Exercises

1. Derive an expression for the free lattice scalar propagator in momentum space, given the action

$$S = -\frac{a^2}{2}\sum_{x,\mu}\varphi(x)[\varphi(x+a\hat{\mu})+\varphi(x-a\hat{\mu})] \;+\; \frac{a^2}{2}(8+m^2a^2)\sum_x \varphi(x)\varphi(x)\,.$$

2. Show that the plaquette action

$$S_\Box[U] = \beta\sum_{x,\mu,\nu>\mu}\left(1 \;-\; \frac{1}{2n}\mathrm{Tr}\,[P_{\mu\nu}(x) \;+\; P^\dagger_{\mu\nu}(x)]\right)$$

   has the correct classical continuum limit.

3. In the strong coupling limit of $U(1)$ gauge theory, explicitly show that Wilson loops obey an area law. Explain why the next-to-leading order term is $O(\beta^{RT+4})$. We know continuum QED is not a confining theory, i.e. static charges have a Coulomb potential $V(R) \propto 1/R$, not linear $V(R) \propto R$. How can we reconcile this fact with the strong coupling lattice result?

4. [Advanced] In the case of $SO(3)$ gauge theory, show that the strong coupling expansion for the Wilson loop obeys a perimeter law. [Recall that $SO(3)$ is the group of rotations in 3 dimensions and that its irreducible representations are of dimension $2\ell+1$, where $\ell = 0, 1, 2, \ldots$ and represents angular momentum in some contexts. The key fact is that, in this case, there is a singlet contribution to $\mathbf{3}\otimes\mathbf{3}$ and $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ representations. There is no distinct $\overline{\mathbf{3}}$ representation, so Wilson lines do not need arrows in the way they do for $SU(3)$.]

5. Sketch versions of the step scaling procedure for the cases of (a) a nontrivial fixed point and (b) the nonexistence of a continuum limit.

6. In the weak coupling limit of Yang–Mills theory, the following perturbative expression for the potential between a static quark–antiquark pair can be obtained

$$V(r, g_0, a) = \frac{C}{4\pi r}\left(g_0^2 \;+\; \frac{22}{16\pi^2}\,g_0^4\log\frac{r}{a} \;+\; O(g_0^6)\right)$$

   where C is a constant. Determine what $g_0(a)$ must be in order for the potential is independent of a (assume a is small).

7. Let $\rho_{eq}(C)$ be the desired "equilibrium" distribution and let $\rho_0(C)$ be an initial non-equilibrium distribution. Given an updating algorithm which has transition probability density $P(C \to C')$, let us consider the case where C and

C' differ by only one link variable, say $U_\mu(x)$ vs $U'_\mu(x)$. Then we can write everything in terms of U and U'; in particular for this single-variable update, define the update probability $p(U \to U') = P(C \to C')$. If we repeatedly update only that one variable, the distribution evolves from Monte Carlo step N to step $N+1$

$$\rho_{N+1}(U') \;=\; \int dU\, \rho_N(U)\, p(U \to U') \qquad (*)$$

(with x-dependence suppressed). Explain why $\int dU'\, p(U \to U') = 1$. Show that detailed balance is sufficient to ensure $\rho_{eq}(U)$ is a fixed point of $(*)$. (In our case $\rho_{eq}(U) = e^{-S(U)}$, but this is not relevant here.) This argument can be generalized to updates which involve all variables.

8. Show that the Metropolis algorithm satisfies detailed balance for the case of $U(1)$ gauge theory.