Some problems for BUSSTEPP

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Abstract

These problems for the BUSSTEPP 2025 AdS/CFT course are designed as starting points for exploration and discussion in our tutorial sessions. They go beyond what we cover in the lectures, inviting you to think creatively, make connections, and push your understanding further. Use them as an opportunity to engage with me, the postdoc fellows, and your classmates. Bring your ideas, your questions, and your thoughts. Let us try to turn these problems into a lively exchange of insights.

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1 Problems for the lecture 1

1–Consider the Thirring and Sine-Gordon models with lagrangians,

$$L_{Th} = \bar{\psi}(i\gamma^\mu \partial_\mu + M)\psi - \frac{g}{2}\left(\bar{\psi}\gamma_\mu\psi\right)^2, \tag{1.1}$$

$$L_{SG} = -\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{\beta}\left[\cos(\beta\phi) - 1\right]. \tag{1.2}$$

Show that the Thirring model has a global invariance under the transformation $\psi \to e^{i\alpha}\psi$ and the conjugate for $\bar{\psi}$. Show that $j_\mu = \bar{\psi}\gamma_\mu\psi$ is a Noether current. As such, it is conserved using the equations of motion. The infinitesimal transformations (for parameter α) being

$$\delta\psi = i\alpha\psi, \quad \delta\bar{\psi} = -i\alpha\bar{\psi}. \tag{1.3}$$

Show that $J^\mu = \epsilon^{\mu\nu}\partial_\nu\phi$ is conserved, without the need to use the equations of motion following from eq.(1.2).

2–Consider AdS_5 space time written in the coordinates,

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2\right). \tag{1.4}$$

Show that the units $[z] = [x^\mu] = [L] =$ length. Show that under the change

$$t \to \lambda t, \quad \vec{x} \to \lambda\vec{x}, \quad z \to \lambda z. \tag{1.5}$$

The metric is left invariant.

Show that other coordinates to write AdS_5 are (find the coordinates change!),

$$ds^2 = \frac{\rho^2}{L^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2 d\rho^2}{\rho^2}, \tag{1.6}$$

$$ds^2 = e^{\frac{2r}{L}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + dr^2, \tag{1.7}$$

Check that the units $[r] = [\rho] =$ length. In the case of eq.(1.4), the boundary of AdS_5 is at $z = 0$. Where is the boundary for the coordinates in eqs.(1.6)-(1.7)?

3–Consider the configuration for D3 branes in Type IIB supergravity. It contains a metric, a constant dilaton (scalar) and a Ramond five form (a Kalb-Ramond field).

$$\begin{aligned} &ds^2 = H(r)^{-\frac{1}{2}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H(r)^{\frac{1}{2}}\left(dr^2 + r^2 d\Omega_5\right), \\ &e^\Phi = e^{\Phi_0}, \quad F_5 = F_{t,x_1,x_2,x_3,r} + F_{S^5}. \quad F_{t,x_1,x_2,x_3,r} = \partial_r \frac{1}{H(r)}. \\ &H(r) = 1 + \frac{4\pi\alpha'^2 g_s N}{r^4}. \end{aligned} \tag{1.8}$$

Consider the variable $u = \frac{r}{\alpha'}$. Show that it has units of energy.
Write the whole configuration in eq.(1.8) in the u-variable. Take the limit $\alpha' \to 0$ keeping u fixed. Write the configuration in eq.(1.8) after the scaling. Show that the metric reads,

$$\frac{ds^2}{\alpha'} = \frac{u^2}{L^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{L^2}{u^2}du^2 + L^2 d\Omega_5^2. \quad L^4 = 4\pi g_s N. \tag{1.9}$$

Write a vielbein for the metric in eq.(1.9). Use any coordinate system you like for S^5.
Write F_5 in term of your vielbein basis. Show that this is (after the scalings above),

$$\frac{1}{\alpha'}F_{tx_1x_2x_3u} = \frac{u^3}{\pi g_s N}dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge du = \frac{4}{L}e^t \wedge e^{x_1} \wedge e^{x_2} \wedge e^{x_3} \wedge e^u. \tag{1.10}$$

Now, we have that a self duality condition must be imposed on F_5. Do you see how easily one can find the Hodge dual in this way? Show that

$$F_{\theta_1 \ldots \theta_5} = \frac{4}{L}e^{\theta_1} \wedge e^{\theta_2} \wedge e^{\theta_3} \wedge e^{\theta_4} \wedge e^{\theta_5}. \tag{1.11}$$

2 Problems for the lectures 2 and 3

1–Consider a generic geometry,

$$ds^2 = -g_{tt}(r)dt^2 + g_{xx}(r)d\vec{x}^2 + g_{rr}(r)dr^2 + g_{ij}(r,\theta^i)d\theta^i d\theta^j. \tag{2.1}$$

Propose that a fundamental string probe (parametrised by the variables (τ,σ)) in the configuration,

$$t = \tau, \quad x = \sigma, \quad r = r(\sigma), \quad \theta^i = \text{constant}. \tag{2.2}$$

Calculate the induced metric

$$g_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \quad a,b = (\tau,\sigma), \quad \mu,\nu = 1,....,10. \tag{2.3}$$

Show that

$$ds^2_{ind} = -g_{tt}d\tau^2 + (g_{xx} + g_{rr}r'^2)d\sigma^2. \tag{2.4}$$

Calculate the determinant of the induced metric. Show that

$$-\det[g_{ab}] = g_{tt}(g_{xx} + g_{rr}r'^2). \tag{2.5}$$

Calculate the Nambu-Goto action,

$$S_{NG} = \frac{1}{2\pi\alpha'}\int_0^T d\tau \int_0^{2\pi} d\sigma \sqrt{-\det[g_{ab}]} = T \times E_{QQ}. \tag{2.6}$$

Show that

$$S_{NG} = \frac{T}{2\pi\alpha'}\int d\sigma \sqrt{F^2 + G^2 r'^2}, \quad F^2 = g_{tt}g_{xx}, \quad G^2 = g_{tt}g_{rr}. \tag{2.7}$$

Find the conserved Hamiltonian. Show that

$$H = -\frac{F^2}{\sqrt{F^2 + G^2 r'^2}} \tag{2.8}$$

Using that the Hamiltonian is conserved (that is $H = -C$), find $\frac{dr}{d\sigma}$. Show that,

$$\frac{dr}{d\sigma} = \frac{F}{CG}\sqrt{F^2 - C^2}. \tag{2.9}$$

Find an expression for $L_{QQ} = \int_0^{2\pi} d\sigma$. Find an expression for E_{QQ}.

2–Consider a massive scalar field $\Phi(t,\vec{x},z)$ propagating in a fixed background AdS_{D+1} with metric

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dx_1^2 + ... + dx_{D-1}^2 + dz^2\right). \tag{2.10}$$

The lagrangian is

$$L = \sqrt{-\det[g]}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{m^2}{2}\Phi^2\right]. \tag{2.11}$$

Write the equation of motion for this scalar. Show that it is

$$\Box\Phi = m^2\Phi, \qquad \frac{1}{\sqrt{-\det g}}\partial_\mu\left[\sqrt{-\det g}g^{\mu\nu}\partial_\nu\Phi\right] = m^2\Phi. \tag{2.12}$$

Now, using the equation of motion, show that the action in eq.(2.11) can be written as a boundary term of the form,

$$S = \frac{1}{2}\int d^Dxdz\partial_\mu\left[\sqrt{-\det g}g^{\mu\nu}\Phi\partial_\nu\Phi\right] = \frac{1}{2}\int d^Dx\left[\sqrt{-\det g}g^{z\nu}\Phi\partial_\nu\Phi\right]_{z=\epsilon}. \tag{2.13}$$

Now, let us come back to eq.(2.12). Show that more explicitly, it reads,

$$\frac{z^2}{L^2}\nabla^2\Phi + \frac{z^2}{L^2}\partial_z^2\Phi - \frac{(D-1)}{L^2}z\partial_z\Phi - m^2\Phi = 0. \tag{2.14}$$

Propose a solution of the form $\Phi = e^{i\vec{k}\vec{x}-i\omega t}\phi(z)$. Use Mathematica to solve the equation of motion for $\phi(z)$.

For $z\sim 0$ (close to the boundary!), we can propose that $\phi(z) = Az^\Delta$. Replace this in eq.(2.14) and find that (neglecting terms that go to zero faster than z^Δ!),

$$\Delta_\pm = \frac{D \pm \sqrt{D^2 + 4m^2L^2}}{2}. \tag{2.15}$$

Show that $\Delta_+ + \Delta_- = D$.
Show that for $m = 0$, $\Delta = D$. The field Φ in gravity is dual to a marginal operator.
Show that for $m^2 > 0$, $\Delta > D$. In this case the field Φ is dual to an irrelevant operator.
On the other hand, if $-\frac{D^2}{4L^2} < m^2 < 0$, the field Φ is dual to a relevant operator.
Why is the lower limit at that value? This is called the Breinthenloner-Freedman bound!