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Stability of PPT in equilibrium states

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Publication: Canadian Journal of Physics • 15 September 2023 • https://doi.org/10.1139/cjp-2023-0086

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Department of Mathematics and Statistics

Canadian **Science** Publishing

Canadian Journal of **Physics**

Quantum systems

Hilbert space $\mathcal{H}$, $\quad \dim \mathcal{H} \leq \infty$

States are density matrices $\in \mathcal{B}(\mathcal{H})$,

$$\mathcal{S}(\mathcal{H}) = \{ \rho \geqslant 0, \ \mathrm{Tr}\,\rho = 1 \}$$

Entanglement

$(\dim \leq \infty)$

Bipartite quantum system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$

$\rho \in \mathcal{S}(\mathcal{H}_{AB})$ is called **separable** iff $\exists\ \rho_n^A \in \mathcal{S}(\mathcal{H}_A)$, $\rho_n^B \in \mathcal{S}(\mathcal{H}_B)$, $p_n \geq 0$, $\sum_{n \geq 1} p_n = 1$, such that

$$\rho = \sum_{n=1}^{\infty} p_n \, \rho_n^A \otimes \rho_n^B$$

$\rho \in \mathcal{S}(\mathcal{H}_{AB})$ is called **entangled** iff it is not separable.

Entanglement is:

- a sign of quantumness (quantum correlation) between subsystems A & B; "spooky action at a distance"
- a basic resource in quantum computing (superdense coding, cryptography, teleportation ...)
- not easy to determine

BUT there is a useful <u>necessary condition for entanglement.</u>

Suppose $\rho = \rho^A \otimes \rho^B$

Denote by $T_B = \mathbb{1} \otimes T$ the <u>partial transposition</u> operator

$$T_B\left[\rho^A \otimes \rho^B\right] = \underbrace{\rho^A}_{\geq 0} \otimes \underbrace{\left(\underbrace{\rho^B}_{\geq 0}\right)^T}_{\geq 0} \geq 0$$

By linearity: ρ separable $\Rightarrow$ $T_B[\rho] \geq 0$

<u>Def.</u>: ρ is PPT (positive partial transpose) iff $T_B[\rho] \geq 0$.

Hence ρ separable $\Rightarrow$ ρ is PPT

Peres–Horodecki criterion for entanglement

$$\boxed{\rho \text{ not PPT} \Rightarrow \rho \text{ entangled}}$$

[Peres '96, Horodecki^3 '96]

- $\Leftarrow$ ✓ if $\dim \mathcal{H}_A \times \dim \mathcal{H}_B = 2\times 2$ or 2×3 [Horodecki^3 '96]
- Higher dim : $\exists\, \rho$ which are PPT and still entangled

  = <u>bound</u> or <u>not distillable</u> entanglement

(not suitable for use for quantum info. purposes [Bennet et al '96, Horodecki^3 '98].)

Q

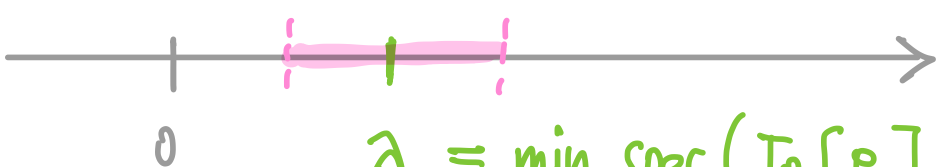
Suppose ρ_0 is PPT. When is $\rho = \rho_0 + \rho'$ still PPT, where ρ' is a perturbation operator?

A1

dim $\mathcal{H}_{AB} < \infty$

- $T_B[\rho] \geq 0 \Leftrightarrow$ all eigenvalues of $T_B[\rho]$ are ≥ 0
- $T_B[\rho] = T_B[\rho_0] + T_B[\rho']$

$\mathrm{spec}(T_B[\rho])$ in $O(\|T_B[\rho']\|_\infty)$ nbhd of $\mathrm{spec}(T_B[\rho_0])$



$\lambda_0 = \min \mathrm{spec}(T_B[\rho_0]) > 0$

PPT stable

What if $\dim \mathcal{H}_{AB} = \infty$?

$\operatorname{Tr} \rho < \infty \Rightarrow T_B[\rho]$ Hilbert schmidt, so compact operator

$\Rightarrow$ eigenvalues of $T_B[\rho]$ accumulate at origin.

Above simple approach does not work.

A2 (Collaboration with Mitch Zagrodnik) equilibrium states

$$\rho_0 = \frac{e^{-\beta H_0}}{\operatorname{Tr} e^{-\beta H_0}}, \qquad \rho = \frac{e^{-\beta(H_0+V)}}{\operatorname{Tr} e^{-\beta(H_0+V)}}$$

$$H_0 = H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B$$

Main idea: 'Remove' accumulating eigenvalues using Dyson expansion

$$e^{-\beta(H_0+V)} = \left[\mathbb{1} + D(\beta)\right] e^{-\beta H_0}$$

$$\sum_{n\geq 1} \int_0^{\beta} ds_1 \cdots \int_0^{s_{n-1}} ds_n \; V(s_n) \cdots V(s_1) = \mathcal{O}(V)$$

$$e^{-s_n H_0} V e^{s_n H_0}$$

$$e^{-\beta(H_0+V)} = \left[e^{-\frac{\beta}{2}(H_0+V)}\right]^{\dagger} e^{-\frac{\beta}{2}(H_0+V)}$$

$$= e^{-\frac{\beta}{2}H_0}\left[\mathbb{1} + F(\beta)\right] e^{-\frac{\beta}{2}H_0}$$

$\mathcal{O}(V)$

Then $\quad T_B\left[e^{-\beta(H_0+V)}\right] = \underbrace{T_B\left[e^{-\frac{\beta}{2}H_0}\right]}_{e^{-\frac{\beta}{2}H_A}\otimes T\left[e^{-\frac{\beta}{2}H_B}\right] \geqslant 0}\left(\mathbb{1} + T_B\left[F(\beta)\right]\right) T_B\left[e^{-\frac{\beta}{2}H_0}\right]$

So $\quad T_B\left[e^{-\beta(H_0+V)}\right] \geqslant 0 \quad \Leftrightarrow \quad \mathbb{1} + T_B\left[F(\beta)\right] \geqslant 0$

Next $\quad \mathbb{1} + T_B\left[F(\beta)\right] \geqslant \mathbb{1} - \|F(\beta)\|_2$

and $\quad \|F(\beta)\|_2 \leq \exp\left[2a\,\frac{e^{\beta b/2}-1}{b}\right] - 1$

<u>Assumption</u>: $\quad \|e^{-sH_0}\,V\,e^{sH_0}\|_2 \leq a\,e^{bs} \qquad (\text{some } a, b \geqslant 0)$

Theorem Suppose $0 < \beta < \beta_* \equiv \frac{2}{b} \ln \left[1 + \frac{b}{a} \frac{\ln 2}{2} \right]$

Then $$\rho_\beta = \frac{e^{-\beta(H_0+V)}}{\mathrm{Tr}\, e^{-\beta(H_0+V)}}$$ is PPT.

- Equilibrium states are PPT at high enough temperature $T \geq T_*$.

- Take $H = H_0 + \lambda V$, λ : coupling constant

$$\longrightarrow \qquad \beta_* \sim \ln\left(\frac{1}{|\lambda|}\right)$$

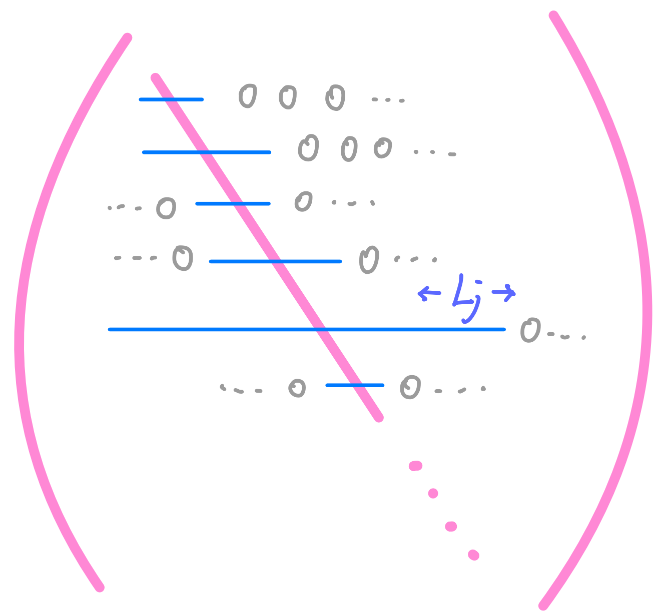
High temp condition $\Leftrightarrow$ small coupling condition

Validity of condition $\| e^{-sH_0} V e^{sH_0} \|_2 \leq a e^{bs}$

- $\dim \mathcal{H}_{A,B} < \infty$: $a = \|V\|_2$, $b = \| H_0 \|_\infty$

- $\operatorname{spec} H_0 = \{E_j\}_{j \geq 1}$, eigenvect. $|\phi_j\rangle$

$$V = \sum_{j \geq 1} \sum_{\ell=1}^{L_j} V_{j,j+\ell} |\phi_j\rangle\langle\phi_{j+\ell}| + h.c.$$



$$a = \sum_{j \geq 1} \sum_{\ell=1}^{L_j} |V_{j,j+\ell}| , \qquad b = \sup_{j \geq 1} \max_{1 \leq \ell \leq L_j} |E_j - E_{j+\ell}|$$

- Interacting harmonics : $H = H_0 + V$

$$H_0 = \omega_0 a^\dagger a + \sum_{j=1}^{N} \omega_j b_j^\dagger b_j$$

$$V = \chi_{H_0 \le \Omega} \left(\sum_{j=1}^{N} g_j a^\dagger b_j + h.c. \right) \chi_{H_0 \le \Omega}$$

form factor $\in \mathbb{C}$

energy cutoff

$$a = 2 \| g \|_2 \sqrt{N} \left(\frac{\Omega}{\omega_{min}} + 1 \right)^{\frac{N+3}{2}}, \qquad b = \max_{1 \le j \le N} | \omega_0 - \omega_j |$$

$\min_{1 \le j \le N} \omega_j$