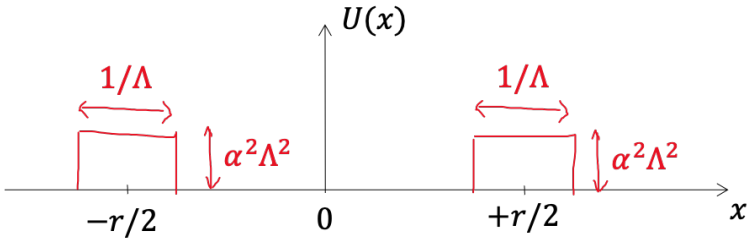
The first approximation at a gravitational wave theorist's disposal is to model compact objects as point particles. In the lectures we saw a simple example of this in 1d quantum mechanics. In this tutorial, we are going to explore a 1d example of a binary (two-body) system.

Consider the 1d potential of two square steps:




where the width of each bump is much smaller than their separation, $1/\Lambda \ll r$. The time-independent Schrodinger equation

$$\left[\partial_x^2 + k^2\right] \psi_k(x) = -U(x)\psi_k(x) \,. \tag{1}$$

can be solved with the boundary condition

$$\psi_k(x) = \begin{cases} e^{+ikx} + r_k\, e^{-ikx} & \text{as } x \to -\infty \,, \\ t_k\, e^{+ikx} & \text{as } x \to +\infty \,, \end{cases} \tag{2}$$

to determine the reflection and transmission of waves off this potential. The full expressions are lengthy, but when the wave has sufficiently low energy ($k \ll \Lambda$) they can be expanded as:

$$r_k = r_k^{(0)} + r_k^{(1)} \frac{k}{\Lambda} + r_k^{(2)} \frac{k^2}{\Lambda^2} + \mathcal{O}\left(\frac{k^3}{\Lambda^3}\right) \,, \qquad t_k = t_k^{(2)}\, \frac{k^2}{\Lambda^2} + \mathcal{O}\left(\frac{k^3}{\Lambda^3}\right) \tag{3}$$

where

$$\begin{aligned} r_k^{(0)} &= -e^{-ikr} \,, & t_k^{(2)} &= \frac{2(-i + \cot(kr))}{i\alpha^2 \sin^2\alpha} \\ r_k^{(1)} &= (1 + \frac{2}{\alpha}\cot\alpha) ie^{-ikr} \,, \\ r_k^{(2)} &= \frac{4e^{2i\alpha}}{\alpha^2}\, \frac{4\cos(kr) - i\sin(kr)(\alpha\sin\alpha + 2\cos\alpha)^2}{(-1+e^{2i\alpha})^2\,(-1+e^{2ikr})} \,. \end{aligned} \tag{4}$$

Given this information, answer the following:

(a) Show that the much simpler effective theory,

$$U_{\rm EFT}(x) = c_-\Lambda\delta(x+a) + c_+\Lambda\delta(x-a) \tag{5}$$

in which the two bumps have been replaced by point-like $\delta$ functions, can reproduce this reflection and transmission at $\mathcal{O}\,(k/\Lambda)$ for an appropriate choice of the parameters $\{c_+, c_-\}$. What would be required to reproduce the higher order terms at $\mathcal{O}\,(k^2/\Lambda^2)$?

(b) Now consider scattering at even lower energies, for which $k/\Lambda \ll kr \ll 1$. Write down an appropriate effective theory in that case, and show how the reflection $r_k$ and transmission $t_k$ can be reproduced in this limit.

(c) These effective descriptions are useful because, once their parameters have been fixed, they can be used to answer a variety of other questions that would be very difficult to tackle using the original $U(x)$. For instance, double well potentials like this one are often used to trap cold atoms. Suppose we initially have an atom inside the trap, with wavefunction sharply peaked around $x = 0$; for instance,

$$\Psi(x, t = 0) = \int_0^\infty dk\; e^{-kb}\, \Psi_k(x) \qquad \left( = \frac{b}{b^2 + x^2} \text{ when } |x| < r \right) \tag{6}$$

where $b$ is a fixed length scale that characterises how localised the atom is within the trap, and $\Psi_k(x)$ is the energy eigenstate defined by (1) and the boundary condition

$$\Psi_k(x) = \cos(kx) \qquad \text{when } |x| < r/2\ . \tag{7}$$

Using (5), show that the wavefunction outside the trap is

$$\Psi(x, t = 0) \approx \frac{b}{b^2 + x^2} + \frac{ic_\pm \Lambda}{2} \left( \arctan\left( \frac{r - |x|}{b} \right) - \arctan\left( \frac{|x|}{b} \right) + \pi \right) \tag{8}$$

when $x > +r/2$ or $x < -r/2$. This tells us the probability of the atom leaking from the trap, without the difficulties of solving the original double well potential.

For what range of $b$ is this approximation (8) reliable?

*Extension.* Try extending the one-body EFT to higher orders in $k/\Lambda$ and show that it can match to any arbitrary order the exact $\{r_k, t_k\}$ coefficients for a single square well:

$$r_k = \frac{(k^2 - q^2)\, e^{-ik/\Lambda}}{(k^2 + q^2) + 2ikq \cot(q/\Lambda)}\ , \qquad\qquad t_k = \frac{2ikq \csc(q/\Lambda)\, e^{-ik/\Lambda}}{(k^2 + q^2) + 2ikq \cot(q/\Lambda)} \tag{9}$$