In the lectures we considered a toy "gravitational" model for a binary:

$$S_{\rm pp} = \int_{t,\mathbf{x}} \left\{ -\frac{1}{2}(\partial_\mu \varphi)^2 - \sum_{A=1}^{2} J_A \left[1 - \alpha_A \frac{\varphi}{M_P} \right] \right\} \tag{1}$$
$$\text{with} \quad J_A = m_A \sqrt{1 - v_A^2(t)}\, \delta(\mathbf{x} - \mathbf{x}_A(t))$$

If "LVK" measures the scalar radiation from a quasi-circular orbit as it inspirals from 10Hz to 1000Hz and finds good agreement with the quadrupolar waveform for $m_1 = m_2 \approx 10 M_\odot$, we showed that this constrains $\alpha_-^2 \ll \alpha_+^4$ (where $\alpha_\pm = \alpha_+ \pm \alpha_-$).

Now consider adding the following non-linear coupling to matter

$$S_{\rm NL} = \int_{t,\mathbf{x}} \sum_{A=1}^{2} J_A \beta_A \frac{\varphi^2}{M_P^2} \,. \tag{2}$$

Compute the $\mathcal{O}(\beta)$ correction to the waveform and the power emitted in scalar radiation, working within a Post-Newtonian expansion. How would the same "LVK" measurement constrain the β_A couplings?

Here is one possible approach, using NREFT:

(a) from (2), write down Feynman rules for the potential modes $\delta\varphi$ and radiation modes $\bar{\varphi}$.

(b) find the conservative potential, and hence show that the energy of circular orbits is

$$E = E_0 + E_\beta$$
$$\text{with} \quad E_0 = -\frac{1}{2}\mu \mathtt{x} + \dots \,, \qquad E_\beta = -\mu \mathtt{x}^2 \left(\frac{\beta_1 \mu}{\alpha_1^2 m_1} + \frac{\beta_2 \mu}{\alpha_2^2 m_2} \right) + \dots \tag{3}$$

where +... denote higher orders in the PN parameter $\mathtt{x} = (\kappa\Omega/\mu)^{2/3} = v^2 = \kappa/(\mu r)$, where $\kappa = 2 G_N \alpha_1 m_1 \alpha_2 m_2$ is the strength of the Newtonian potential.

(c) find the current J to which $\bar{\varphi}$ couples in $S_{\rm NR}[\mathbf{x}_A, \bar{\varphi}]$, and hence determine the $\mathcal{O}(\beta)$ correction to the waveform.

(d) hence, or otherwise, show that the leading PN correction to the transmitted power at order β can be written as

$$P = (C_0 + C_\beta)\, \kappa\, r_0^4\, \Omega_0^6$$
$$\text{with} \quad C_0 = \frac{16\nu}{15\alpha_1\alpha_2} \left(\frac{\alpha_1 m_2 + \alpha_2 m_1}{m_1 + m_2} \right)^2 , \quad C_\beta = \frac{4\nu\alpha_-}{3\alpha_1\alpha_2} \left(\frac{\beta_1 \mu}{\alpha_1 m_1} - \frac{\beta_2 \mu}{\alpha_2 m_2} \right) \tag{4}$$

where $\nu = \mu/M = m_1 m_2/(m_1 + m_2)^2$ is the mass ratio and +... denote higher PN orders and also negligible dipolar contributions (which are suppressed by $\alpha_- \ll \alpha_+$).

(e) using the flux-balance law and assuming that the quadrupole is the dominant source of power, show that the $\mathcal{O}(\beta)$ shift in the accumulated phase of the scalar waveform is

$$\delta\Psi_{22} \approx N_\beta \left(\frac{\mu\, c^3}{\kappa\, \Omega}\right)^{5/3} \tag{5}$$

where N_β is a constant that you should determine.

(f) By demanding that $|\delta\Psi_{22}| < \pi$ for the binary described above, place an "observational" bound on the β_A in terms of the α_A.

Extension: Repeat the same steps for the so-called "disformal" coupling

$$S_{\rm NL} = \int_{t,\mathbf{x}} \sum_A J_A d_A \frac{\dot{\varphi}^2}{M_P^2} \tag{6}$$

to determine how this affects the emitted waveform and phase.