing a number of excellent reviews that would be good starting points for those interested in learning more:

- Goldberger's lecture notes [1]
- Maggiore's book[2]
- Blanchet's review [3],
- Porto's review [4]
- Levi's review [5],
- Schafer and Jaranowski's review [6].

Finally, let me stress that my goal is not to describe the "best" way to do these GW calculations. Instead, I am going to use the two-body problem as an excuse to show off some interesting applications of field theory techniques. Enjoy!

Conventions

Spatial vectors are written in bold font, e.g. $\mathbf{x}$ and $\mathbf{p}$ for position and momentum. The magnitude of a vector will be written without bold font, e.g. $x = |\mathbf{x}|$ and $p = |\mathbf{p}|$. This should not be confused with the four-vectors $x^\mu = (t, \mathbf{x})$ and $p^\mu = (\omega, \mathbf{p})$, or their Lorentz-invariant magnitudes $x^\mu x_\mu = -t^2 + x^2$ and $p^\mu p_\mu = -\omega^2 + p^2$. Integrals over position or momenta are often abbreviated as

$$\int_{\mathbf{x}} = \int d^d\mathbf{x} \,, \qquad \int_{\mathbf{p}} = \int \frac{d^d\mathbf{p}}{(2\pi)^d}$$
$$\int_t = \int_{-\infty}^{+\infty} dt \,, \qquad \int_\omega = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \,, \tag{1.5}$$

where d is the number of spatial dimensions (i.e. usually $d = 3$, except when regulating divergent integrals in dim reg). Natural units $\hbar = c = 1$ are used throughout, and $G_N = 1/(8\pi M_P^2)$.

> *Optional text.* Framed text is optional and can be skipped without causing problems later.

2 Introducing Effective Field Theory

In this section, we introduce some of the basic ideas behind effective field theory (EFT).

2.1 The EFT philosophy

"Effective" means "simpler, but gives the same answer".

Given any QFT, with action S, its *low-energy effective action* is a different action that produces the same observables at low energies. For a simple example, suppose we have a massless scalar field φ and a heavy scalar field χ (with mass M), that can interact via a cubic term in the Lagrangian $g\phi^2\chi$. This gives a scattering amplitude

$$S_{\rm UV} = g\phi^2\chi \;\; \Rightarrow \;\; \mathcal{A}_{\phi\phi\to\phi\phi} = \frac{g^2}{M^2 - p_\mu p^\mu} \tag{2.1}$$

which is the usual Feynman propagator, where p_μ is the total momentum of the internal χ line in the tree-level Feynman diagram. Compare that with the following effective action:

$$S_{\rm EFT} = \frac{g^2}{M^2}\phi^2\left(1 + \frac{\partial^2}{M^2} + \mathcal{O}\left(\frac{\partial^4}{M^4}\right)\right)\phi^2 \;\; \Rightarrow \;\; \mathcal{A}_{\phi\phi\to\phi\phi} = \frac{g^2}{M^2}\left(1 + \frac{p_\mu p^\mu}{M^2} + \mathcal{O}\left(\frac{(p_\mu p^\mu)^2}{M^4}\right)\right) \tag{2.2}$$

The effective action is simpler, because it does not refer to any χ field: it only has φ self-interactions. But it gives the same answer as the original $S_{\rm UV}$, providing we restrict our attention to scattering particles with $p_\mu p^\mu \ll M^2$ and neglect terms of $\mathcal{O}\left(\frac{(p_\mu p^\mu)^2}{M^4}\right)$. The EFT ideology is that, by adding more and more local self-interactions for φ, we can reproduce higher and higher orders terms in $p_\mu p^\mu$. The EFT is therefore a systematic way to capture, order by order in some parameter, the corrections to a simple free theory.

Integrating out. In practice this is how EFTs are often constructed: we simply write down all of the local we can think of, built from the fields we want to scatter and up to some fixed order in derivatives, and then check that its observables (such as scattering amplitudes) indeed match the underlying UV theory. In principle (and sometimes in practice), one could define the low-energy effective action via a path integral

$$e^{iS_{\rm EFT}[\phi_{\rm low}]} = \int \mathcal{D}\phi_{\rm high}\, e^{iS_{\rm UV}[\phi_{\rm high},\phi_{\rm low}]} \tag{2.3}$$

in which $S_{\rm UV}$ depends on both high and low energy degrees freedom (φ and χ in the example above) and the high-energy degrees of freedom are "integrated out" (or "averaged over") to produce $S_{\rm EFT}$. We can always simply write (2.3) as our *definition* of $S_{\rm EFT}$, but the really non-trivial thing is that—for a wide class of field theories at sufficiently low energies—the resulting $S_{\rm EFT}$ is *local* and contains only point-like interactions between the remaining $\phi_{\rm light}$. This is non-trivial because the full $S_{\rm UV}$ contains Feynman diagrams like that above, in which heavy fields can communicate signals between the light particles over extended distances.

The main thing to take away from this is that:

$$S_{\rm UV} \to S_{\rm EFT}$$

$$\text{Extended} \to \text{Point-like.}$$

2.2 Modelling compact objects as point particles

Let's start by illustrating this key concept in 1d quantum mechanics. Suppose a single particle, moving in one spatial dimension, encounters a region in which its potential energy becomes $V(x)$. The probabilities for the particle to be reflected or transmitted are given by $|r|^2$ or $|t|^2$ respectively, where r and t are complex numbers that depend on the shape of the potential and the energy of the incoming particle. We are going to show that the same low-energy physics (i.e. the same r and t at low momenta) can be obtained by replacing $V(x)$ with a simpler *low-energy effective potential* $V_{\rm EFT}(x) \propto \delta(x)$ which is localised at a single point. In a nutshell, we are basically expanding in powers of the wavevector k,

$$\begin{aligned} V(x) &= \int dk\, e^{ikx}\, \tilde{V}(k) \\ \Rightarrow\quad V_{\rm EFT}(x) &= \int dk\, e^{ikx} \left[\tilde{V}(0) + \tilde{V}'(0)k + \tilde{V}''(0)\frac{k^2}{2} + ...\right] = \sum_{j=0} \tilde{V}^{(j)}(0)\, \delta^{(j)}(x) \end{aligned} \tag{2.4}$$

where $\delta^{(j)}(x)$ is the $j^{\rm th}$ derivative of the Dirac delta function.

Energy eigenstates. Recall that the wavefunction $\psi(x,t)$ that describes the position of this particle at time t is the solution of the Schrodinger equation

$$i\hbar\partial_t\psi(x,t) = \left[-\tfrac{\hbar^2}{2m}\partial_x^2 + V(x)\right]\psi(x,t)\ . \tag{2.5}$$

Since the potential is time-independent, the energy of the particle is conserved and so we can simplify the problem by writing

$$\psi(x,t) = e^{-iE_kt/\hbar}\psi_k(x) \ \text{ with } \ E_k = \tfrac{\hbar^2k^2}{2m} \ \text{ and } \ V(x) = -\tfrac{\hbar^2U(x)}{2m} \tag{2.6}$$

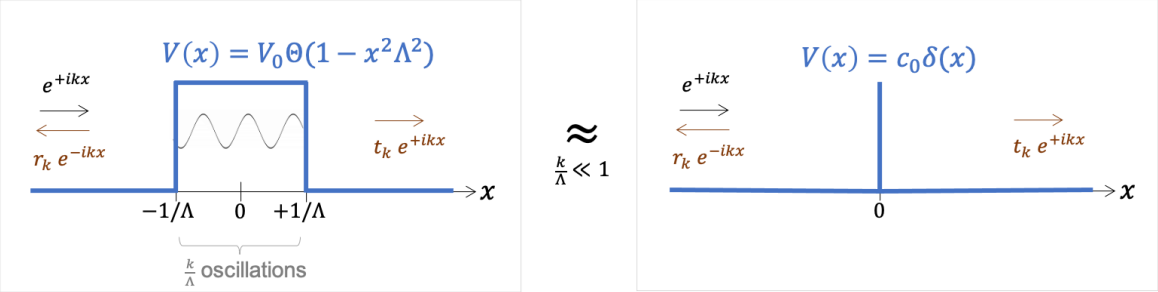


Figure 1: Illustration of point-particle approximation in 1d quantum mechanics.

so that the (time-independent) Schrodinger equation is

$$\left[\partial_x^2 + k^2\right] \psi_k(x) = -U(x)\psi_k(x) \,. \tag{2.7}$$

Assuming that the potential $U(x) \to 0$ as $x \to \pm\infty$, we expect the particle to propagate as a plane wave $\psi(x) \sim e^{\pm ikx}$ when far from the potential. The relevant boundary conditions for our scattering problem are

$$\psi_k(x) = \begin{cases} e^{+ikx} + r_k\, e^{-ikx} & \text{as } x \to -\infty \,, \\ t_k\, e^{+ikx} & \text{as } x \to +\infty \,, \end{cases} \tag{2.8}$$

which corresponds to a particle with wavevector k incident from the left and splitting into reflected and transmitted components.

Square well potential. For instance, suppose we have a square well potential of width and depth characterised by a constant scale Λ (which has the same units as k) and a dimensionless coupling constant α,

$$U(x) = \begin{cases} \alpha^2\Lambda^2 & \text{if } |x| < \frac{1}{2\Lambda} \,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

By demanding that both $\psi(x)$ and $\psi'(x)$ are continuous at $x = \pm 1/2\Lambda$, the reflection and transmission coefficients are found to be:

$$r_k = \frac{\left(k^2 - q^2\right) e^{-ik/\Lambda}}{(k^2 + q^2) + 2ikq\cot(q/\Lambda)} \,, \qquad t_k = \frac{2ikq\csc(q/\Lambda)\, e^{-ik/\Lambda}}{(k^2 + q^2) + 2ikq\cot(q/\Lambda)} \tag{2.10}$$

where $q = \sqrt{k^2 + \alpha^2\Lambda^2}$. These expressions are not particularly transparent, but some noteworthy features are:

(i) the total probability $|r_k|^2 + |t_k|^2 = 1$, since $\csc^2(q/\Lambda) = 1 + \cot^2(q/\Lambda)$,

(ii) as the potential is switched off ($\alpha \to 0$), we have $r_k \to 0 + \mathcal{O}(\alpha)$ and $t_k \to 1 + \mathcal{O}(\alpha)$ since the wave is then entirely transmitted,

(iii) on the other hand, when the energy of the incoming wave is much smaller than the depth of the well ($k \ll \Lambda$), it is almost entirely reflected:

$$r_k = -1 + \left(1 + \frac{2\cot\alpha}{\alpha}\right)\frac{ik}{\Lambda} + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) \,, \qquad t_k = 0 + \frac{2\csc\alpha}{\alpha}\,\frac{ik}{\Lambda} + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) \tag{2.11}$$

(iv) finally, the coefficients (2.10) represent *exact* solutions to the time-independent Schrodinger equation (i.e. valid to all orders in α and k/Λ).

Effective point-particle potential. Let's consider the low-energy limit (2.11) more carefully. When $k \ll \Lambda$, this corresponds to scattering waves with wavelength much larger than the width of the square well. But in that case, the wave does not resolve any of the structure in the potential: as far as the wave is concerned, it receives a kick near the origin, but is otherwise completely free.

Let's see how the intuition can be made mathematically precise. Consider replacing the square well with a δ function spike at the origin,

$$U_{\rm EFT}(x) = g\Lambda\,\delta(x) \tag{2.12}$$

where g is a new coupling constant (and Λ has been included to make g dimensionless). Solving the time-independent Schrodinger equation[3] for this potential gives

$$r_k^{\rm EFT} = \frac{-g\Lambda}{2ik+g\Lambda}\,, \qquad\qquad t_k^{\rm EFT} = \frac{2ik}{2ik+g\Lambda}\,. \tag{2.13}$$

These are clearly very different to (2.10), however in the limit of small k/Λ

$$r_k^{\rm EFT} = -1 + \frac{2ik}{g\Lambda} + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right)\,, \qquad\qquad t_k^{\rm EFT} = \frac{2ik}{g\Lambda} + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right)\,. \tag{2.14}$$

If we choose

$$g = \alpha \sin\alpha \tag{2.15}$$

we therefore find that this δ function potential gives the *same* physical probabilities at leading order in k/Λ,

$$|t_k|^2 = \frac{4k^2}{\alpha^2\sin^2\alpha\Lambda^2} + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) = |t_k^{\rm EFT}|^2 + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) \tag{2.16}$$

and similarly for $|r_k^{\rm EFT}|^2 = |r_k|^2$.

Phases. However, the alert among you will notice that while the probabilities match, the complex coefficients differ by a phase. It is tempting to sweep this under the carpet, with words like "phases are unobservable, so don't worry". But there's an important lesson to be learned here. If our effective potential does not match a *complete* set of observables from the original potential, then it is *not* reproducing the same physics. In this example, $|r_k|^2$ and $|t_k|^2$ are not a complete set of observables because we can consider other kinds of scattering. For instance, suppose that instead of waves striking this potential from the left, we consider sending in waves from both left and right in equal amounts.

$$\psi_k(x) = \begin{cases} e^{+ikx} + \left(S_k^+ + S_k^-\right) e^{-ikx} & \text{as } x \to -\infty \\ e^{-ikx} + \left(S_k^+ - S_k^-\right) e^{+ikx} & \text{as } x \to +\infty \end{cases} \tag{2.17}$$

This is a useful boundary condition to use because now the asymptotic in- and out-states have definite parity. Some simple manipulations[4] gives

$$S_k^\pm = t_k \pm r_k \tag{2.18}$$

[3] Recall that with a δ function potential, $\psi(x)$ is continuous but $\psi'(x)$ has a jump: $\psi'(+\epsilon) - \psi'(-\epsilon) = -g\Lambda\psi(0)$.

[4] see e.g. David Tong's nice notes on scattering theory.

These observables cannot match between our simple EFT and the original $U(x)$ because we have found that

$$t_k^{\rm EFT} - r_k^{\rm EFT} = 1 \tag{2.19}$$

for *any* value of g, while the original potential had

$$t_k - r_k = 1 + \frac{2\csc\alpha - 2\cot\alpha - \alpha}{\alpha}\frac{ik}{\Lambda} + \mathcal{O}\left(\frac{k^2}{\Lambda^2}\right) \ . \tag{2.20}$$

Complete basis. So what has gone wrong? The issue is that our ansatz for $U_{\rm EFT}$ is not general enough. For instance, if we want to replace $U(x)\psi(x)$ with terms proportional to $\delta(x)$, we could have used:

$$\left[\partial_x^2 + k^2\right]\psi_k(x) = c\,\Lambda\,\delta(x)\psi_k(x) + \frac{d}{\Lambda}\partial_x\left(\delta(x)\partial_x\psi_k(x)\right) \ . \tag{2.21}$$

where we now allowed *derivatives* of the Dirac delta. This leads to discontinuous jumps at the origin of:

$$\Delta\psi(0) = -\frac{d}{\Lambda}\psi'(0) \ , \qquad\qquad \Delta\psi'(0) = -c\Lambda\psi(0) \ . \tag{2.22}$$

where we define $\Delta\psi(0) = \lim_{\epsilon\to 0}\left[\psi(\epsilon) - \psi(-\epsilon)\right]$ and $\psi(0) = \frac{1}{2}\lim_{\epsilon\to 0}\left[\psi(\epsilon) + \psi(-\epsilon)\right]$. This gives

$$r_k^{\rm EFT} = \frac{2\left(c\Lambda^2 + dk^2\right)}{(2k + ic\Lambda)(dk + 2i\Lambda)} \ , \qquad\qquad t_k^{\rm EFT} = \frac{k\Lambda(4 - cd)}{(c\Lambda - 2ik)(dk + 2i\Lambda)} \tag{2.23}$$

which can now be expanded at small k/Λ and will match the reflection and transmission coefficients of $U(x)$ if we fix

$$c = -\frac{4\alpha}{\alpha + 2\cot\alpha + 2\csc\alpha} \ , \qquad\qquad d = \frac{-\alpha - 2\cot\alpha + 2\csc\alpha}{\alpha} \ . \tag{2.24}$$

Redundant operators. To build a complete basis, it is often easiest to work with a Lagrangian. To preserve the linearity of quantum mechanics, we may only include terms that are quadratic in ψ. So the most general Lagrangian is

$$S[\psi] = \int_{t,x}\psi^\dagger(x)\left(i\hbar\partial_t + \partial_x^2\right)\psi(x) + \sum_{a,b,c}\frac{c_{abc}}{\Lambda^{a+b+c-1}}\partial_x^a\delta(x)\partial_x^b\psi^\dagger(x)\partial_x^c\psi(x) \ . \tag{2.25}$$

However, most of these terms are *redundant*. They can be removed by integrating by parts and redefining the wavefunction. Furthermore, if the underlying $U(x)$ is parity-invariant, then every c_{abc} with an odd number of spatial derivatives $a + b + c$ must vanish. So to find a minimal set, we can do the following:

(i) first integrate by parts to remove all derivatives from the δ function. This produces $\delta(x)\partial_x^A\psi^\dagger\partial_x^B\psi$, where $A + B$ is even.

(ii) Notice that ψ can be redefined to replace any $\partial_x^{2j}\psi^\dagger$ with $(i\hbar\partial_t)^j\psi^\dagger$ plus higher order terms[5]. Take for instance

$$\mathcal{L} = \tfrac{1}{2}\psi^\dagger\left(i\hbar\partial_t + \partial_x^2\right)\psi + \frac{c}{\Lambda}\delta(x)\partial_x^2\psi^\dagger\psi \ . \tag{2.26}$$

[5]This is often referred to as "substituting the equations of motion into the action", but do not be misled: one should never simply evaluate the Lagrangian on-shell.

If we redefine

$$\psi(x) \to \psi(x) - \frac{c}{\Lambda}\delta(x)\psi(x) \tag{2.27}$$

then this Lagrangian transforms into:

$$\mathcal{L} \to \tfrac{1}{2}\psi^\dagger\left(i\hbar\partial_t + \partial_x^2\right)\psi + \frac{c}{\Lambda}\delta(x)i\hbar\partial_t\psi^\dagger\psi + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) . \tag{2.28}$$

This new Lagrangian is equivalent to the previous one, but with $\partial_x^2\psi^\dagger \to i\hbar\partial_t\psi^\dagger$ (plus a redefinition of the higher-order terms). The same manipulations can be used for any ∂_x^{2j}, and so we end up with only $\delta(x)\partial_t^{A/2}\psi^\dagger\partial_x^B\psi$ if A is even or $\delta(x)\partial_t^{A/2}\partial_x\psi^\dagger\partial_x^B\psi$ if A is odd.

(iii) Integrating by parts in time can move all ∂_t from $\psi^\dagger$ to ψ, and then a similar field redefinition can be used to replace $\partial_t^{A/2}\psi \to \partial_x^A\psi$. This leaves us with only two possibilities:

$$\delta(x)\psi^\dagger\partial_x^{2j}\psi\ , \qquad \delta(x)\partial_x\psi^\dagger\partial_x^{2j-1}\psi \tag{2.29}$$

for any integer j.

A minimal EFT basis is then

$$\mathcal{L} = \tfrac{1}{2}\psi^\dagger\left(i\hbar\partial_t + \partial_x^2\right)\psi + \sum_j \delta(x)\left(\frac{c_j}{\Lambda^{2j-1}}\psi^\dagger\partial_x^{2j}\psi + \frac{d_j}{\Lambda^{2j-1}}\partial_x\psi^\dagger\partial_x^{2j-1}\psi\right) \tag{2.30}$$

where the $\{c_j, d_j\}$ are dimensionless coupling constants called the "EFT (or Wilson) coefficients".

EFT Universality. In the above problem, we had a square well $U(x)$ which was parity even (namely $U(-x) = U(x)$), and this symmetry immediately removed half of all possible operators from the theory. For the remaining coefficients, these can be fixed by *matching* $r_k^{\rm EFT}$ and $t_k^{\rm EFT}$ to the full r_k and t_k order-by-order in k/Λ. The EFT miracle is that: *any* $U(x)$ with finite support can be represented as an infinite sum of δ functions! In the tutorial, you will do this for a pair of square wells.

Exercise. Another example, which is a little harder, is the Yukawa potential

$$U(x) = -e^{-M|x|} . \tag{2.31}$$

If we scatter waves with $k \ll M$, they do not have enough energy to penetrate into the potential and resolve its internal structure. For these low-energy waves, this $U(x)$ also behaves like a δ-function obstruction at the origin. Find the r_k and t_k from (2.31) and show that it can be reproduced to $\mathcal{O}(k/\Lambda$ by choosing c_0 and d_1 appropriately. The main takeaway here is: for low enough energy particles, the potential is always ultra-local.

2.3 Modelling radiation from compact objects

We have now established that an extended potential can be replaced by a δ function if we "zoom out" and consider only low energy waves. To see this in action, let's imagine that the motion of particles in a binary system (or indeed any accelerating distribution of accelerating charges) has been completely solved, so that (neglecting non-linearities for now) the effective action for φ is

$$S_J[\varphi] = \int d^4x\left[-\frac{1}{2}(\partial_\mu\varphi)^2 + \frac{1}{M_P}\,J(x)\varphi(x)\right] . \tag{2.32}$$

where J is some (potentially very complicated) function of time and space. How can we determine the scalar radiation that would be measured by a distant observer? Well, suppose that J is spatially localised near the origin of our co-ordinates and we wish to determine φ at spatial infinity (a good approximation for gravitational wave sources, which are typically binary systems many light years away from Earth). The EFT expansion of the source corresponds to writing down all possible terms that are local in $\mathbf{x}$,

$$J_{\rm EFT}(t,\mathbf{x}) = \left[I(t) - I^i(t)\partial_i + \frac{1}{2}I^{ij}(t)\partial_i\partial_j + ...\right]\delta^3(\mathbf{x}) = \left[\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}I^{i_1...i_\ell}(t)\partial_{i_1}...\partial_{i_\ell}\right]\delta^3(\mathbf{x})\ . \quad (2.33)$$

This EFT expansion is universal, in the sense that *any* source $J(t,\mathbf{x})$, no matter how complicated the binary's internal motions, can be represented in this way (providing we observe from sufficiently far away). Different underlying theories, i.e. different $J(x,\mathbf{x}_A,\mathbf{x}_B)$ and resulting $\{\mathbf{x}_A(t),\mathbf{x}_B(t)\}$ dynamics, will correspond to different EFT coefficients $\{I, I^i, I^{ij}, ...\}$. Notice that we have not assumed any symmetry here: the binary motion will generically break both time translations and spatial rotations, so there is no requirement that these EFT coefficients be time-independent or isotropic.

2.3.1 Multipoles

The EFT expansion (2.33) is secretly over-complete. The trace of each $I^{i_1...i_\ell}$ tensor is a redundant operator that can be removed by field redefinitions and integration-by-parts.

Proof. To see this, consider separating out the trace $\delta_{ij}I^{ij} = I_{\rm tr}(t)$ and massaging as follows:

$$\begin{aligned} I_{\rm tr}(t)\partial_i^2\delta^3(\mathbf{x})\varphi &\overset{\rm IBP}{=} I_{\rm tr}(t)\delta(\mathbf{x})\partial_i^2\varphi \\ &\overset{\rm eom}{=} I_{\rm tr}(t)\delta^3(\mathbf{x})\partial_t^2\varphi + \text{higher order} \\ &\overset{\rm IBP}{=} \ddot{I}_{\rm tr}(t)\delta^3(\mathbf{x})\varphi + \text{higher order}\ . \end{aligned} \quad (2.34)$$

This shows that, up to total derivatives and field redefinitions, the following two EFT expansions are *equivalent*:

$$\left[I + \frac{1}{2}I^{ij}\partial_i\partial_j\right]\delta^3(\mathbf{x}) \approx \left[\left(I + \frac{1}{6}\ddot{I}_{\rm tr}\right) + \frac{1}{2}\left(I^{ij} - \frac{1}{3}I_{\rm tr}\delta^{ij}\right)\partial_i\partial_j\right]\delta^3(\mathbf{x}) \quad (2.35)$$

modulo higher order corrections that can be absorbed into the remaining $\{I^{ijk}, ...\}$. The same argument can be applied to the trace of any $I^{i_1...i_\ell}$, which can be absorbed by shifting the traceless parts of both lower and higher order tensors.

We can therefore choose our EFT basis so that the $I^{i_1...i_\ell}$ are all *symmetric, trace-free* tensors[6]. They are referred to as the "multipole moments" (multipoles, for short) of $J(t,\mathbf{x})$, where I is the "monopole", I^i is the "dipole", I^{ij} is the "quadrupole", etc.

Matching. Each multipole moment can be written explicitly as an integral (average over) the underlying $J(x)$ source by matching the on-shell actions

$$\int d^4x\, J(x)\varphi(x) \overset{!}{=} \int d^4x\, J_{\rm EFT}(t,\mathbf{x})\varphi(x)\ . \quad (2.36)$$

[6] i.e. they are irreducible representations of the rotation group (for those that like that kind of thing).

Taylor expanding $\varphi(\mathbf{x})$ about $\mathbf{x} = 0$ on the left-hand-side and matching each $\partial_{i_1}...\partial_{i_\ell}\varphi$ with the corresponding term on the right-hand-side leads us to the following EFT coefficients:

$$I^{i_1\ldots i_\ell}(t) = \int d^3\mathbf{x}\, J(t,\mathbf{x})\, \mathbf{x}^{i_1}...\mathbf{x}^{i_\ell} \,. \tag{2.37}$$

However, these are not traceless and so contain some redundancy. For instance, performing the steps in the box above, we can write an equivalent $\{I, I^{ij}\}$ which are traceless

$$I(t) = \int d^3\mathbf{x} \left[J(t,\mathbf{x}) + \frac{1}{6} x^2 \ddot{J}(t,\mathbf{x}) \right] \tag{2.38}$$

$$I^{ij}(t) = \int d^3\mathbf{x}\; J(t,\mathbf{x}) \left(\mathbf{x}^i\mathbf{x}^j - \frac{1}{3}|\mathbf{x}|^2\delta^{ij} \right) \,. \tag{2.39}$$

This procedure can be repeated to re-organise every (2.37) into traceless combinations, where the kth trace of $I^{i_1\ldots i_\ell}$ will be replaced with a $\partial_t^{2k} J$ term in $I^{i_1\ldots i_{\ell-2k}}$.

Exercise. Show that the leading-order octopole and next-to-leading dipole are given by

$$I^i(t) = \int d^3\mathbf{x} \left[J(t,\mathbf{x}) + \frac{1}{10} x^2 \ddot{J}(t,\mathbf{x}) \right] \mathbf{x}^i \tag{2.40}$$

$$I^{ijk}(t) = \int d^3\mathbf{x}\; J(t,\mathbf{x}) \left[\mathbf{x}^i\mathbf{x}^j\mathbf{x}^k - \frac{1}{5}|\mathbf{x}|^2 \left(\delta^{ij}x^k + \delta^{ik}x^j + \delta^{jk}x^j \right) \right] \,. \tag{2.41}$$

Notice that all the multipole moments take the same form

$$I^{i_1\ldots i_\ell}(t) = \int d^3x \left[\sum_{k=0} c_{\ell k}\; \partial_t^{2k} J(t,\mathbf{x}) \right] Q^{i_1\ldots i_\ell}(\mathbf{x}) \tag{2.42}$$

where $c_{\ell k} = \frac{(2\ell+1)!!}{(2k)!!(2\ell+2k+1)!!}$ are model-independent constants and the traceless tensors, $Q^{i_1\ldots i_\ell}(\mathbf{x}) = \mathbf{x}^{i_1}...\mathbf{x}^{i_\ell}$ − traces, provide a complete basis.[7]

2.3.2 Waveform

This multipolar expansion is convenient because the equation of motion for the field

$$\left(\partial_t^2 - \partial_i^2\right)\varphi = J_{\rm EFT} \tag{2.44}$$

is readily solved. The standard Green function approach is to first solve $(\partial_t^2 - \partial_i^2)G(x - x') = \delta^4(x - x')$ with appropriate boundary conditions, since the solution of (2.44) can then be written as

$$\varphi(x) = \int d^4x' G(x - x') J_{\rm EFT}(x') \,. \tag{2.45}$$

[7] Explicitly, if $X^{i_1\ldots i_\ell} = \mathbf{x}^{i_1}...\mathbf{x}^{i_\ell}$ is the overcomplete basis of monomials, then the traceless basis is given by [7]

$$Q^{i_1\ldots i_\ell}(x) = \sum_{k=0}^{k=[\ell/2]} \frac{(2\ell-4k+1)!!}{(\ell-2k)!(2\ell-2k+1)!!(2k)!!} \delta^{(i_1 i_2}...\delta^{i_{2k-1}i_{2k}} X^{i_{2k+1}\ldots i_\ell)} |\mathbf{x}|^{2k} \,. \tag{2.43}$$

where $(i_1...i_\ell)$ denotes a symmetrisation over these indices, e.g. $X^{(i_1 i_2)} = X^{i_1 i_2} + X^{i_2 i_1} = 2x^{i_1}x^{i_2}$ and $X^{(i_1 i_2 i_3)} = 3! x^{i_1}x^{i_2}x^{i_3}$.

The physically relevant Green function in this case is the retarded one[8]:

$$G_R(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip_0(t-t')}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{-(p_0+i\epsilon)^2+\mathbf{p}^2} = -\Theta(t-t')\frac{\delta\left(t-t'-|\mathbf{x}-\mathbf{x}'|\right)}{4\pi|\mathbf{x}-\mathbf{x}'|} \,. \tag{2.46}$$

The retarded boundary condition required by causality, since it ensures that the field $\varphi(t)$ depends only on the source $J(t')$ at earlier times. The solution is then[9]:

$$\frac{\varphi(t,\mathbf{x})}{M_P} = -\frac{2G_N}{|\mathbf{x}|}\left[I(t_R) + \hat{\mathbf{x}}_i \dot{I}^i(t_R) + \frac{1}{2}\hat{\mathbf{x}}_i\hat{\mathbf{x}}_j \ddot{I}^{ij}(t_R) + ...\right] \tag{2.47}$$

where $t_R = t - |\mathbf{x}|$ is the retarded time, and $G_N = 1/(8\pi M_P^2)$. In practice, we often treat each term in this sum individually and decompose into a (time-dependent) amplitude and phase:

$$\frac{\varphi(t,\mathbf{x})}{M_P} = -\frac{2G_N}{|\mathbf{x}|}A(t)e^{i\Psi(t)} \,. \tag{2.48}$$

In this toy Universe, "LIGO" would be a device that can measure these scalar waveforms and determine the values of $A(t)$ and $\Psi(t)$ over a particular range of frequencies.

Partial waves. Another convenient way to represent this waveform is via spherical harmonics; this amounts to writing $\mathbf{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and performing an angular Fourier transform from the spherical angles (θ,ϕ) to the angular momentum numbers (ℓ, m). This gives

$$\frac{\varphi(t,r,\theta,\phi)}{M_P} = -\frac{2G_N}{r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} a_{\ell m}(t-r)Y_\ell^m(\theta,\phi) \tag{2.49}$$

where $Y_\ell^m \propto P_\ell^m(\cos\theta)e^{im\phi}$ are the usual spherical harmonics. It is convenient to normalise these with an additional $\sqrt{4\pi}/\sqrt{2\ell+1}$, so that $Y_\ell^0 = P_\ell(\cos\theta)$. Then comparing with (2.47)[10],

$$\begin{aligned}
&a_{00} = I \\
&a_{10} = I_3 \,, \qquad a_{1,\pm1} = \frac{1}{2}\left(I_1 \mp iI_2\right) \\
&a_{20} = I_{33} \,, \qquad a_{2,\pm1} = \frac{1}{2}\left(I_{23} \mp iI_{13}\right) \qquad a_{2,\pm2} = \frac{1}{4}\left(I_{11} - I_{22} \mp 2iI_{12}\right)
\end{aligned} \tag{2.50}$$

This is convenient because each $a_{\ell m}(t)$ is a complex number with amplitude and phase which can now be studied separately:

$$a_{\ell m}(t) = A_{\ell m}(t)e^{i\Psi_{\ell m}(t)} \,. \tag{2.51}$$

[8] Notice that the $+i\epsilon$ shifts both poles into the lower-half of the complex plane, which ensures that this integral vanishes whenever $t' > t$ by Cauchy's residue theorem (since then the convergence of $e^{-ip_0(t-t')}$ as $\mathrm{Im}(p_0) \to \infty$ allows us to close the contour in the upper-half plane, encircling neither pole).

[9] To derive this, notice that $\partial_i = -\hat{\mathbf{x}}_i\partial_t$ when acting on retarded propagator.

[10] Notice that each $I^{i_1\ldots i_\ell}$ contributes only to the ℓth coefficients. This is only true because we used traceless tensors as our basis: otherwise objects like $\delta_{ij}I^{ij}$ would show up in a_{00}.

2.3.3 Power emitted

In addition to the waveform, another quantity of interest is the total power emitted through radiation. There are several ways to compute this, and here I will describe three:

from Noether's theorem. The first is the classical approach. Noether's theorem identifies the energy of the scalar field with the charge that generates time translations[11]

$$E_\varphi(t) = \int d^3\mathbf{x}\, \mathcal{J}^0(x) \quad \text{where} \quad \mathcal{J}^\mu = -\dot\varphi \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi} = \dot\varphi\partial_\mu\varphi\ . \tag{2.52}$$

To find the total power P_φ that has been emitted from the origin in some time interval $[t-T,t]$, we integrate the current at time t over a spatial sphere of radius T (which contains all of the emitted radiation),

$$TP_\varphi = \int_{|\mathbf{x}|<T} d^3\mathbf{x}\ \dot\varphi^2\ . \tag{2.53}$$

One reason the multipole expansion is so powerful is that, when computing quadratic scalars like $\dot\varphi^2$, the ℓth multipole can only appear contracted with the ℓth multipole (the only other options would be δ_{ij} or ϵ_{ijk}, but these contractions give zero since $I^{i_1\ldots i_\ell}$ is traceless and symmetric). For instance, consider the dipole contribution:

$$\begin{aligned} TP_\varphi^{\text{dipole}} &= \frac{G_N}{2\pi}\int_0^T dr \int d^2\hat{\mathbf{x}}\,\hat{\mathbf{x}}_i\hat{\mathbf{x}}_j\,\ddot I^i(t-r)\ddot I^j(t-r) \\ &= \frac{2G_N}{3}\int_{t-T}^t dt'\ \ddot I^i(t')\delta_{ij}\ddot I_j(t')\ . \end{aligned} \tag{2.54}$$

In practice, we are often interested in the average power emitted over several orbits. We denote this using angled brackets:

$$P_\varphi^{\text{dipole}} = \frac{2G_N}{3}\langle \ddot I^i\delta_{ij}\ddot I^j\rangle_{\text{orbit}} \qquad \text{where } \langle\mathcal{O}\rangle_{\text{orbit}} = \frac{1}{T}\int_{t-T}^t dt'\ \mathcal{O}(t') \tag{2.55}$$

where T is assumed to be much larger than the orbital period. The angular integral is straightforward for any multipole[12], leading to the general result [8]:

$$\begin{aligned} P_\varphi &= 2G_N\sum_{\ell=0}^\infty \frac{1}{\ell!(2\ell+1)!!}\left\langle \partial_t^{\ell+1}I^{i_1\ldots i_\ell}\partial_t^{\ell+1}I^{i_1\ldots i_\ell}\right\rangle_{\text{orbit}} \\ &= 2G_N\left\langle \dot I\dot I + \frac{1}{3}\ddot I^i\ddot I^i + \frac{1}{30}\dddot I^{ij}\dddot I^{ij} + \ldots\right\rangle_{\text{orbit}} \end{aligned} \tag{2.56}$$

where the repeated spatial indices are contracted with δ_{ij}'s. In practice, due to the increasing number of time derivatives, the higher ℓ multipole contributions are often suppressed.

[11] Note that E_φ need not be conserved since φ can exchange energy with the binary.

[12] using the identity $\int d^2\hat{\mathbf{x}}\,\hat{\mathbf{x}}^{i_1}\ldots\hat{\mathbf{x}}^{i_\ell} = \frac{4\pi}{(\ell+1)\ell!}\delta^{(i_1i_2}\ldots\delta^{i_{\ell-1}i_\ell)}$ if ℓ is even, and zero otherwise.

from QFT amplitudes. Here is a second way to compute the emitted power, where we start to make use of quantum mechanics. The amplitude of φ emission from the ℓth multipole moment is given by a single Feynman diagram in this EFT

$$\mathcal{A}^{(\ell)}_{J\to J\varphi}(k_\mu) = \quad = \frac{(-i)^\ell}{\ell! M_P}\tilde{I}^{i_1\ldots i_\ell}(k_0)\mathbf{k}_{i_1}\ldots\mathbf{k}_{i_\ell} \tag{2.57}$$

where $k_\mu(k_0, \mathbf{k})$ is the 4-momentum of the emitted scalar and $\tilde{I}$ is the Fourier transform of the multipole moment. The total amplitude for emission is then $\sum_{\ell=0}^{\infty}\mathcal{A}^{(\ell)}$. The magnitude-squared of this amplitude represents the probability that, at some point between $t=-\infty$ and $t=+\infty$, a φ particle with energy k_0 and momentum $\mathbf{k}$ was emitted from the source J. To find the total power emitted, we can therefore integrate over this probability, multiplied by its energy (remembering to impose the on-shell condition $k_0 = +|\mathbf{k}|$ in a Lorentz-invariant way)

$$\begin{aligned} TP_\varphi &= \int \frac{d^4k}{(2\pi)^4}\, k_0 |\mathcal{A}_{J\to J\varphi}|^2\, 2\pi\delta(k_0^2 - |\mathbf{k}|^2)\Theta(k^0) \\ &= \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, |\mathcal{A}_{J\to J\varphi}|^2 \text{ with } k_0 = |\mathbf{k}|\,, \end{aligned} \tag{2.58}$$

where now T is the very large time interval over which the emission can take place[13]. To convince ourselves this is the same as the classical result above, let's take the monopole contribution as an example:

$$TP_\varphi^{\text{monopole}} = \frac{1}{2M_P^2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,|\tilde{I}(|\mathbf{k}|)|^2 = 4G_N\int_0^\infty\frac{d\omega}{2\pi}\omega^2|\tilde{I}(\omega)|^2 \tag{2.59}$$

where we relabelled the $|\mathbf{k}|$ integration variable to ω. Returning the multipole to the time domain[14]

$$\begin{aligned} P_\varphi^{\text{monopole}} &= \frac{2G_N}{T}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty} dt\, dt'\, e^{i\omega(t-t')}\dot{I}(t)\dot{I}(t') \\ &= \frac{2G_N}{T}\int_{-\infty}^{+\infty} dt\, \dot{I}(t)\dot{I}(t) = 2G_N\langle\dot{I}\dot{I}\rangle_{\text{orbit}} \end{aligned} \tag{2.60}$$

This can be repeated for any multipole, with the angular integrals carried out as above, to find a frequency-domain representation of the total power

$$P_\varphi = \frac{4G_N}{T}\int_0^\infty\frac{d\omega}{2\pi}\sum_{\ell=0}^{\infty}\frac{\omega^{2\ell+2}}{\ell!(2\ell+1)!!}\left|\tilde{I}^{i_1\ldots i_\ell}(\omega)\right|^2 \tag{2.61}$$

Transforming back to the time-domain reproduces the classical result (2.56).

from effective action. Now that we convinced the same classical information can be extracted from a quantum mechanical amplitude, here is a third and final way to compute the emitted power. This is a path integral approach that makes clever use of unitarity, and in fact it can be applied to

any QFT. Consider the general path integral

$$Z[J] = \int \mathcal{D}\varphi \; e^{iS[\varphi] + i\int d^4x \, J\,\varphi} \equiv e^{iW[J]} \; . \tag{2.62}$$

This represents the overlap between the vacuum in the far past $|0_-\rangle$ and the vacuum in the far future $|0\rangle_+$ in the presence of a source J. The corresponding probability for this transition is

$$|\langle 0_+|0_-\rangle_J|^2 = |e^{iW[J]}|^2 = e^{-2\mathrm{Im}\, W[J]} \; . \tag{2.63}$$

and is crucially is *not equal to unity* if the effective action $W[J]$ has an imaginary part. This imaginary part therefore encodes the probability that some φ particles will be created. Suppose $\Gamma_J(\omega)$ is the probability that particles with a total energy ω are created by the source J. Then

$$\int_0^\infty d\omega \, \Gamma_J(\omega) = 1 - |\langle 0_+|0_-\rangle_J|^2 \approx 2\,\mathrm{Im}W[J] \tag{2.64}$$

is the total probability that one or more φ particles have been emitted between $t = -\infty$ and $t = +\infty$. The total emitted power in this time interval T then corresponds to the average energy carried by these particles

$$TP_\varphi = \int_0^\infty d\omega \, \Gamma_J(\omega) \, \omega \; . \tag{2.65}$$

This is an exact result that is true for any QFT. To use it in practice, notice that $W[J]$ is the generating function of connected time-ordered correlators (for the usual reason that $\delta^n Z/\delta J^n \propto \langle 0_+|T\varphi^n|0_-\rangle$), and so we have

$$iW[J] = \underbrace{\langle 1\rangle}_{=1} + \int d^4x \, J(x) \underbrace{\langle\varphi(x)\rangle}_{=0} + \int d^4x \, d^4x' \, J(x)J(x') \underbrace{\langle T\varphi(x)\varphi(x')\rangle}_{=-iG_F(x-x')} + ... \tag{2.66}$$

where $\langle\mathcal{O}\rangle = \langle 0_+|\mathcal{O}|0_-\rangle_{J=0}$ and as usual we renormalise so that $\langle 1\rangle = 1$ and $\langle\varphi\rangle = 0$. The leading contribution to $\mathrm{Im}W[J]$ therefore comes from the two-point function, which in the free theory is the usual Feynman Green function

$$G_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip_0(t-t')}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{-p_0^2 + \mathbf{p}^2 + i\epsilon} = \frac{i}{4\pi^2(-(t-t')^2 + |\mathbf{x}-\mathbf{x}'|^2 - i\epsilon)} \tag{2.67}$$

which again solves $(\partial_t^2 - \partial_i^2)G_F(x-x') = \delta^4(x-x')$, but notice the different $+i\epsilon$ compared with the retarded (2.46)—this is to implement the time-ordered boundary condition relevant for $\langle T\varphi\varphi\rangle$.[15]

Now let's specialise to the J_{EFT} given above, which is localised at $\mathbf{x} = 0$. For illustration, focus on the monopole contribution[16]:

$$\begin{aligned}\mathrm{Im}\,[W[J]] &= \frac{1}{M_P^2}\int_{-\infty}^{+\infty} dt_1 dt_2 \, I(t_1)\, I(t_2)\, \mathrm{Im}\left[\int \frac{d^4p}{(2\pi)^4}\, \frac{e^{-ip_0(t-t')}}{p_0^2 - \mathbf{p}^2 - i\epsilon}\right] \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{|\tilde{I}(|\mathbf{p}|)|^2}{2|\mathbf{p}|M_P^2} = 4G_N \int_0^\infty d\omega \, |\tilde{I}(\omega)|^2 \, \omega \; .\end{aligned} \tag{2.68}$$

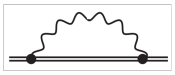
where we have performed the p_0 integral using the handy relation

$$\frac{1}{z-i\epsilon}-\frac{1}{z+i\epsilon}=2\pi i\,\delta(z)\ . \tag{2.69}$$

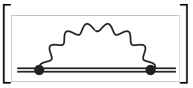
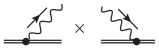
We can therefore read off $\Gamma_J^{\text{monopole}}(\omega)=4G_N\,|\tilde{I}(\omega)|^2\omega$, and hence write the power as

$$\begin{aligned}P_\varphi^{\text{monopole}}&=\frac{4G_N}{T}\int_0^\infty d\omega|\tilde{I}(\omega)|^2\,\omega^2\\&=\frac{2G_N}{T}\int_{-\infty}^{+\infty}dt\ \dot{I}(t)\dot{I}(t)=2G_N\langle\dot{I}\dot{I}\rangle_{\text{orbit}}\ .\end{aligned} \tag{2.70}$$

Repeating these steps for any ℓ, we find that this recipe for extracting the power from the two-point term in $\text{Im}\,W$ exactly reproduces our earlier (2.61) from the scattering amplitude (and hence the classical result (2.56)). In fact, there's a nice diagrammatic way to understand this. The two non-zero terms in (2.66) correspond to

$W[J]=$ $+$  $+\ldots$ (2.71)

where the two horizontal lines are supposed to represent the binary system (they do not contribute any factors, they are just drawn for aesthetic/traditional reasons), the black vertex represents an insertion of $J(x)$ and the wiggly line represents the propagation of a φ field. When we take the imaginary part of this loop diagram, unitarity guarantees that

$2\text{Im}\left[\right.$  $\left.\right]=$  $=|\mathcal{A}|^2\ .$ (2.72)

This makes it clear why these two calculations of the power must agree at this order in perturbation theory. It also highlights why (2.65) is the exact expression: if we continue to the next order, we see that $\text{Im}W[J]$ will contain not just the $\mathcal{A}_{J\to J\varphi}$ amplitude to emit a single φ particle, but also the $\mathcal{A}_{J\to J\varphi\varphi}$ amplitude to emit two φ particles, and in general every $\mathcal{A}_{J\to J\varphi^n}$ will appear at sufficiently high orders in perturbation theory.

In summary, by replacing the source $J(x)$ with an EFT expansion J_{EFT} in terms of symmetric, traceless multipole moments, we have been able to solve for the scalar waveform $\varphi(t,\mathbf{x})$ that would be detected by a distant observer, and also calculate the power carried by these waves. At the end of section part 3.3, we will repeat the same steps in General Relativity to find the gravitational waveform and gravitational wave power.

3 Point particle EFT for a binary and its fields

From here on, we will model our binary system of two compact objects as two point particles which move along approximately Newtonian orbits. The idea behind "Post-Newtonian" perturbation theory is to systematically add corrections to the orbital motion (and the emitted radiation) in powers of v^2/c^2, noting that $v^2/c^2\sim\kappa/r$ for bound orbits. As the two particles orbits, they lose energy via the

emission of radiation and so (i) the orbit shrinks and (ii) the particles speed up (since $v^2/c^2 \sim \kappa/r$). At some point the Post-Newtonian expansion breaks down because v^2/c^2 or κ/r are no longer small. In practice, this part is therefore mostly relevant for the "inspiral" phase of a binary's evolution, and once the two objects approach merger our expansions will inevitably break down (at which point we typically turn to numerical relativity).

Let us start by calculating the motion of the point particles and the emitted radiation using traditional (purely classical) field theory techniques—i.e. solving the Euler-Lagrange equations perturbatively in v^2/c^2. We will then, in section 4, introduce a systematic "Feynman diagram" expansion for this problem, motivated by non-relativistic QFT.

3.1 Scalar field toy model

To remove the computational difficulties of gravity (a spin-2 tensor field), we will illustrate the key physics concepts with a spin-0 scalar field φ. The fundamental field theory takes the form

$$S_{\rm pp}[\mathbf{x}_A, \mathbf{x}_B, \varphi] = S_0[\mathbf{x}_A] + S_0[\mathbf{x}_B] + S_2[\varphi] + \int d^4x \, \frac{\varphi(x)}{M_P} \, J(x; \mathbf{x}_A, \mathbf{x}_B) + S_{\rm nl}[\mathbf{x}_A, \mathbf{x}_B, \varphi] \tag{3.1}$$

where $\mathbf{x}_A(t)$ and $\mathbf{x}_B(t)$ are the worldlines of two point particles, which source $\varphi(x)$ via the current J. The constant scale M_P is the analogue of the Planck scale, related to Newton's constant by $G_N = 1/(8\pi M_P^2)$. S_0 and S_2 are exactly solvable free theories (containing 0 and 2 scalar fields respectively). $S_{\rm nl}$ contains higher-order non-linearities: namely self-interactions of the field and non-linear couplings between $\varphi^{n>1}$ and $\{\mathbf{x}_A, \mathbf{x}_B\}$. The free propagation of a massless scalar field is controlled by

$$S_2[\varphi] = -\frac{1}{2}\int d^4x \, (\partial_\mu \varphi)^2 \qquad \Rightarrow \qquad \left(\partial_t^2 - \partial_i^2\right)\varphi = 0 \tag{3.2}$$

while the inertial motion of each point particle is controlled by

$$S_0[\mathbf{x}_A] = -m_A \int dt \, \sqrt{1 - \dot{\mathbf{x}}_A^2} \qquad \Rightarrow \qquad \partial_t \left(\frac{m_A \dot{\mathbf{x}}_A}{\sqrt{1 - \dot{\mathbf{x}}_A^2}} \right) = 0 \tag{3.3}$$

where $x^\mu = (t, \mathbf{x}^i)$.[17] These are the usual Klein-Gordon equation for null rays and the conservation of spatial momentum $\mathbf{p} = \gamma m \dot{\mathbf{x}}$ for each particle.

In this toy Universe, the massless scalar φ plays the role of gravity. A "gravitational wave" is then φ radiation that can escape from the binary system and carry energy off to spatial infinity (where it is measured by our detector). In an ideal world, we would solve this problem in two steps:

(i) given $J(x, \mathbf{x}_A, \mathbf{x}_B)$, solve for $\mathbf{x}_A(t)$ and $\mathbf{x}_B(t)$, the motion of the binary,

(ii) from the resulting $J(x) = J(x, \mathbf{x}_A, \mathbf{x}_B)|_{\rm on-shell}$, solve for $\varphi(x)$, the scalar radiation produced at large distances from the binary.

In practice however, (i) can only be solved approximately. We then proceed to find an approximate solution for (ii), which turns out to be slightly incompatible with our (i) solution. So we iterate, improving our solution (i) and then finding a more accurate (ii), etc.

[17] Recall that Greek spacetime indices run over $\{0, 1, 2, 3\}$ and Latin spatial indices run over $\{1, 2, 3\}$.

3.2 Newtonian limit

The expressions (2.47) and (2.56) for the scalar waves and their power are only useful if we can specify the multipole moments for a binary system of two orbiting compact objects. In this subsection, we do the simplest thing and treat these objects as non-relativistic point particles moving very slowly at large separations: in this Newtonian limit, their dynamics is very simple.

In Newtonian limit, we treat all time derivatives as small quantities. The lowest order part of each action is[18]

$$S_N[\mathbf{x}_1, \mathbf{x}_2, \varphi] = \int dt \, \left[-\frac{1}{2} m_1 \dot{\mathbf{x}}_1^2 + -\frac{1}{2} m_2 \dot{\mathbf{x}}_2^2 \right] + \int d^4x \, \left[\frac{1}{2} \varphi \partial_i^2 \varphi + \varphi J_N \right] \tag{3.4}$$

where the Newtonian currents are

$$J_A = q_A \delta(\mathbf{x} - \mathbf{x}_A(t)) \, . \tag{3.5}$$

Since this simple theory is linear in φ, we can write the solution as $\varphi = \varphi_1 + \varphi_2$, where φ_A is the field sourced by particle A:

$$\partial_i^2 \varphi_A(t, \mathbf{x}) = J_A(t, \mathbf{x}) \qquad \Rightarrow \qquad \varphi_A(t, \mathbf{x}) = \frac{q_A}{|\mathbf{x} - \mathbf{x}_A(t)|} \tag{3.6}$$

Particle $\mathbf{x}_1$ then experiences a force from particle $\mathbf{x}_2$ via the field φ_2 (and vice versa)[19],

$$\begin{aligned} m_1 \ddot{\mathbf{x}}_1^i(t) &= -q_1 \partial^i \varphi_2(t, \mathbf{x})|_{\mathbf{x}=\mathbf{x}_1(t)} = +\frac{q_1 q_2}{|\mathbf{r}_{12}(t)|^3} \mathbf{r}_{12}^i(t) \\ m_2 \ddot{\mathbf{x}}_2^i(t) &= -q_2 \partial^i \varphi_1(t, \mathbf{x})|_{\mathbf{x}=\mathbf{x}_2(t)} = -\frac{q_1 q_2}{|\mathbf{r}_{12}(t)|^3} \mathbf{r}_{12}^i(t) \end{aligned} \tag{3.7}$$

This is the standard Kepler problem; it can be solved by changing variables from $\{\mathbf{x}_1, \mathbf{x}_2\}$ to the centre-of-mass position and relative separation $\{\mathbf{X}_{12}, \mathbf{r}_{12}\}$, defined by

$$\mathbf{X}_{12} = m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2 \, , \qquad \mathbf{r}_{12} = \mathbf{x}_1 - \mathbf{x}_2 \, . \tag{3.8}$$

Since the action S_N is invariant under translations, the total momentum $\mathbf{P}$ of the two-particle system is conserved. $\mathbf{X}$ is the co-ordinate conjugate to this total momentum,

$$\dot{\mathbf{X}}_{12}(t) = \mathbf{P}_{12} = \text{constant} \tag{3.9}$$

which can be verified by adding the equations of motion (3.7). Imposing the initial conditions $\mathbf{P}_{12} = \mathbf{X}_{12}(t_{\text{in}}) = 0$ then produces solutions in which $\mathbf{X}_{12}(t) = 0$ for all times (this is referred to as working in the "centre-of-mass frame"). The remaining equation

$$\mu \ddot{\mathbf{r}}_{12} = \frac{q_1 q_2}{|\mathbf{r}_{12}|^3} \mathbf{r}_{12} \tag{3.10}$$

describes a single particle of mass μ moving in the Coulomb potential $V(r) = \kappa/r$, where $\kappa = q_1 q_2/(4\pi)$ in this example.

The general solution for $|r|$ is a conic; either an ellipse for bound orbits or a hyperbola for unbound scattering. In this post-Newtonian section, we focus on bound orbits. Furthermore, since binary orbits

[18] The $\int dt m_A$ terms are a constant that does not affect the dynamics.

[19] The "self-force" that a particle exerts on itself through $\partial^i \varphi_A(t, \mathbf{x})|_{\mathbf{x}=\mathbf{x}_A}$ can be safely neglected in this Newtonian limit. We will return to this point below.

are believed to circularise before they emit a detectable level of gravitational waves, we focus on circular orbits defined by the conditions $\hat{\mathbf{r}} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{a} = -\Omega_0^2 \boldsymbol{r}$, where Ω_0 is a constant frequency. The explicit solution can be written as

$$m_1 \mathbf{x}_1(t) = -m_2 \mathbf{x}_2(t) = \mu r_0 \begin{pmatrix} \cos(\Omega_0 t) \\ \sin(\Omega_0 t) \\ 0 \end{pmatrix} \tag{3.11}$$

where r_0 is a constant radius related to the frequency by $\Omega_0 = \sqrt{q/\mu r_0^3}$. Notice that we have a constant speed $\boldsymbol{v}^2 = r_0^2 \Omega_0^2$. For circular and quasi-circular orbits, it is common to introduce the post-Newtonian variable

$$\mathtt{x} \equiv (\kappa\, \Omega_0/\mu)^{2/3} \quad (= \frac{\kappa}{\mu r_0} = v^2 \text{ at leading PN order}). \tag{3.12}$$

where $\kappa = q_1 q_2/4\pi$ above (or would $= G_N m_1 m_2$ in GR). Since we have that $\mathtt{x} = \kappa/(\mu r_0) = v^2$ on account of the virial theorem, performing a post-Newtonian expansion is then tantamount to expanding in powers of this single variable $\mathtt{x}$. Using the equation of motion and the definition of $\mathtt{x}$, we can replace[20] every (r, Ω) with just $\mathtt{x}$.

The conservative binding energy is

$$E = -\frac{\kappa}{2r} = -\frac{1}{2}\mu \mathtt{x} \tag{3.14}$$

and the current that sources φ is

$$I^{i_1 \dots i_\ell} = \sum_A q_A Q^{i_1 \dots i_\ell}(\mathbf{x}_A) + \mathcal{O}(v^2)\,. \tag{3.15}$$

Monopolar power. Since the leading order $I = q_1 + q_2$ is a constant, it does not contribute to the emitted power (it contributes only a constant shift to $\varphi_{\rm far}$). The next-to-leading order is

$$I(t) = \int_{\mathbf{x}} \left[J_{v^2}(t, \mathbf{x}) + \frac{1}{6} x^2 \partial_t^2 J_{v^0}(t, \mathbf{x}) \right] = \sum_A \left(-\frac{1}{2} q_A v_A^2(t) + \frac{1}{6} q_A \partial_t^2 x_A^2(t) \right)\,. \tag{3.16}$$

In the centre-of-mass frame, this becomes

$$I = -\frac{\mu}{3} \left(\frac{q_1}{m_1^2} + \frac{q_2}{m_2^2} \right) \left[\frac{1}{2} \mu v^2 + \frac{\kappa}{r} \right] \tag{3.17}$$

Notice that it is proportional to the total leading order energy: it therefore also does not radiate, since it is a constant (up to even higher order corrections). This means that, unless the charges q_A themselves are given a non-trivial time dependence, the monopolar power is suppressed by several powers of v. In almost all cases of interest, it is the dipole or quadrupole that dominates the waveform and emitted power at large distances.

[20] Useful identities for this are:

$$\Omega_0^2 = \frac{\mu^2}{\kappa^2} \mathtt{x}^3\,, \quad \frac{\kappa}{r_0} = \mu \mathtt{x}\,, \quad r_0^2 \Omega_0^2 = \mathtt{x} \tag{3.13}$$

Dipolar power. In the centre-of-mass frame, the leading dipole moment is

$$I^i(t) = q_1 \mathbf{x}_1^i(t) + q_2 \mathbf{x}_2^i(t) = \frac{q_1 m_2 - q_2 m_1}{M} \mathbf{r}^i(t) \,. \tag{3.18}$$

For a circular orbit, the magnitude of the dipolar acceleration is a constant

$$\ddot{I}^i(t)\ddot{I}^i(t) = a_1^2 \Omega_0^4 r_0^2 \qquad \text{with} \qquad a_1 = \frac{q_1 m_2 - q_2 m_1}{M} \tag{3.19}$$

and so averaging over one orbit is trivial and immediately gives

$$P^{\text{dipole}} = \frac{1}{3\pi} \langle \ddot{I}^i \ddot{I}^i \rangle_{\text{orbit}} = \frac{a_1^2}{3\pi} r_0^2 \Omega_0^4 \tag{3.20}$$

for the radiated power.

Quadrupolar power. In the centre-of-mass frame, the leading quadrupole moment is

$$I^{ij}(t) = \frac{q_1 m_2^2 + q_2 m_1^2}{M^2} Q^{ij}(\mathbf{r}(t)) \,. \tag{3.21}$$

For this circular orbit,

$$Q^{ij}(\mathbf{r}(t)) = r_0^2 \begin{pmatrix} \cos^2(\Omega_0 t) - \frac{1}{3} & \sin(\Omega_0 t)\cos(\Omega_0 t) & 0 \\ \sin(\Omega_0 t)\cos(\Omega_0 t) & \sin^2(\Omega_0 t) - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \,, \tag{3.22}$$

and hence the acceleration which determines the waveform is

$$\ddot{I}^{ij}(t) = -2 a_2 r_0^2 \Omega_0^2 \begin{pmatrix} \cos(2\Omega_0 t) & \sin(2\Omega_0 t) & 0 \\ \sin(2\Omega_0 t) & -\cos(2\Omega_0 t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{with} \qquad a_2 = \frac{q_1 m_2^2 + q_2 m_1^2}{M^2} \,. \tag{3.23}$$

This gives a waveform with only $\ell = 2 = \pm m$ modes:

$$\varphi_{\text{far}}(t, R, \theta, \phi) = \frac{a_2 r_0^2 \Omega_0^2}{4\pi R} \sin^2\theta \cos\left(2(\Omega_0 t - \phi_0)\right) \tag{3.24}$$

$$= \frac{1}{R} A_{22}(t) e^{i\Psi_{22}(t)} Y_{2,2}(\theta, \phi) \tag{3.25}$$

in which the amplitude and phase are given by[21]

$$A_{22}(t) = \sqrt{\tfrac{2\pi}{15}} a_2 r_0^2 \Omega_0^2 \qquad\qquad \Psi_{22}(t) = 2\Omega_0 t \,. \tag{3.26}$$

This wave transmits a power:

$$P^{\text{quadrupole}} = \frac{1}{30\pi} \langle \dddot{I}^{ij} \dddot{I}^{ij} \rangle_{\text{orbit}} = \frac{32 a_2^2}{30\pi} r_0^4 \Omega_0^6 \tag{3.27}$$

[21] It is a general result that $\Psi_{\ell m}(t) = m\Omega t$ at leading order.

Power from circular orbits. We can repeat the above steps for the multipole moment from a circular orbit at any ℓ, and then exploit the properties of symmetric, trace-free tensors to evaluate the power (see Appendix C of 2107.10841 for details). Explicitly, the multipoles are:

$$I^{i_1\ldots i_\ell} = a_\ell \, Q^{i_1\ldots i_\ell}(\mathbf{r}) \qquad \text{where} \qquad a_\ell = \frac{q_1 m_2^\ell + (-1)^\ell q_2 m_1^\ell}{M^\ell} \tag{3.28}$$

and the general expression for the power is[22]

$$P = \frac{4\mu^2}{\kappa^2} \sum_{\ell=0}^{\infty} N_\ell \, a_\ell^2 \, \mathtt{x}^{\ell+3} \tag{3.29}$$

where the normalisation is given by

$$N_\ell = \sum_{m=-\ell}^{+\ell} \left| \frac{Y_{\ell m}(\mathbf{d})}{(2\ell+1)!!} \right|^2 m^{2\ell+2} \tag{3.30}$$

where $\mathbf{d}$ is any reference vector in the plane of the orbit (e.g. $\mathbf{d} = (1,0,0)$ gives $Y_{\ell m}(\frac{\pi}{2}, 0)$). The first few are $N_0 = 0, N_1 = \frac{1}{12\pi}, N_2 = \frac{4}{15\pi}$, which give

$$\begin{aligned} P^{\text{monopole}} &= 0 + \mathcal{O}(\mathtt{x}^4) \\ P^{\text{dipole}} &= \frac{\mu^2}{3\pi\kappa^2 M^2} \left(q_1 m_2 - q_2 m_1\right)^2 \mathtt{x}^4 + \mathcal{O}(\mathtt{x}^5) \\ P^{\text{quadrupole}} &= \frac{16\mu^2}{15\pi\kappa^2 M^4} \left(q_1 m_2^2 + q_2 m_1^2\right)^2 \mathtt{x}^5 + \mathcal{O}(\mathtt{x}^6) \end{aligned} \tag{3.31}$$

Backreaction. We have assumed that Ω_0 is constant; this solves the Newtonian (0PN) equation of motion for $\mathbf{x}_A$, but clearly it cannot be an exact solution since the binary is *losing energy* through scalar waves. The particles recoil slightly when they emit radiation, an effect called "radiation reaction" that shows up at higher PN orders, and this causes the orbit to shrink. Later we will compute this radiation reaction force explicitly. Here, we make use of a neat trick: since the energy is conserved, the power emitted in radiation must exactly balance the energy lost by the orbit:

$$-\frac{dE}{dt} = P \, . \tag{3.32}$$

This "flux-balance law" can then be used to find the time dependence of any quantity $\mathcal{O}$

$$\dot{\mathcal{O}}(t) = -\frac{P}{\partial_{\mathcal{O}} E} \, . \tag{3.33}$$

For instance, let us now relax our assumption to merely $\dot{\Omega} \ll \Omega^2$, i.e. that the change in the orbital frequency over one orbit is very small (often referred to as the "adiabatic approximation"). Since the energy of the bound orbit can be written as $E = -\frac{1}{2}\mu \left(\kappa\Omega/\mu\right)^{2/3}$ at leading order, the flux-balance law

[22] This is consistent with the expectation that $P \sim \Omega_0^2 = \mathtt{x}^3 \mu^2/\kappa^2$ plus higher PN corrections in $\mathtt{x} = v^2$.

immediately tells us that

$$\dot{\Omega} = \frac{3\Omega P(\Omega)}{\mu\,(\kappa\Omega/\mu)^{2/3}} + ... \tag{3.34}$$

Typically, it is a single multipole that dominates the power emission in the PN expansion: for instance in electromagnetism it is the dipole, and in General Relativity it is the quadrupole. Assuming that the ℓ-pole dominates, i.e. that $P \sim \Omega^{\frac{2\ell+6}{3}}$, gives a solution

$$\Omega \approx \Omega_0 \left(1 - \frac{t}{t_c}\right)^{-\frac{3}{2(\ell+2)}} \tag{3.35}$$

where t_c is the time of coalescence, at which $r \to 0$ (although our approximations break down much earlier). The right way to view this equation is that t_c depends implicitly on the initial Ω_0 of the orbit and goes to $+\infty$ in the Newtonian limit where radiation is neglected.

$$\frac{1}{t_c} \approx \frac{2(\ell+2)}{\mu} \frac{P^{\ell\text{-pole}}(\mathtt{x})}{\mathtt{x}} \,. \tag{3.36}$$

Phase shift. So what is the effect of this backreaction on our observed amplitude and phase? We are more sensitive to changes in this phase than the amplitude. It accumulates as orbit changes.

For instance, suppose we measure a quadrupolar signal in $\varphi_{\rm far}$, with phase Ψ_{22}. It can be found from

$$\Psi_{22}(t) = 2\int dt\,\Omega(t) = 2\int d\Omega\,\frac{\Omega}{\dot{\Omega}} = 2\int d\Omega\,\frac{\Omega\,E'(\Omega)}{P(\Omega)} \tag{3.37}$$

In this Newtonian theory, as the binary inspirals from $\Omega_{\rm in}$ to $\Omega_{\rm out}$ (the frequencies at which the waveform enters and leaves the detector's sensitivity band), the phase changes by:

$$P = \frac{32a_2^2}{30\pi} r_0^4 \Omega_0^6 + ... \qquad \Rightarrow \qquad \Psi_{22} = \frac{3\pi}{8a_2^2}\left(\frac{\mu^5}{\kappa^2\Omega^5}\right)^{1/3}\Bigg|_{\Omega_{\rm in}}^{\Omega_{\rm out}} + ... \tag{3.38}$$

Although this quadrupolar power is suppressed by v^2 relative to the dipolar contribution, here we are assuming that (like in our Universe) the combination $a_1 = (q_1 m_2 - q_2 m_1)/M$ is sufficiently small that $P^{\rm dipole} \ll P^{\rm quadrupole}$. We can ask: how small does a_1 need to be? If Ψ_{22} is measured and agrees well with (3.38), what is the experimental upper bound placed on $|a_1|$? To answer this, we expand the power and phase as[23]

$$\begin{aligned} P &= P^{\rm quadrupole}\left(1 + \frac{P^{\rm dipole}}{P^{\rm quadrupole}} + ...\right) \\ \Psi_{22} &= 2\int d\Omega\,\frac{E'(\Omega)}{P^{\rm quadrupole}}\left(1 - \frac{P^{\rm dipole}}{P^{\rm quadrupole}} + ...\right) \,. \end{aligned} \tag{3.39}$$

The effect of the dipolar emission is to change the accumulated phase in the detector band by

$$P^{\rm dipole} = \frac{a_1^2}{3\pi} r_0^2 \Omega_0^4 \qquad \Rightarrow \qquad -\delta\Psi_{22} = 2\int d\Omega\,\frac{E'(\Omega)P^{\rm dipole}}{\left(P^{\rm quadrupole}\right)^2} = \frac{75\pi a_1^2}{896 a_2^4}\left(\frac{\mu^7}{\kappa^4\Omega^7}\right)^{1/3}\Bigg|_{\Omega_{\rm in}}^{\Omega_{\rm out}} \tag{3.40}$$

[23] More precisely, here I am describing the "TaylorT2" approximant for the phase. There is an art to expanding these quantities in v^2 and different expansions/resummations can lead to different accuracy, see [9].

Although this is a scalar toy model, let's complete the story by assuming "LIGO"-like values for our observation: suppose the waveform first enters our detector at frequency $f = \Omega/\pi = 10$Hz and remains in-band[24] and reliably described by PN perturbation theory until around $f = 1000$Hz. From the chirp and other features of the waveform, we disentangle the mass of the two compact objects as $m_1 \approx m_2 \approx 10M_\odot$, where $M_\odot \approx 4.7 \times 10^{38} M_P$ is the mass of the Sun. Our measurement of the phase will have some uncertainty, but the most conservative thing is to demand that

$$|\delta\Psi_{22}| \overset{!}{<} \pi \tag{3.41}$$

so the wave is not completely dephased by the extra dipolar contribution. Parameterising the charges as $q_A = \alpha_A m_A/(2M_P)$, so that for equal masses

$$a_1 = \frac{m\alpha_-}{4M_P} \qquad\qquad a_2 = \frac{m\alpha_+}{8M_P} \tag{3.42}$$

this experimental data places the following constraint on $\alpha_\pm = \alpha_1 \pm \alpha_2$,[25]

$$\alpha_-^2 \overset{!}{<} 10^{-5}\, \alpha_+^4 \tag{3.43}$$

which amounts to

$$P^{\rm dipole} \overset{!}{<} 10^{-8}\, \alpha_+^2\, P^{\rm quadrupole}\,. \tag{3.44}$$

So we see that $P^{\rm dipole}$ must be *hugely* smaller than the $P^{\rm quadrupole}$, or else it will lead to large changes in the waveform.

Exercise. When comparing with experimental values, we need to convert our clean natural unit expressions into something with proper SI units. Show that, for equal masses and GR-type couplings of $q_A = \alpha_A m_A/(2M_P)$, the phase and its shift can be written as

$$\Psi_{22} = \frac{3}{\alpha_+^2}\left(\frac{c^3}{2G_N m\,\Omega}\right)^{5/3}, \qquad\qquad \delta\Psi_{22} = \frac{75\alpha_-^2}{224\alpha_+^4}\left(\frac{c^3}{2G_N m\,\Omega}\right)^{7/3} \tag{3.45}$$

which now have the correct units, and can be evaluated using $G_N = 6.67 \times 10^{-11}$ kg^{-1} m^3 s^{-2}, $c = 3 \times 10^8$m/s and $m = 20 \times 10^{30}$kg.

In summary, in the Newtonian limit the particles can undergo circular orbits around their centre-of-mass, which produces multipoles for every ℓ that trasmit power:

$$P = \kappa\, \Omega_0^2 \sum_{\ell=0} C_\ell\, \mathtt{x}^{2\ell} \tag{3.46}$$

where C is a dimensionless constant (dimensionless in SI units). For instance the quadrupole is

$$C_2 = \frac{32}{30\pi}\frac{a_2^2}{\kappa} \tag{3.47}$$

[24] "in-band" means "at a frequency the detector can pick up".

[25] Note that the phase change is dominated by the low frequency $\Omega_{\rm in} = 10\pi$ Hz, which for these masses gives $r_0 = 1400$km as the initial size of the binary; significantly larger than their 30km Schwarschild radii and at speeds $v/c \approx 0.1$.

and in GR-type theories is the dominant emission channel. Then we have predominantly quadurpole radiation (the $\ell = m = 2$ mode), in which the phase is given by

$$\Psi_{22} = \frac{2}{5C_2}\left(\frac{\mu\, c^3}{\kappa\, \Omega_0}\right)^{5/3} \tag{3.48}$$

where I restored the factors of c so that it can evaluated in SI units[26].

3.3 Post-Newtonian corrections

Now suppose we go beyond the Newtonian limit by including

$$S_0[\mathbf{x}_A] = -m_A \int dt\big[1 + \tfrac{1}{2}|\dot{\mathbf{x}}_A|^2 \overbrace{+\tfrac{1}{8}|\dot{\mathbf{x}}_A|^2 + ...}^{(c)}\big] \ , \quad S_2[\varphi] = \frac{1}{2}\int d^4x \varphi(\partial_i^2 \overbrace{-\partial_t^2}^{(a)})\varphi \ ,$$
$$J_A = q_A \delta(\mathbf{x} - \mathbf{x}_A(t))\big[1 + \underbrace{\frac{1}{2}|\dot{\mathbf{x}}_A|^2 + ...}_{(b)}\big] \tag{3.49}$$

There are three differences:

(a) The scalar field is now dynamical: this leads to both retardation effects (it takes time for signals to propagate between the two bodies) and radiation effects (scalar waves can carry energy away from the binary).

(b) The particles' potential energy is modified; this arises from coupling to the fields in a Lorentz-invariant way, otherwise the resulting force law would not be Lorentz-covariant.

(c) The particles' kinetic energy is modified; this amounts to replacing the momentum $m_A\dot{\mathbf{x}}_A$ with the Lorentz-dilated $m_A\dot{\mathbf{x}}_A/\sqrt{1-|\dot{\mathbf{x}}_A|^2}$.

Of these, (a) is the most difficult. So we'll start with that one.

(a) Relativistic fields

Suppose we only turn on (a), i.e. consider a theory in which the particles move according to the usual Newtonian second law

$$m_A\ddot{\mathbf{x}}_A(t) = -\partial^i\varphi(t,\mathbf{x})|_{\mathbf{x}=\mathbf{x}_A(t)} \tag{3.50}$$

but now the field mediating the force is fully relativistic

$$\left(\partial_t^2 - \partial_i^2\right)\varphi_A(t,\mathbf{x}) = J_A(t,\mathbf{x}) \tag{3.51}$$

Using the retarded Green function $G_R(x-x') = \delta((t-t') - |\mathbf{x}-\mathbf{x}'|^2)$, we find

$$\varphi_A(t,\mathbf{x}) = \int d^4x'\, G_R(x-x')J_A(x')$$
$$= \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_A(t_A)|(1-\mathbf{v}_A(t_A)\cdot\mathbf{n}(t_A,\mathbf{x}))} \qquad \text{with} \qquad \mathbf{n}(t,\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}_A(t)}{|\mathbf{x}-\mathbf{x}_A(t)|} \tag{3.52}$$

[26] In this theory, $\kappa = G_N\alpha_1\alpha_2 m_1 m_2$ and should be fixed to the value $G_N^{\rm obs}m_1m_2$ where $G_N^{\rm obs} \approx 6.67\times10^{-11}{\rm kg}^{-1}{\rm m}^3{\rm s}^{-2}$.

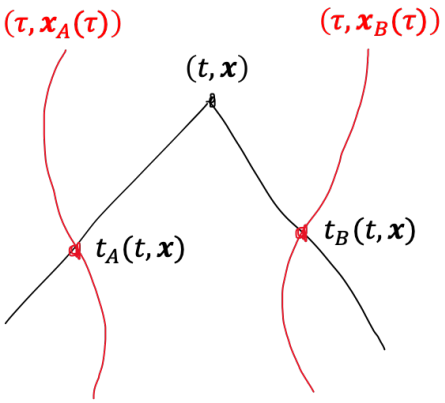


Figure 2: Spacetime diagram showing the definition of t_A from (3.53).

where t_A is the solution of

$$t - t_A - |\mathbf{x} - \mathbf{x}_A(t_A)| = 0 \tag{3.53}$$

i.e. it is the time at which a null ray leaving the worldline $(t, \mathbf{x}_A(t))$ in order to arrive at $(t, \mathbf{x})$. This can be understood as $\varphi \sim 1/(4\pi R)$, where R is the distance that φ needs to travel if the source were at rest, multiplied by a boost factor to account for the motion of the source (relativistic Doppler effect: leads to "beaming"/headlight effect of the signal in the forward direction).

Retardation. There are two consequences of this. The first is that, if we assume the particles are moving slowly, we can expand

$$t_A = t - \left(1 + \mathbf{n} \cdot \mathbf{v}_A(t) + \frac{|\mathbf{v}_A(t)|^2}{2} + \frac{(\mathbf{n} \cdot \mathbf{v}_A(t))^2}{2} + \mathcal{O}\left(v^3\right)\right) |\mathbf{x} - \mathbf{x}_A(t)| \tag{3.54}$$

which gives

$$\varphi_A(t, \mathbf{x}) = \frac{q_A}{4\pi |\mathbf{x} - \mathbf{x}_A(t)|} \left[1 + \frac{|\mathbf{v}_A(t)|^2}{2} - \frac{(\mathbf{n} \cdot \mathbf{v}_A(t))^2}{2} + \mathcal{O}\left(v^4\right)\right] . \tag{3.55}$$

The equation of motion is then

$$m_1 \ddot{\mathbf{x}}_1(t) = \frac{q_1 q_2}{4\pi |\mathbf{r}_{12}|^3} \left[\left(1 + \frac{|\mathbf{v}_2(t)|^2}{2} - \frac{3(\mathbf{n} \cdot \mathbf{v}_2(t))^2}{2}\right) \mathbf{r}_{12} + (\mathbf{v}_2 \cdot \mathbf{r}_{12}) \mathbf{v}_2 + \mathcal{O}\left(v^4\right)\right] \tag{3.56}$$

These are relativistic corrections to the magnitude of the Newtonian force, plus an additional force which points along the velocity of the other particle.

Radiation. The second consequence is that φ now carries energy off to infinity.

$$\varphi_1(t, \mathbf{x}) = \frac{1}{|\mathbf{x}|} \left(I_1(t - |\mathbf{x}|) + ...\right) \tag{3.57}$$

where the multipole expansion is

$$\begin{aligned} J_A(t, \mathbf{x}) &= q_A \delta\left(\mathbf{x} - \mathbf{x}_A(t)\right) \\ &= q_A \left[\delta(\mathbf{x}) + \mathbf{x}_A^i \partial_i + ...\right] \delta(\mathbf{x}) \end{aligned} \tag{3.58}$$

$$I_A(t) = q_A \left(1 - \frac{1}{6}\partial_t^2 |\mathbf{x}_A(t)|^2 + ...\right)$$
$$I_A^i(t) = q_A \left(1 + ...\right) \mathbf{x}_A^i(t)$$
$$I_A^{ij}(t) = q_A \left(1 + ...\right) Q^{ij}(\mathbf{x}_A(t)) \tag{3.59}$$

We do this because $\mathbf{x}_A^i \partial_i \sim rk \sim v$ is a small quantity in the PN expansion. Notice that, if the coupling to the source is constant, then $\dot{I}_A(t) = 0 + \mathcal{O}(v^2)$ and the monopole does not radiate power. In electromagnetism, $q_A \mathbf{x}_A^i$ is the dipole and this is the leading source of power. There's also monopolar power if a charge accelerates. In GR, both $I_{\rm LO}^i$ and $I_{\rm NLO}$ do not radiate, so the leading power comes from the quadrupole.

Radiation reaction. To account for the changing energy, the classical approach has historically been to balance energy: emitting the radiation must push the particles in such a way that the work done balances the energy carried away by the radiation:

$$\int_{\mathbf{x}} \mathbf{F}^{\rm rad} = PT \ . \tag{3.60}$$

The radiation reaction force is called the *Abraham-Lorentz force* in EM, or the *Burke-Thorne force* in GR. Nowadays, it is viewed as "self-energy" effect: while the gravitational field exerted by a particle on itself is formally infinite (i.e. $\varphi_A(t, \mathbf{x}) \to \infty$ as $\mathbf{x} \to \mathbf{x}_A$), if carefully renormalised there is a finite remainder that satisfies (3.60). This is how to derive the radiation reaction force from first principles.

(b) Relativistic couplings

The additional terms in (b) are required by Lorentz invariance. They inclusion is straightforward, since it amounts to adding an additional

$$\varphi_A \supset -\frac{q_A v_A^2}{4\pi |\mathbf{x} - \mathbf{x}_A(t)|} \tag{3.61}$$

to the field. To see that this is consistent with Lorentz invariance, notice that the full field profile is now

$$\varphi_A(t, \mathbf{x}) = \frac{q_A}{4\pi R_A} \tag{3.62}$$

where

$$R_A = |\mathbf{x} - \mathbf{x}_A(t)| \left(1 - \frac{(\mathbf{n} \cdot \mathbf{v}_A)^2}{2} + ...\right) = \frac{1}{\gamma} |\mathbf{x} - \mathbf{x}_A(t)| \ . \tag{3.63}$$

is the length contracted separation from source and receiver, where $\gamma = 1/\sqrt{1 - (\mathbf{n} \cdot \mathbf{v})^2}$.

(c) Relativistic kinetic energy

These do not modify the coupling to the field, but they do modify the force law

$$\partial_t \left(m_A \mathbf{v}_A \left(1 + \frac{v_A^2}{2} + ...\right)\right) = \mathbf{F}_A \tag{3.64}$$

This can be written as

$$m_A \dot{\mathbf{v}}_A = \mathbf{F}_A + \mathbf{F}_A^{(c)} \qquad \text{with} \qquad \mathbf{F}_A^{(c)} = -\mathbf{F}_A \frac{v_A^2}{2} - m_A \mathbf{v}_A (\mathbf{v}_A \cdot \mathbf{F}_A) \,. \tag{3.65}$$

This amounts to a rescaling of the strength of the force, but also a new force in the direction of the particle's motion. In practice, since these terms mostly decouple from the effective potential, they are usually treated separately (as part of the problem of solving for the $\mathbf{x}_A(t)$ trajectories rather than the problem of determining their effective equation of motion).

Summary

So overall, we learn that the 1PN correction to the motion of a point particle coupled to a scalar field can be written as

$$m_A \ddot{\mathbf{x}}_A(t) = \mathbf{F}_A^N(t) + \underbrace{\mathbf{F}_A^{(a)}(t) + \mathbf{F}_A^{(b)}(t) + \mathbf{F}_A^{(c)}(t)}_{\text{conservative}} + \underbrace{\mathbf{F}_A^{\text{rad}}(t)}_{\text{dissipative}} \tag{3.66}$$

which includes corrections to the usual Keplerian force law coming from (a) dynamical field effects (retardation and radiation reaction forces), (b) relativistic corrections to the potential, (c) relativistic corrections to the kinetic energy. As we go to higher PN orders, there are further corrections to both the kinetic energy of the particles and their coupling to the fields. Ideally we'd have a systematic way to compute these various corrections, and extend to higher orders in perturbation theory. That will be the focus of Section 4.

From scalar fields to General Relativity

But first, let us comment on what this toy model teaches us about GR.

In General Relativity, we can repeat all of the preceding steps. Beginning with the multipolar EFT, we find that replacing

$$\varphi J \to h_{\mu\nu} T^{\mu\nu} \tag{3.67}$$

where $T_{\mu\nu}$ is the stress-energy tensor, leads to the famous quadrupole formula at leading PN order

$$P = \frac{G_N}{5} \langle \dddot{\mathcal{I}}^{ij} \dddot{\mathcal{I}}^{ij} \rangle_{\text{orbit}} \tag{3.68}$$

where these (mass) multipoles are moments of the energy density T^{00},

$$\mathcal{I}^{ij}(t) = \int_{\mathbf{x}} T^{00}(t, \mathbf{x}) Q^{ij}(\mathbf{x}) \tag{3.69}$$

There are analogous (current) multipoles from taking moments of T^{0i}, and these are important beyond the leading PN order.

Note that the equivalence principle fixes $q_A = m_A/2M_P$. As a result, the dipolar power vanishes and the quadrupolar power at fixed Ω (i.e. with $\mathbf{x} = (G_N M \Omega)^{2/3}$) depends on the single combination

$$M_c = \left[\frac{\mu^2}{M^4} (m_1 m_2^2 + m_2 m_1^2)^2 M^{\frac{10}{3}} \right]^{3/10} = \mu^{3/5} M^{2/5} \tag{3.70}$$

which is called the "chirp mass". This gives a key result used in LIGO/Virgo/KAGRA data analysis:

$$P^{\text{quadrupole}} = \frac{32}{5 G_N} (G_N M_c \Omega)^{10/3} \,. \tag{3.71}$$

Finally, unlike for the scalar field, the only diffeomorphism invariant couplings between $h_{\mu\nu}$ and the world-line of the particle start at 5PN. This leads to the *effacement theorem*, which says that finite size effects for compact objects in GR only show up at 5PN order. This is one reason why 5PN is such an important threshold in these perturbative calculations. Below 5PN, there is literally *no difference* between the predictions for a binary black hole system or a binary neutron star system. But from 5PN and onwards, specifying finite-size operators (i.e. the nature of the compact object) becomes necessary; or put another way, before 5PN order observations can be used to probe the finite-size structure of these merging objects.

4 Non-relativistic EFT for binary radiation

In order to use the technology developed for QFT, we will now recast the post-Newtonian problem in momentum space:

$$\varphi(t,\mathbf{x}) = \int_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\varphi(t,\mathbf{k}) = \int_{\omega\mathbf{k}} e^{-i\omega t}e^{-i\mathbf{k}\cdot\mathbf{x}}\varphi(\omega,\mathbf{k}) \,. \tag{4.1}$$

In particular, the propagator for $(\partial_t^2 - \partial_i^2)\varphi$ becomes simply

$$G = \frac{1}{k^2-\omega^2} \tag{4.2}$$

and loop integration techniques will allow us to efficiently compute very high PN corrections.

4.1 Separating near zone and far zone

Inconsistent limits? Using the definition (4.1), the multipole expansion of the source becomes

$$\int_{t,\mathbf{x}} J(t,\mathbf{x})\varphi(t,\mathbf{x}) = \int_{t,\mathbf{k}} \left(I(t) - i\mathbf{k}^i I^i(t) - \mathbf{k}^i\mathbf{k}^j I^{ij}(t) + ...\right)\varphi(t,\mathbf{k}) \,. \tag{4.3}$$

and the EFT expansion in derivatives becomes an expansion in powers of $\mathbf{k}$. This makes sense because, since time derivatives are suppressed by $\sim v/c$ in the PN expansion, if φ satisfies the free equation of motion $(\partial_t^2 + k^2)\varphi = 0$ then $k^2 \sim v^2/c^2$ is also PN suppressed.

However, there is an important subtlety here. Notice that, for a single particle A,

$$J(t,\mathbf{x}) = m_A\delta(\mathbf{x}-\mathbf{x}_A(t)) \qquad \Rightarrow \qquad J(t,\mathbf{k}) = m_A e^{i\mathbf{k}\cdot\mathbf{x}_A(t)} \tag{4.4}$$

and to find the Newtonian potential, we argued exactly the opposite limit, namely that time derivatives are much smaller than spatial derivatives for a slowly moving source:

$$\begin{aligned} J &= (\partial_t^2 + k^2)\varphi \approx k^2\varphi \\ \Rightarrow \varphi(t,\mathbf{x}) &\approx \int_{\mathbf{k}} \frac{m_A e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_A(t))}}{k^2} = \frac{m_A}{4\pi|\mathbf{x}-\mathbf{x}_A(t)|} \,. \end{aligned} \tag{4.5}$$

So there is a conceptual puzzle here: once we go to momentum space, should we expand in small $\mathbf{k} \sim v/c$, or should we expand in large $\mathbf{k} \gg v/c$?

Splitting the field. The solution is to split the field into two parts:

$$\varphi = \bar{\varphi} + \delta\varphi \tag{4.6}$$

where $\bar{\varphi}$ are the *radiation* modes (obeying $(\partial_t^2 - \partial_i^2)\bar{\varphi} \approx 0$ for a propagating wave) and $\delta\varphi$ are the *potential* modes (obeying $\partial_i^2 \delta\varphi \approx J$ for a Newtonian potential). We then carry out perturbation theory as if there are two scalar fields in the theory, and should treat the momentum of every $\bar{\varphi}$ as small and the momentum of every $\delta\varphi$ as large. Physically, this corresponds to separating small-scale fluctuations of the field ($\delta\varphi$) from the large-scale radiation that we measure ($\bar{\varphi}$). Mathematically, this corresponds to using

$$\begin{aligned} \frac{1}{k^2 - \omega^2} &= \frac{1}{k^2} + \frac{\omega^2}{k^2} + ... \qquad \text{if } \delta\phi \text{ propagator,} \\ J(t, \mathbf{k}) &= I(t) + i\mathbf{k}_i I_i(t) + ... \quad \text{if } \bar{\varphi} \text{ coupling .} \end{aligned} \tag{4.7}$$

Method of regions. This trick of splitting the field into two contributions with complementary expansions is known as the *method of regions*. It is a very useful technique that underpins essentially every EFT. To illustrate why it is necessary to include both expansion regions, consider the example integral

$$\int_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2 - \omega^2 + i\epsilon} = \frac{e^{-i|\omega|x}}{4\pi x} \tag{4.8}$$

If we naively expand the integrand in powers of ω, we find a series of IR divergent integrals. In dimensional regularisation, these become (see (A.10))

$$I_{\text{large } k} = \int_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2} \left(1 + \frac{\omega^2}{k^2} + \frac{\omega^4}{k^4} + ...\right) = \frac{1}{4\pi x}\left(1 - \frac{1}{2}\omega^2 x^2 + \frac{1}{24}\omega^4 x^4 + ...\right) . \tag{4.9}$$

This does not match the expansion of the right-hand-side of (4.8). For one thing, (4.9) is even in ω. Another alarm bell is that the $i\epsilon$ dependence has disappeared and so any propagator would give the same expansion (4.9) (but only the Feynman prescription gives (4.9)). This alarm bell also provides the hint we need to resolve the issue. The problem with (4.9) is that we are supposed to integrate over all k, including the integration region $k \approx \omega$ in which we may not expand $1/(k^2 - \omega^2)$. However, in that region k itself is small: we can therefore expand the numerator in a series of UV divergent integrals, which evaluate in dimensional regularisation to:

$$I_{\text{small } k} = \int_{\mathbf{k}} \frac{1 - i\mathbf{k}\cdot\mathbf{x} - (\mathbf{k}\cdot\mathbf{x})^2 + ...}{k^2 - \omega^2 + i\epsilon} = \frac{1}{4\pi x}\left(-i|\omega|x + \frac{i|\omega|^3 x^3}{3!} + ...\right) . \tag{4.10}$$

These are precisely the terms missing from (4.9), so that $I_{\text{large } k} + I_{\text{small } k} = e^{-i|\omega|x}/(4\pi x)$. The moral of this example is:

$$\text{expansion of integral} \neq \text{expansion of integrand}$$

unless we carefully add up all of the relevant expansion regions.

Layers of EFT. Our goal is then to remove ("integrate out") the $\delta\varphi$ fluctuations from S_{pp} to produce an even simpler EFT for just the radiation $\bar{\varphi}$. This is called a non-relativistic EFT (NREFT),

which we will parameterise as:

$$S_{\rm NR}[\bar{\varphi}, \mathbf{x}_A] = \int_{t,\mathbf{x}} \left\{ -\frac{1}{2}(\partial_\mu \bar{\varphi})^2 - V - (I + I_i \partial_i + I_{ij}\partial_i\partial_j + ...)\,\bar{\varphi} + ... \right\} \tag{4.11}$$

where V and the multipole moments depend on the $\mathbf{x}_A(t)$ and the $+...$ captures non-linear self-interactions (see section 4.3.3) or non-linear couplings to the binary.

Finally, one might wonder what happens if the radiation modes of the field are also integrated out. This produces the so-called *effective action* $W[\mathbf{x}_A]$, which is the simplest possible description of the binary motion (but no longer captures information about the waveform). From a path integral perspective, we are describing the following:

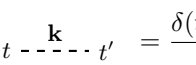
$$e^{iW[\mathbf{x}_A]} = \int \mathcal{D}\bar{\varphi}\; e^{iS_{\rm NR}[\bar{\varphi},\mathbf{x}_A]} = \int \mathcal{D}\bar{\varphi} \left[\mathcal{D}\delta\varphi\; e^{iS_{\rm PP}[\varphi=\bar{\varphi}+\delta\varphi,\mathbf{x}_A]} \right] \tag{4.12}$$

where $\int \mathcal{D}X$ represents (the classical limit of) a path integral over the modes X.

4.2 Feynman rules

We will now make this NREFT picture calculationally precise by providing simple Feynman rules with which to calculate all PN corrections. In order to implement (4.7) diagrammatically, we must introduce an infinite series of propagators for $\delta\varphi$ and an infinite series of worldline couplings for $\bar{\varphi}$. To evaluate a diagram, we use the following rules[27]:

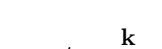
- $\delta\varphi$ propagators are represented by dashed lines and typically labelled by two times and a single momentum. The leading order one is:



$$= \frac{\delta(t-t')}{k^2} \tag{4.13}$$

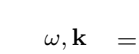
  and subleading ones are denoted by treating $\varphi\partial_t^2\varphi$ as a quadratic vertex insertion, for instance:

$$= \frac{-\partial_t^2\delta(t-t')}{k^4}\,, \qquad \text{etc.} \tag{4.14}$$



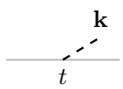
$$= \frac{\partial_t^4\delta(t-t')}{k^6}\,,$$

- $\bar{\varphi}$ propagators are represented by wavy lines and typically represented by a single frequency and momentum:



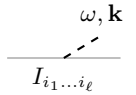
$$= \frac{1}{k^2 - \omega^2} \tag{4.15}$$

- $\delta\varphi$ couples to each wordline source J_A



$$= J_A(t, \mathbf{k}) \tag{4.16}$$

- $\bar{\varphi}$ couples to each worldline via a vertex for each multipole moment:



$$= I_{i_1\ldots i_\ell}(\omega)\mathbf{k}_{i_1}\ldots\mathbf{k}_{i_\ell} \tag{4.17}$$

[27] Note that some sources include i's in the propagator and vertex factors. These combine to give i^{L-1}, and since $L = 0$ we have simply removed them.

- an external $\bar{\varphi}$ line represents a factor of $\bar{\varphi}(t, \mathbf{k})$.
- any non-linearity in the action that $\sim \varphi^n$ corresponds to an interaction vertex with n legs and contributes a vertex factor $\delta^n S/\delta\varphi^n$.
- Finally, all times/momenta/frequency are integrated over.

Potential. With these rules, the potential is given by summing all diagrams with no external or radiation $\bar{\varphi}$ lines:

$$\int_t V = \quad \text{(- - - only)} \tag{4.18}$$

including any symmetry factors which account for the different ways a diagram can be permuted without changing. For instance, without any non-linearities

$$\int_t V = \quad + \frac{1}{2} \quad + \frac{1}{2} \tag{4.19}$$

Multipole moments. The multipole moments are given by summing all diagrams with just one external radiation $\bar{\varphi}$ line:

$$\int_{t,\mathbf{k}} \left(I(t) - i\mathbf{k}^i I^i(t) - \mathbf{k}^i\mathbf{k}^j I^{ij}(t) + ...\right) \bar{\varphi}(t, \mathbf{k}) = \quad \text{(- - - only)} \tag{4.20}$$

For instance, without any non-linearities, the matching between $S_{|rmpp}$ and $S_{\rm NR}$ is trivial since:

$$\int_{t,\mathbf{k}} \left(I(t) - i\mathbf{k}^i I^i(t) - \mathbf{k}^i\mathbf{k}^j I^{ij}(t) + ...\right) \bar{\varphi}(t, \mathbf{k}) = \quad + \tag{4.21}$$

and therefore $I_{i_1\ldots i_\ell}$ is simply the total multipole moment of each point particle. In section 4.3.3 we will consider a cubic interaction and find non-trivial corrections to the multipole moments from diagrams such as:

$$\tag{4.22}$$

Effective action. The effective action is given by summing all diagrams with no external lines:

$$\int_t W = \tag{4.23}$$

For instance, without any interactions

$$\int_t W = \int_t V + \quad + \frac{1}{2} \quad + \frac{1}{2} \tag{4.24}$$

Collapsed worldlines. Finally, a condensed notation is often used for the sum over all relabellings of the worldlines,

$$\quad = \quad + \frac{1}{2} \quad + \frac{1}{2} \tag{4.25}$$

This is useful because radiation in the far zone will always depend on only the total multipoles of the binary (since it couples to the $\mathbf{x}$-independent multipoles of $S_{\rm pp}$).

4.3 Applications of NREFT

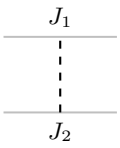
We will now draw and compute the various diagrams that contribute at the first few PN orders. We will denote the sources in $S_{\rm pp}$ as

$$J_A(t,\mathbf{x}) = E_A(t)\delta(\mathbf{x}-\mathbf{x}_A(t)) \tag{4.26}$$

where $E_A = m_A\sqrt{1-v_A^2}$.

4.3.1 Potential

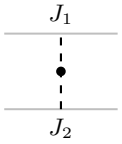
The simplest diagram is the exchange of a potential $\delta\varphi$ fluctuation between the two bodies:



$$= \int_{t,\mathbf{k}} \frac{J_1(t,\mathbf{k}))J_2(t,-\mathbf{k})}{k^2} = \int_t \frac{E_1E_2}{4\pi r}\,. \tag{4.27}$$

It is the exchange of this field that mediates the Newtonian potential, and this single diagram now also conveniently contains all relativistic corrections to the masses of the objects.

Retardation. To incorporate retardation effects from the motion of the sources, we can evaluate a similar diagram with a subleading propagator [4, (2.10)]:



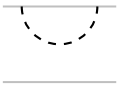
$$= \int_{t,\mathbf{k}} \frac{J_1(t,\mathbf{k}))\partial_t^2 J_2(t,-\mathbf{k})}{k^4} = \int_t \frac{E_1E_2}{8\pi r}\left[\mathbf{v}_1\cdot\mathbf{v}_2 - \frac{\mathbf{v}_1\cdot\mathbf{r}\;\mathbf{v}_2\cdot\mathbf{r}}{r^2}\right]\,. \tag{4.28}$$

In general, this diagram with $(\omega^2/k^2)^n$ factors inserted into the propagator will generate a series in $\frac{E_1E_2}{4\pi r}v^{2n}$.

Self energy. Finally, there is a diagram in which a worldline feels its own potential:

$$= \int_{t,\mathbf{k}} \frac{J_1(t,\mathbf{k}))J_1(t,-\mathbf{k})}{k^2} = \int_{t,\mathbf{k}} \frac{E_1^2}{k^2} \tag{4.29}$$

This integral diverges in the UV. This represents finite size effects we have neglected in our point particle approximation. Luckily, (4.29) vanishes in dim reg, but it is instructive to consider another scheme to see what becomes of the divergence. For instance with a hard UV cutoff, $\int_0^\Lambda dk$, we have



$$= \frac{E_1^2\Lambda}{2\pi^2}\,. \tag{4.30}$$

This divergence can be absorbed into a renormalised energy:

$$E_A^{\rm ren} = E_A + \frac{E_1^2\Lambda}{2\pi^2}\,. \tag{4.31}$$

Point particle limit. To explain this shift, let us momentarily return to position space. Since an energy density ρ would create a field,

$$\delta\varphi = \int_{\mathbf{y}} \frac{\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \tag{4.32}$$

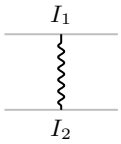
it will have a potential energy in that field of:

$$E_{\text{self}} = \int_{\mathbf{x}} \left(\int_{\mathbf{y}} \frac{\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} \right) \rho(\mathbf{x}) \tag{4.33}$$

This integral need not diverge if ρ is an extended mass distribution. But if we now take our point particle limit, $\rho(\mathbf{x}) \to E_A \delta(\mathbf{x}-\mathbf{x}_A) = \int_{\mathbf{k}} E_A\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_A)}$, we find precisely that $E_{\text{self}} = \frac{E_1^2 \Lambda}{2\pi^2} \to \infty$.

4.3.2 Radiation

Radiation reaction. The emission of radiation exerts a force back on the object. To find this radiation reaction force, we can integrate out $\bar{\varphi}$ to find the effective action W. There is a contribution from the diagram

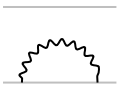


$$= \int_{\omega,\mathbf{k}} \frac{I_1(\omega) I_2(-\omega)}{k^2-\omega^2} \propto \int_{\omega} I_1(\omega) I_2(-\omega)\, \omega \tag{4.34}$$

where we have focussed on the monopole contribution. In the time domain, this gives

$$W \propto I_1 \dot{I}_2 \tag{4.35}$$

In total,

$$\int_t W_{\text{rad}} = \frac{1}{2}$$ $$+$$ $$+ \frac{1}{2}$$ 

$$\propto \int_t I\dot{I}\,. \tag{4.36}$$

where $I = I_1 + I_2$.

Since $E_A = m_A\sqrt{1-v_A^2}$, the change in the total energy is

$$\dot{E} = \sum_A m_A \mathbf{a}_A \cdot \mathbf{v}_A \tag{4.37}$$

Using the equations of motion from the effective action[28],

$$\sum_A \left(\frac{\delta W_{\text{rad}}}{\delta \mathbf{x}_A} \right) \cdot \mathbf{v}_A \propto -\dot{I}\dot{I} \tag{4.38}$$

This precisely matches the emitted power found above, and confirms the flux-balance relation.

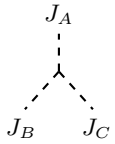
[28] Note that, as written, W_{rad} is a total time derivative and therefore does not contribute to the equations of motion. The resolution is to apply retarded boundary conditions more carefully, for instance using the in-in formalism, see [10].

4.3.3 Non-linearities

To explore the effect of self-interactions, consider now adding the interaction $g\varphi^3$ to $S_{\rm pp}$. Diagrammatically, this introduces cubic vertices $\{\delta\varphi^3, \delta\varphi^2\bar{\varphi}, \delta\varphi\bar{\varphi}^2, \bar{\varphi}^3\}$ with vertex factor

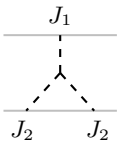
$$F_{\varphi^3}(\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3) = g(2\pi)^d\delta^d\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right) \ . \tag{4.39}$$

Conservative potential. Potential energy gravitates. This is captured by diagrams with three potential modes



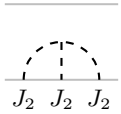
$$= \int_t \int_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \frac{J_A(t,\mathbf{k}_1)J_B(t,\mathbf{k}_2)J_C(t,\mathbf{k}_3)}{k_1^2k_2^2k_3^2} F_{\varphi^3}(\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3) \ . \tag{4.40}$$

where $J_A(t,\mathbf{k}) = E_A(t)e^{i\mathbf{k}\cdot x_A(t)}$ and again $\{A,B,C\}$ are either worldline 1 or 2. The momentum integrals can be evaluated in d dimensions using the identities (A.3) and (A.10). There is a potential energy contribution that diverges in the IR [4, (2.19)],



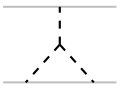
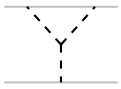
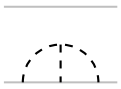
$$= \int_t gE_1E_2^2 \int_{\mathbf{kq}} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{k^2q^2|\mathbf{k}-\mathbf{q}|^2} = \int_t \frac{gE_1E_2^2}{16\pi^2}\left(\frac{1}{(d-3)_{\rm IR}} + \log\, r + ...\right) \tag{4.41}$$

where the +... denote finite constants and terms that vanish as $d \to 3$. There is also a self energy contribution that diverges in both the IR and UV:



$$= \int_t gE_2^3 \int_{\mathbf{kq}} \frac{1}{k^2q^2|\mathbf{k}-\mathbf{q}|^2} = \int_t \frac{gE_2^3}{16\pi^2}\left(\frac{1}{(d-3)_{\rm IR}} - \frac{1}{(d-3)_{\rm UV}}\right) \tag{4.42}$$

where we have used (A.6) to separate the divergence coming from the $\mathbf{k}\to 0$ and $\mathbf{k}\to\infty$ limits of the integration. In total,

$$\int_t V_{\varphi^3} = \frac{1}{3!}$$ $$+\frac{1}{2}$$  $$+\frac{1}{2}$$  $$+\frac{1}{3!}$$ 

$$= \int_t \frac{g}{3!(4\pi)^2}\left[\frac{E^3}{(d-3)_{\rm IR}} - \frac{E_1^3+E_3^3}{(d-3)_{\rm UV}} + 3EE_1E_2\log r\right] \ . \tag{4.43}$$

Notice that:

- The UV divergences from the self energy diagrams can be absorbed into mass counter-terms as before.
- The IR divergences have combined into a function of the total binary energy $E = E_1 + E_2$. We will see below how they cancel against other divergences from radiation diagrams.
- The physical effect of this interaction is the $\log r$ term. This leads to a force that $\sim 1/r$, so would dominate over the Newtonian force (that $\sim 1/r^2$) at large distances. This is one reason why massless scalar fields cannot have such interactions in our Universe[29]. Fortunately, in GR the cubic coupling that arises from $\sqrt{-g}R$ is of the form ∂^2h^3 and contains two derivatives. It therefore leads to a force correction that $\sim 1/r^3$, which is indeed consistent with what we observe.

[29] At least not without some screening mechanism that hides these large forces.

Multipole corrections. Potential energy radiates. This is captured by diagrams with one radiative mode and two potential modes, such as:

$$= \int_{t,\mathbf{q}} \bar{\varphi}(t,\mathbf{k}) \frac{J_1(t,\mathbf{k}-\mathbf{q})J_2(t,\mathbf{q})}{|\mathbf{k}-\mathbf{q}|^2 q^2} \tag{4.44}$$

Since the momentum $\mathbf{k}$ carried by $\bar{\varphi}$ is PN suppressed, we should expand in it:

$$= \int_{t,\mathbf{q}} E_1 E_2 \left(\frac{1 - i\mathbf{k}\cdot\mathbf{x}_1}{q^4} - \frac{2\mathbf{q}\cdot\mathbf{k}}{q^6} + \mathcal{O}(\mathbf{k}^2) \right) e^{-i\mathbf{q}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \tag{4.45}$$

from which we can read off the correction to the first two multipole moments [4, (4.9)]:

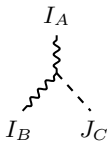
$$(I)_{\varphi^3} = -\frac{E_1 E_2 r}{4\pi} \,, \qquad (I_i)_{\varphi^3} = \frac{E_1 E_2 r}{8\pi} \left(\mathbf{x}_1 + \mathbf{x}_2\right) \,, \qquad \ldots \tag{4.46}$$

The self-energy would also contribute to the multipole moments via the diagrams

(4.47)

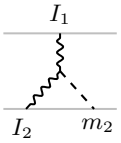
in any regularisation scheme where it is non-zero, however they vanish in dim reg [4, (4.5)].

Tail effect. Radiation gravitates. This is captured by diagrams with two radiative modes; the leading such diagram comes from two monopoles:



$$= \int_{t,\mathbf{k}_3} \int_{\substack{\omega_1,\mathbf{k}_1 \\ \omega_2,\mathbf{k}_2}} \frac{J_C(t,\mathbf{k}_3) I_A(\omega_1) I_B(\omega_2) e^{i(\omega_1+\omega_2)t}}{k_3^2 \left((\omega_1 + i\epsilon)^2 - k_1^2\right)\left((\omega_2 + i\epsilon)^2 - k_2^2\right)} F_{\varphi^3}(\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3) \,. \tag{4.48}$$

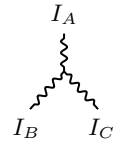
Since $\mathbf{k}_1$ and $\mathbf{k}_2$ belong to radiative modes, they $\sim v/c$ and should be treated as small. By momentum conservation, $\mathbf{k}_3$ is therefore also small and so we expand $J_C = m_C + \mathcal{O}(\mathbf{k})$. The time integral then sets $\omega_1 = -\omega_2$, and we can use the identity (A.4) to evaluate the momentum integrals. This gives



$$= \int_{\omega} \frac{g m_2 I_1(\omega) I_2(-\omega)}{32\pi^2} \left[-\frac{1}{(d-3)_{\text{UV}}} - \log\omega^2 + i\pi \,\text{sign}(\omega) + \ldots \right] \tag{4.49}$$

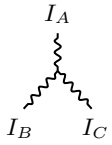
where $+\ldots$ denotes finite constants and terms that vanish as $d \to 3$. Until now, all of our corrections have been analytic functions of ω and therefore correspond to sources and their time derivatives. The non-analytic function of ω in this result represents something *non-local* in time—see [11] for the analogous result in GR.

Memory effect. Radiation radiates. This is captured by diagrams with three radiation modes. The leading diagram comes from three monopoles:



$$= \int_{\omega_j,\mathbf{k}_j} \frac{I_A(\omega_1) I_B(\omega_2) I_C(\omega_3)}{(k_1^2 - \omega_1^2)(k_2^2 - \omega_2^2)(k_3^2 - \omega_3^2)} F_{\varphi^3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\Big|_{\omega_3 = -\omega_1 - \omega_2} \tag{4.50}$$

The momentum integrals can be evaluated using the identities (A.1) and (A.2). They are UV divergent, however the divergence is particularly simple:



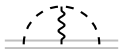
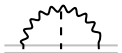
$$= \int_t \frac{g I_A(t) I_B(t) I_C(t)}{32\pi^2} \left[\frac{-1}{(d-3)_{\rm UV}} + \text{finite} \right] \tag{4.51}$$

If we again introduce the double worldline to denote the sum over all permutations of $\{A, B, C\}$, together with the appropriate symmetry factors, we arrive at

$$= \int_t \frac{g}{3!} \frac{I^3}{2(4\pi)^2} \left[\frac{-1}{(d-3)_{\rm UV}} + \text{finite} \right] . \tag{4.52}$$

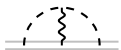
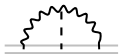
The UV divergences have combined into a function of the total multipole moment only.

Effective potential. The effective potential at leading order in the post-Newtonian expansion is the total of all four effects discussed above:

$$\int_t W_{\varphi^3} = \int_t V_{\varphi^3} + \quad \underset{J\ I\ J}{\text{  }} \quad + \quad \underset{I\ m\ I}{\text{  }} \quad + \quad \underset{I\ I\ I}{\text{ }} \quad . \tag{4.53}$$

One particularly interesting aspect is the cancellation that takes place between the divergences. Recall that V_{φ^3} had UV divergences that can be removed by mass counter-terms, plus an IR divergence that $\sim E^3 = m^3 + ...$ at leading PN order. The radiative corrections to W_{φ^3} contain a UV divergence that also $\sim m^3$, since $I = m + ...$ at leading PN order. Pleasingly, these two remaining divergences exactly:

$$\int_t V_{\varphi^3} + \text{counter-terms} = \int_t \frac{g\, m^3}{3!(4\pi)^2} \left[+ \frac{1}{(d-3)_{\rm IR}} + \text{finite} \right]$$

$$\underset{J\ I\ J}{\text{  }} \quad + \quad \underset{I\ m\ I}{\text{  }} \quad + \quad \underset{I\ I\ I}{\text{ }} \quad = \int_t \frac{g\, m^3}{3!(4\pi)^2} \left[- \frac{1}{(d-3)_{\rm UV}} + \text{finite} \right] \tag{4.54}$$

Divergences. You may recall from your QFT courses that Feynman diagrams with loops typically diverge and require renormalisation. Mathematically, this is due to integrating over all momenta that can flow through the loop, since for certain daigrams/interactions the $\sim 1/p^2$ behaviour of the propagators may not be enough to ensure converge at large $p^2 \to \infty$ (in "the UV") or at small $p^2 \to 0$ (in "the IR"). Physically, these divergences reflect the fact that our field theory description is not valid at arbitrary length scales, but must break down at sufficiently high energies (where new degrees of freedom can become important) or at sufficiently low energies (where classicalisation can take place).

An important lesson is that these divergences are *not* inherently quantum in nature. When computing the NREFT Feynman diagrams that capture the classical potential and radiation from a two-body system, we must similarly integrate over all momentum that can be carried by internal lines[30]. These integrals can diverge and require renormalisation. Just as in QFT, these divergences reflect the fact that our field theory description is only reliable over a limited range of scales.

[30] We usually do QFT on a translationally-invariant background: the total momenta of the fields is therefore conserved at each vertex so the only way to have an intergration over an undetermined momenta is to form a closed loop. However, since we do NREFT on a background with two points particles that spontaneously break translations, we encounter divergent momenta integrals already for tree-level diagrams.

This has caused some difficulties, historically, and led to the introduction of various "ambiguity" parameters in the first 4PN potentials. A nice description of various divergent diagrams in GR can be found in [12].

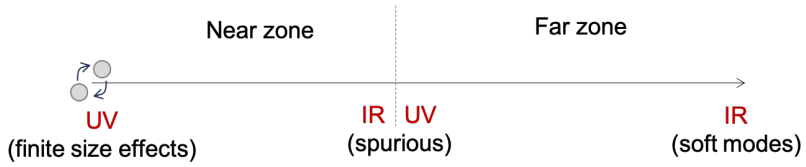


Figure 3: Divergences in the PN expansion are either due to (i) the finite extent of the two compact bodies (⇒ physical UV divergences), (ii) the long-range nature of a massless graviton (⇒physical IR divergences), (iii) the split into near and far zones (⇒ spurious UV/IR pairs of divergences that must cancel, as in (4.54)).

A Dim reg integrals

Once we have constructed Feynman diagrams, we can evaluate them efficiently using dimensional regularisation. Typically, this involves first using Feynman parameters to combine all propagators into a single momenutm-dependent factor:

$$\frac{1}{X_1^{a_1}X_2^{a_2}...X_n^{a_n}} = \frac{\Gamma(a_1+...+a_n)}{\Gamma(a_1)...\Gamma(a_n)}\int_0^1 \frac{dx_1}{x_1}...\int_0^1 \frac{dx_n}{x_n}\frac{x_1^{a_1}...x_n^{a_n}\delta(x_1+...+x_n-1)}{(x_1X_1+...+x_nX_n)^n} \tag{A.1}$$

and then applying the master integral identity:

$$\int_p \frac{1}{(p^2+2\mathbf{p}\cdot\mathbf{q}+m^2)^a} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(a-\frac{d}{2})}{\Gamma(a)}(m^2-q^2)^{\frac{d}{2}-a}\,. \tag{A.2}$$

For instance, applying these leads to

$$\int_{\mathbf{p}} \frac{1}{p^{n_1}|\mathbf{p}-\mathbf{q}|^{n_2}} = \frac{1}{(4\pi)^{d/2}q^{n_1+n_2-d}}\frac{\Gamma\left(\frac{d-n_1}{2}\right)\Gamma\left(\frac{d-n_2}{2}\right)\Gamma\left(\frac{n_1+n_2-d}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(d-\frac{n_1+n_2}{2}\right)} \tag{A.3}$$

which can be applied iteratively to evaluate any conservative sector contribution to V, since the potential modes $\delta\varphi$ have propagators $\propto 1/k^{2n}$. Integrals with radiation mode $\bar{\varphi}$ propagators can also be evaluated in this way, for instance

$$\int_{\mathbf{k},\mathbf{q}} \frac{1}{q^2\left((\omega+i\epsilon)^2-k^2\right)\left((\omega+i\epsilon)^2-|\mathbf{k}+\mathbf{q}|^2\right)} = \frac{1}{d-3}\frac{\Gamma\left(\frac{2-d}{2}\right)\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^d}\left(-(\omega+i\epsilon)^2\right)^{d-3}\,. \tag{A.4}$$

Anther noteworthy integral that can be evaluated in this way is

$$\int_p \frac{1}{p^{2a}(p^2-\omega^2)^b} = \frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(a+b-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)\Gamma(b)}(-\omega^2)^{\frac{d}{2}-a-b} \tag{A.5}$$

since it allows us to separate the UV and IR divergences in any scale-free integral using a trick called "zero-bin subtraction" [13, 14],

$$\begin{aligned}\int_{\mathbf{p}} \frac{1}{p^3} &= \int_{\mathbf{p}} \frac{1}{p(p^2-\Lambda^2)} - \int_{\mathbf{p}} \frac{\Lambda^2}{p^3(p^2-\Lambda^2)} \\ &= -\frac{1}{2\pi^2}\left[\frac{1}{(d-3)_{\rm UV}} - \frac{1}{(d-3)_{\rm IR}}\right]\end{aligned} \tag{A.6}$$

where notice that the first integral diverges only in the UV and the second integral diverges only in the IR (so we have added subscripts to keep track of whether $d \neq 3$ is regulating a large-scale or short-scale effect).

Finally, if the integrand also contains an exponential factor, we can often change integration variables so that it depends on a single momentum $e^{i\mathbf{q}\cdot\mathbf{r}}$ and evaluate the $\int_{\mathbf{q}}$ at the end using[31]

$$\int_{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^n} = \frac{r^{n-d}}{2^n \pi^{d/2}} \frac{\Gamma\left(\frac{d-n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \,. \tag{A.9}$$

For instance, setting $n = 2$ and $d = 3$ gives the well-known result,

$$\int_{\mathbf{q}} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2} = \frac{1}{4\pi r} \,. \tag{A.10}$$

B NREFT with multiple fields

Here I describe the NREFT approach pioneered by Goldberger and Rothstein [15], closely following the mini-review given in [16] that includes additional degrees of freedom such as a light scalar. See Figure 4 for an overview of this EFT pipeline.

Separation of scales. The decisive property that allows us to make analytic predictions about how compact binaries evolve is the presence of large separations of scales that are inherent during the early inspiral. In the NRGR approach [1, 4, 5, 10, 13, 15, 17–24], these separations of scales are used to establish a tower of effective field theories (EFTs) that — when matched onto another — provide a coherent description of the system and facilitate the systematic computation of observables, like the phase of the emitted gravitational-wave signal.

Scalar-tensor theories. Our very first step, then, is to be precise about the class of theories under consideration. We will here be interested specifically in metric theories of gravity that can be captured by the general action

$$S = S_{\rm fields}[g, \phi] + S_{\rm matter}[\tilde{g}(g, \phi)]. \tag{B.1}$$

[31] This follows from the fact that the d-dimensional Fourier transform of a radial function coincides with the following Hankel transform,

$$\int d^d\mathbf{r}\, e^{i\mathbf{q}\cdot\mathbf{r}} f(r) = \int_0^\infty dr\, q \left(\frac{2\pi r}{q}\right)^{d/2} f(r) J_{\frac{d}{2}-1}(qr) \tag{A.7}$$

where J_n is the usual Bessel function of the first kind. This more general relation allows other exponential integrals to also be evaluated in dim reg, for instance

$$\int_{\mathbf{q}} \frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{q^2 - t + i\epsilon} = \frac{1}{2\pi}\left(\frac{\sqrt{t}}{2\pi r}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(\sqrt{t}r\right) = \frac{e^{-\sqrt{t}r}}{4\pi r} + \mathcal{O}(d-3) \tag{A.8}$$

where K_n is the modified Bessel function of the second kind.

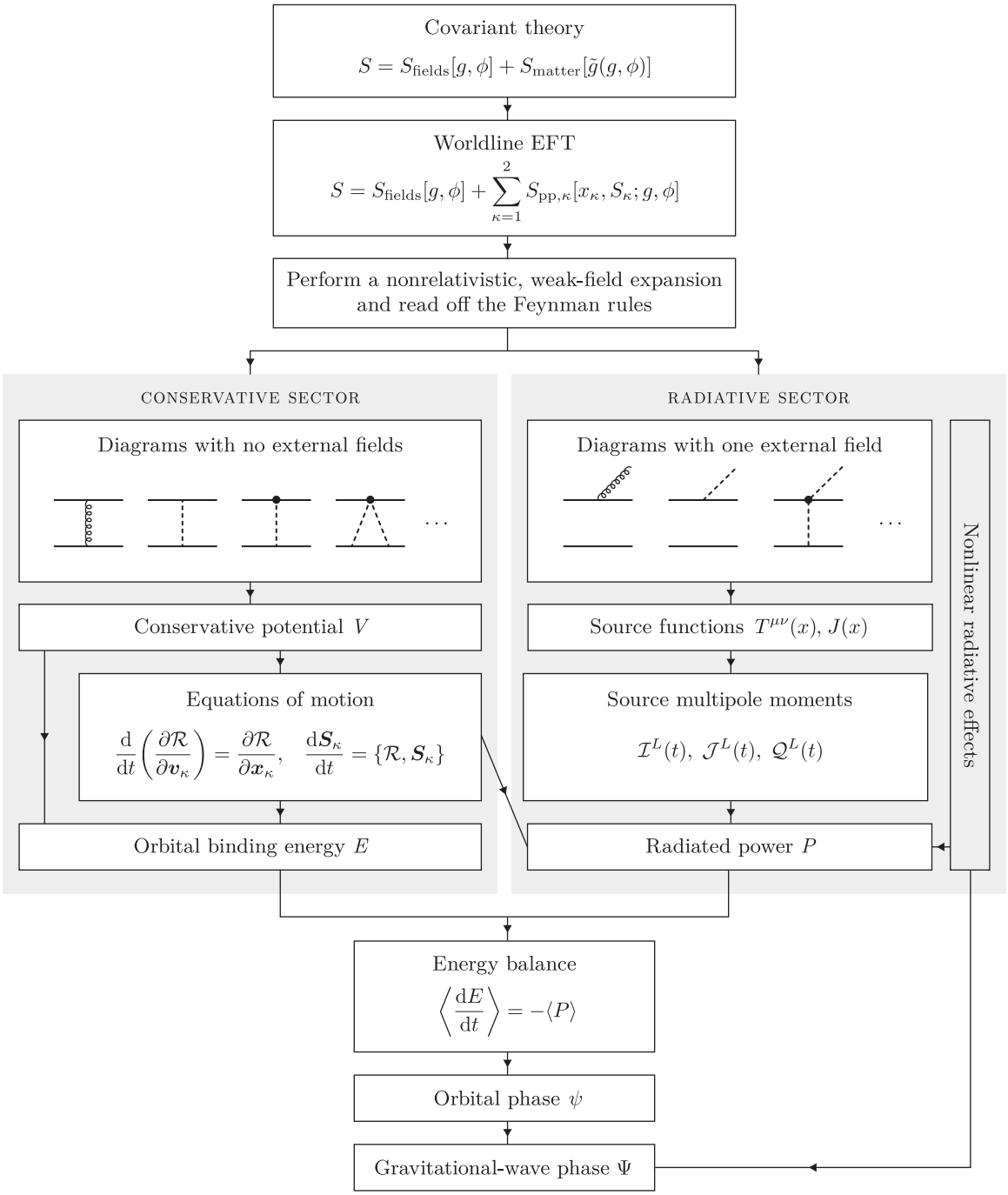


Figure 4: The general pipeline for computing the phase of the gravitational-wave signal emitted by a compact binary in the effective field theory approach. This figure makes allusions to the need for additional steps if one wishes to incorporate nonlinearities like tail effects in the radiative sector.

The kinetic terms for and the interactions between the Einstein-frame metric $g_{\mu\nu}$ and an additional scalar field ϕ are contained in the field action S_{fields}, while the dynamics of the matter fields and their couplings to $g_{\mu\nu}$ and ϕ are encoded in the matter action S_{matter}.

Point-particle approximation. When modelling the inspiral of a (widely separated) two-body system, this surplus of irrelevant information is discarded by replacing (B.1) with the effective action

$$S_{\text{eff}} = S_{\text{fields}}[g, \phi] + \sum_{\kappa=1}^{2} S_{\text{pp},\kappa}[x_\kappa, S_\kappa; g, \phi], \tag{B.2}$$

which is valid on length scales greater than the individual radii of the binary's constituents. In this regime, the two extended bodies (labelled by $\kappa \in \{1, 2\}$) have been replaced by effective point particles whose centres of energy travel along the worldlines $x^\mu_\kappa(\lambda)$. The details about how these point particles move in the bulk of the spacetime and how they interact with the metric and scalar are encoded in the point-particle actions $S_{\text{pp},\kappa}$, while the dynamics of the bulk fields $(g_{\mu\nu}, \phi)$ continue to be captured by the same field action S_{fields} as before.

We may regard this coarse-grained description of the system as emerging directly from (B.1) after having integrated out all of the irrelevant short-wavelength degrees of freedom, although in practice it is much simpler to construct these point-particle actions from the bottom up. In a Lagrangian approach, the relevant objects available for contraction with $g_{\mu\nu}$, ϕ, and their derivatives are the particle's 4-velocity[32] v^μ.

It follows that the action $S_{\text{pp},\kappa}$ is composed of an infinite series of terms, which one can order by relevancy, that couple v^μ_κ to the bulk fields. Multiplying most of these terms are Wilson coefficients, whose values encode all of the salient information about the underlying scalar-tensor theory, as well as the internal structure of the extended objects. One can then determine these values by matching appropriate observables computed within this point-particle EFT to results obtained in the full theory (B.1).

Potential and radiation modes. Two more simplifications are necessary to render the problem analytically tractable. We first make the weak-field approximation by writing

$$(g_{\mu\nu}, \phi) = (\eta_{\mu\nu}, \phi_0) + (h_{\mu\nu}/m_{\text{Pl}}, \varphi), \tag{B.3}$$

where the part $(h_{\mu\nu}, \varphi)$ that is sourced by the binary is to be treated as a weak perturbation about some fixed background $(\eta_{\mu\nu}, \phi_0)$. Implicit in this choice of background is the assumption that the binary's immediate neighbourhood is mostly just empty space, albeit permeated by some ambient value ϕ_0 of the scalar that we take to be a constant.[33]

We then split the field perturbations further into different parts based on their kinematics. In pure general relativity, the metric perturbation (or "graviton") $h_{\mu\nu}$ splits into two parts [15]: there are *potential modes* that are always off shell and are responsible for mediating the attractive forces holding the two-body system together, and there are *radiation modes* that can go on shell and transport energy and momentum away from the binary. The former varies on length scales on the order of the distance r between the two bodies, while the latter varies on length scales $\sim r/v$, where v is the binary's orbital velocity. If we now suppose that the scalar perturbation φ is sufficiently light that it is

[32] Plus the inclusion of the particle's angular momentum, if it is spinning (see e.g. [18, 19, 25]).

[33] If the scalar is a candidate for dark energy or fuzzy dark matter, this background value ϕ_0 will exhibit some mild time dependence, although the large hierarchy between the typical timescales in a binary and those of cosmology mean that any effects resulting from this time-dependent background are usually small [26–30], and so will be neglected here.

effectively massless on length scales $\lesssim r/v$, then it too admits this simple decomposition into potential and radiation modes. We shall make this assumption throughout, and so are justified in writing

$$(h_{\mu\nu}, \varphi) = (\bar{h}_{\mu\nu}, \bar{\varphi}) + (\hat{h}_{\mu\nu}, \hat{\varphi}), \tag{B.4}$$

where $(\hat{h}_{\mu\nu}, \hat{\varphi})$ denote the potential modes, while $(\bar{h}_{\mu\nu}, \bar{\varphi})$ are the radiation modes.

That the radiation modes have wavelengths λ ($\sim r/v$) much larger than the characteristic size r of the binary during the early inspiral allows us to "zoom out" on this system even further by integrating out the potential modes. The result is a new EFT,

$$S'_{\text{eff}} = -i \log \int D[\hat{h}, \hat{\varphi}] \, \exp(iS_{\text{eff}}), \tag{B.5}$$

whose action can be organised into a conservative and radiative sector:

$$S'_{\text{eff}} = S_{\text{con}}[x_\kappa, S_\kappa] + S_{\text{rad}}[x_\kappa, S_\kappa; \, \bar{h}, \bar{\varphi}]. \tag{B.6}$$

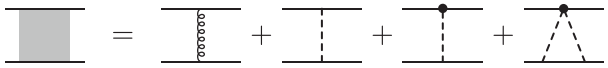
Most of the hard work in the NRGR approach is in the evaluation of (B.5). It goes without saying that, for astrophysical binaries, we are ultimately interested only in the classical limit of this result, but framing the problem in this kind of quantum field theoretic language allows us to use the machinery of Feynman diagrams to our advantage when computing (B.6) systematically to some prescribed order in the post-Newtonian (PN) expansion; i.e., to some order in v. For classical processes, only the tree-level Feynman diagrams are needed [15].

Conservative sector. The conservative sector of (B.6) encodes information about the orbital dynamics of the binary in the absence of any outgoing radiation, and may be written as the integral of the two-body Lagrangian

$$\mathcal{L} = -\sum_\kappa m_\kappa \sqrt{1 - \boldsymbol{v}_\kappa^2} - V(x_\kappa, v_\kappa, S_\kappa) \tag{B.7}$$

over the coordinate time t (i.e., $S_{\text{con}} = \int \mathrm{d}t \, \mathcal{L}$). The conservative potential V arises from summing over all Feynman diagrams involving internal potential lines only and no external ones:

$-\int \mathrm{d}t\, V$ =



$+ \cdots .$ (B.8)

Radiative sector. Turning now to the radiative sector, we find that the remaining terms in S'_{eff} naturally group themselves into two categories:

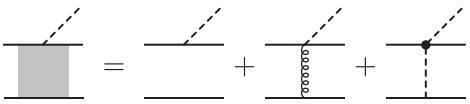
$$S_{\text{rad}} = S_{\text{fields}}[\eta + \bar{h}/m_{\text{Pl}}, \phi_0 + \bar{\varphi}] + S_{\text{binary}}[\mathcal{M}^L(x_\kappa); \bar{h}, \bar{\varphi}]. \tag{B.9}$$

The first term gives us the propagators for and the interactions between the radiation modes, while the second accounts for how these fields are sourced by the binary. As we discussed previously, the radiation modes vary on length scales much greater than the orbital separation between the two bodies, and so are unable to resolve either of them individually — instead perceiving the entire two-body system as behaving like a single point particle. Mathematically, this notion is reflected in the way the interactions are organised in S_{binary}; namely, in terms of multipole moments $\mathcal{M}^L = \{\mathcal{I}^L, \mathcal{J}^L, \mathcal{Q}^L, \dots\}$ for the binary as a whole.

To go from (B.2) to the multipole moments in (B.9), one proceeds as follows. First, we integrate out the potential modes while holding the radiation modes fixed to determine the source functions $J(x)$

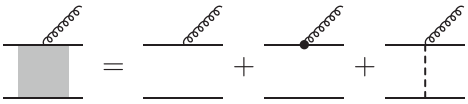
and $T^{\mu\nu}(x)$. In the scalar sector, we sum over all Feynman diagrams with one external radiation-mode scalar to obtain

$$\int \mathrm{d}^4x\, J(x)\bar{\varphi}(x) \;=\; = \; + \; + \; + \;\cdots\,, \tag{B.10}$$



from which $J(x)$ can be deduced. (Any internal lines in these diagrams correspond to potential modes.) Similarly, the binary's energy-momentum tensor $T^{\mu\nu}(x)$ follows from summing over all Feynman diagrams with one external radiation-mode graviton:

$$\frac{1}{2m_{\rm Pl}}\int \mathrm{d}^4x\, T^{\mu\nu}(x)\bar{h}_{\mu\nu}(x) \;=\; = \; + \; + \; + \;\cdots\,. \tag{B.11}$$



Next, we perform a Taylor expansion of the radiation fields about the binary's centre of energy, which we may take to coincide with the origin of our coordinate system without any loss of generality. Then regrouping terms into symmetric and trace-free (STF) operators, the result in the scalar sector is[34]

$$\int \mathrm{d}^4x\, J(x)\,\bar{\varphi}(x) = \int \mathrm{d}^4x\, J(x)\bigg(\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\boldsymbol{x}^L\partial_L\bar{\varphi}(t,\boldsymbol{0})\bigg) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\int \mathrm{d}t\,\mathcal{Q}^L(t)\partial_L\bar{\varphi}(t,\boldsymbol{0}), \tag{B.12}$$

where the binary's scalar multipole moments $\mathcal{Q}^L(t)$ are given by [31]

$$\mathcal{Q}^L(t) = \sum_{p=0}^{\infty}\frac{(2\ell+1)!!}{(2p)!!(2\ell+2p+1)!!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{2p}\int \mathrm{d}^3x\, J(t,\boldsymbol{x})\,\boldsymbol{x}^{2p}\boldsymbol{x}^{\langle L\rangle}. \tag{B.13}$$

Identical steps can then be taken in the gravitational sector to obtain analogous formulae relating $T^{\mu\nu}(x)$ to the binary's mass-type and current-type multipoles, $\mathcal{I}^L(t)$ and $\mathcal{J}^L(t)$; see, e.g., eqs. (7.44) and (7.45) of ref. [4] for the explicit expressions. With these multipole moments in hand, it is then a straightforward exercise to solve the wave equations (as derived from $S_{\rm rad}$) for $\bar{\varphi}$ and $\bar{h}_{\mu\nu}$ to determine the rate at which energy is carried off to infinity. Ignoring nonlinear effects like tail terms, the power radiated away into scalar waves is [31]

$$P_\phi = \sum_{\ell=0}^{\infty}\frac{\langle(^{(\ell+1)}\mathcal{Q}^L)^2\rangle}{4\pi\ell!(2\ell+1)!!}, \tag{B.14a}$$

while the power radiated into gravitational waves is [7, 31]

$$P_g = \sum_{\ell=2}^{\infty}\frac{G_{\rm N}}{\ell!(2\ell+1)!!}\bigg(\frac{(\ell+2)(\ell+1)}{\ell(\ell-1)}\langle(^{(\ell+1)}\mathcal{I}^L)^2\rangle + \frac{4\ell(\ell+2)}{(\ell+1)(\ell-1)}\langle(^{(\ell+1)}\mathcal{J}^L)^2\rangle\bigg). \tag{B.14b}$$

In writing these formulae, we have used the short hand $^{(\ell)}X \equiv \mathrm{d}^\ell X/\mathrm{d}t^\ell$ to denote the action of multiple time derivatives, and note also that the angled brackets around the multipoles denote a time average over several orbital periods.

[34] We are employing standard multi-index notation: we write $\mathcal{Q}^L \equiv \mathcal{Q}^{i_1\cdots i_\ell}$ to denote any tensor with ℓ spatial indices, while any vector repeated ℓ times is abbreviated to read $\boldsymbol{x}^L \equiv \boldsymbol{x}^{i_1}\cdots\boldsymbol{x}^{i_\ell}$, and likewise $\partial_L \equiv \partial_{i_1}\cdots\partial_{i_\ell}$. Lastly, angled brackets around indices instruct us to keep only the STF part of that tensor.

Phase evolution. The two key outputs from the above calculations — the binding energy E from the conservative sector, which follows from the equations of motion and the total power $P = P_\phi + P_g$ computed in the radiative sector — can now be combined into the balance equation

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = -\langle P \rangle \tag{B.15}$$

to tell us how the binary's orbit evolves as it emits gravitational and scalar waves. For circular nonprecessing orbits, both E and P depend on time only through the orbital frequency Ω, and so (B.15) can in this case be recast into a differential equation for the binary's orbital phase $\psi = \int \mathrm{d}t\, \Omega$.

After solving this equation, all that remains is to relate the orbital phase ψ to the gravitational-wave phase Ψ. At the detector, the gravitational-wave signal can be decomposed into spin-weighted spherical harmonics labelled by the familiar integers $(\mathfrak{l}, \mathfrak{m})$, and the phase of the $(\mathfrak{l}, \mathfrak{m})$ mode is simply given by $\Psi_{\mathfrak{l}\mathfrak{m}} = \mathfrak{m}\psi$, up to corrections from nonlinear effects like tail terms [3, 32]. Any scalar waves that reach the detector can also be decomposed into spherical harmonics, and the corresponding $(\mathfrak{l}, \mathfrak{m})$ modes will possess the same phases $\mathfrak{m}\psi$ up to nonlinear corrections. However, in viable scalar-tensor theories, for which the strength of the scalar-matter couplings in the weak-field regime are strongly constrained, these scalar waves leave a much smaller imprint on the detector than their gravitational counterparts [33, 34].

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