Constructing ALF and ALE gravitational instantons

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Outline

1. Toric gravitational instantons

2. Existence and uniqueness for asymptotically ALF and ALE instantons (in progress J Lucietti and M Khuri)

Gravitational instantons

Definition
A gravitational instanton is a smooth, geodesically complete (non-compact) connected Ricci flat Riemannian four-manifold.

- characterized by their asymptotic geometry and topology.
- We say a gravitational instanton (M, g) is **toric** if it admits the action of $\mathbb{T}^2$ as isometries.

Many explicit examples exist by 'Euclideanizing' black hole metrics.

Gravitational instantons

- Euclidean quantum gravity: [GIBBONS-HAWKING-PERRY]: analogy with Yang-Mills instantons: i.e. $F = \star F$ minimizes action

$$S = \int_{\mathbb{R}^4} \mathrm{Tr}|F|^2, \qquad D \star F = 0, \qquad DF = 0$$
$$= \frac{1}{2} \int_{\mathbb{R}^4} \mathrm{Tr}|F - \star F|^2 + \text{boundary term}$$

  on Euclidean $\mathbb{R}^4$.

- gravity: minimize the Einstein-Hilbert action: natural to impose

$$R_{abcd} = (\star R)_{abcd} = \frac{1}{2} \epsilon_{abmn} R^{mn}{}_{cd}$$

  Such (M, g) are **hyperKähler** (Ricci flat, Kähler)
  But the gravitational action is not positive definite so we relax this requirement.

Schwarzschild instanton

Consider the Schwarzschild exterior metric with $t \to i\tau$:

$$g_S = \left(1 - \frac{2m}{r}\right) \mathrm{d}\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 g_{S^2}$$

Extends to a global metric on $\mathbb{R}^2 \times \mathbb{S}^2$ provided we identify:

$$\tau \sim \tau + \beta, \qquad \beta = 8\pi m$$

ensures $T = \partial_\tau$ smoothly degenerates at $r = r_+$.
Asymptotically flat: as $r \to \infty$,

$$g \to \mathrm{d}\tau^2 + g_{\mathbb{R}^3}$$

the canonical flat metric on $\mathbb{S}^1 \times \mathbb{R}^3$. Note $|T|$ is bounded as $r \to \infty$.

Taub-NUT instanton

Hyperkähler Taub-NUT metric;

$$\mathbf{g}_{TN} = H^{-1}L^2 \left(\mathrm{d}\psi + k\cos^2\left(\frac{\theta}{2}\right)\mathrm{d}\phi \right)^2 + H(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)),$$

$$H = 1 + \frac{kL}{2r}, \qquad r > 0,\ \theta \in (0, \pi)$$

Surfaces of constant r are $\mathbb{S}^1$ bundles over $\mathbb{S}^2$ provided $k \in \mathbb{Z}$ and we identify the (ψ, ϕ)-plane as:

$$(\psi, \phi) \sim (\psi + 2\pi, \phi) \sim (\psi, \phi + 2\pi)$$

Regularity as $r \to 0$ fixes $k = 1 \Rightarrow \mathbf{g}_{TN}$ is a smooth metric on $\mathbb{R}^4$:.

Eguch-Hanson instanton

Complete hyperkähler 1-parameter family of metrics on $T^*\mathbb{S}^2$

$$\mathbf{g}_{EH} = \frac{\mathrm{d}r^2}{U} + \frac{r^2}{4}\left[U(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi)^2 + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right],$$

$$U = 1 - \frac{a^4}{r^4}, \quad a \geq 0$$

with $r \in (a, \infty)$ and (θ, ϕ) standard coords on $\mathbb{S}^2$.

- If $a = 0$ then $\mathbf{g}_{EH}$ is Euclidean metric on $\mathbb{R}^4$ when $\psi \sim \psi + 4\pi$.
- if $a > 0$, ∂_ψ degenerates at $r = a$. Regularity fixes $\psi \sim \psi + 2\pi$. Surfaces of constant r have topology $L(2, 1)$. Asymptotically $\mathbf{g}_{EH} \to$ flat metric on $\mathbb{R}^4 \setminus \mathbb{Z}_2$.

ALF and ALE manifolds

We construct new gravitational instantons with prescribed asymptotic behaviour. Roughly:

1. (M, g) is asymptotically locally Euclidean (ALE) if it has an asymptotic end with the topology $\mathbb{R} \times L(p, q)$ and $g \to g_E$ (Euclidean metric).

2. (M, g) is asymptotically locally flat (ALF) if it has an asymptotic end with the topology $\mathbb{R} \times L$ where L is an $\mathbb{S}^1$-bundle over $\mathbb{S}^2$.

3. if (M, g) is ALF and $L = \mathbb{S}^1 \times \mathbb{S}^2$ (trivial bundle) then we say it is asymptotically flat (AF).

ALE spaces have volume growh $\sim r^4$ whereas ALF and AF spaces have volume growth $\sim r^3$ as $r \to \infty$.

Toric gravitational instantons

Let $(M, \mathbf{g})$ satisfy

$$\mathrm{Ric}(\mathbf{g}) = 0$$

with isometry subgroup $\mathbb{T}^2 \simeq U(1)^2$ with generators $\eta_{(i)}$, $i = 1, 2$.

$$\omega_i := \star(\eta_1 \wedge \eta_2 \wedge \mathrm{d}\eta_i) \in C^\infty(M), \qquad \mathrm{d}\omega_i = 0$$

If the axis set

$$\mathcal{A} := \{p \in M | \det \mathbf{g}(\eta_i, \eta_j)|_p = 0\} \neq \emptyset \ \text{, then } \omega_i = 0$$

By Froebenius theorem the distribution orthogonal to $\mathrm{span}(\eta_1, \eta_2) \subset TM$ is integrable. Then in $M \setminus \mathcal{A}$,

$$\mathbf{g} = g_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j + \hat{g}, \qquad \eta_i = \frac{\partial}{\partial\phi^i}, \quad \phi^i \sim \phi^i + 2\pi$$

where the matrix $g_{ij} = \mathbf{g}(\eta_i, \eta_j)$, $\hat{g}$ is a metric on $\hat{M} = M/\mathbb{T}^2$.

$\mathrm{Ric}(\mathbf{g}) = 0$ equivalent to system no $(\hat{M}, \hat{g})$:

$$\mathrm{div}_{\hat{g}}(\rho J) = 0, \qquad \hat{R}_{ab} = \hat{\nabla}_a \hat{\nabla}_b \log \rho + \frac{1}{4}\mathrm{Tr}(J_a J_b)$$
$$J := g^{-1}\mathrm{d}g, \qquad \rho := \sqrt{\det g}$$

Theorem ([Hollands-Yazadjiev])
$\hat{M}$ is simply connected with boundaries and corners (homeomorphic to upper-half plane)

- interior $\hat{M}$ is $M \setminus \mathcal{A}$: g_{ij} is rank-2 $(\rho > 0)$
- boundary segments: g_{ij} is rank-1 $(\rho = 0)$
- corners: g_{ij} is rank-0 $(\rho = 0)$

- global chart for int $\hat{M}$:

$$\hat{g} = e^{2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2), \ \rho > 0, z \in \mathbb{R}$$

identify (ρ, z) with cylindrical coords in $\mathbb{R}^3$

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Orbit space

Axis $\mathcal{A}$ ($\rho = 0$) divides into intervals

$$(-\infty, z_1), \qquad (z_1, z_2), \qquad \ldots \qquad (z_N, \infty)$$

where g_{ij} is rank-1. Each such rod $I_A = (z_{A-1}, z_A)$ is separated by corners $z_A, A = 1, 2 \ldots, N$ with length $\ell_A = z_A - z_{A-1}$ and associated $v_A \subset\in \mathrm{Ker}(g)$. These generate $2\pi-$periodic flows iff $v_A = q^i_A \eta_i$, $q^i \in \mathbb{Z}$.
interval structure: collection of the boundary data

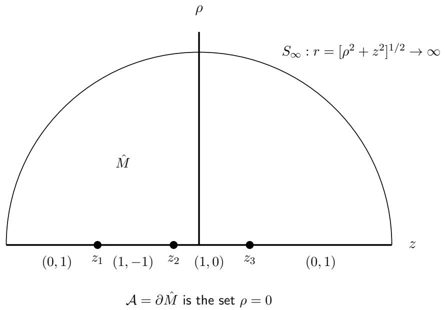
$$\mathcal{R} := \{(\ell_A, v_A) | A = 1, 2, \ldots, N+1\}$$

To avoid **orbifold singularities** at z_A, require adjacent rods I_{A-1}, I_A satisfy *admissibility condition*

$$\det (v_{A-1}, v_A) = \pm 1$$

.

Main Results



Theorem (HK, Lucietti,Khuri)

Let $(M, \mathbf{g})$ be an ALE/ALF gravitational instanton admitting $\mathbb{T}^2-$isometry. Then it is *uniquely* characterized its rod data set $\mathcal{R}$ (and asymptotic parameter $L > 0$ in the ALF case). Moreover given *admissible* data a **unique, smooth instanton exists**.

Harmonic map reduction with Riemannian target

Proposition (Riemannian target space)

Define the symmetric, positive definite, unimodular matrix

$$\Phi := \rho^{-1} g \Rightarrow \det \Phi = 1$$

$$\text{Ricci flat equation } \Rightarrow \mathrm{d}\hat{\star}(\rho \Phi^{-1} \mathrm{d}\Phi) = 0$$

Φ defines **harmonic map** with target $SL(2,\mathbb{R})/SO(2) = \mathbb{H}^2$.
$\mathcal{R}$ is encoded in singular behaviour of Φ as $\rho \to 0$.

$$E[\Phi] := \int_{\mathbb{R}^3 \setminus \Gamma} \mathrm{Tr}[\Phi^{-1}\mathrm{d}\Phi \cdot \Phi^{-1}\mathrm{d}\Phi] \, \mathrm{dVol}$$

$$\text{tension : } \tau = \vec{\Delta}\Phi - \Phi^{-1}\vec{\nabla}\Phi \cdot \Phi^{-1}\vec{\nabla}\Phi$$

- apply existence theory for harmonic maps with *prescribed singularities* [Weinstein, Khuri-Weinstein-Yamada]

Open problems

- Regularity on axis using integrability [Neugebauer-Hennig, Lucietti-Tomlinson]
- Positive mass theorem for ALF/ALE gravitational instantons? [Alaee,Khuri, HK]
- Remarkably, harmonic map structure extends to $n > 4$ (in progress)