Spectrum of the Laplacian on the Page Metric

at Theory Canada 17 (2025)
June 09, 2025

Kam To Billy Sievers

Spectrum of the Laplacian on the Page metric [math.SP 2412.19879] submitted to J.Phys.A.

Hari Kunduri (McMaster U)

Yiqing (Mia) Wang (McMaster U)

Robie Hennigar (Durham U)

McMaster University

NSERC CRSNG

Page Metric

- Born out of Euclidean quantum gravity
  - Euclideanize Kerr-de Sitter to obtain this twisted S^2 bundled over S^2
  - The metric covering this has $SU(2) \times U(1)$ (for $x \in (-1,1)$)

$$g = S\left[\frac{\mathrm{d}x^2}{A(x)} + 4\alpha^2 A(x)\left(\mathrm{d}\psi + \frac{\cos\theta}{2}\mathrm{d}\phi\right)^2 + B(x)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right]$$

$$A(x) = \frac{(3-\nu^2-\nu^2(1+\nu^2)x^2)(1-x^2)}{1-\nu^2x^2}, \qquad B(x) = \frac{1-\nu^2x^2}{3+6\nu^2-\nu^4},$$

$$S = \frac{3(1+\nu^2)}{\Lambda}, \qquad \alpha = (2(3+\nu^2))^{-1}.$$

Laplacian on the Page Metric

- Recall the classical problem of finding spherical harmonics on S^2

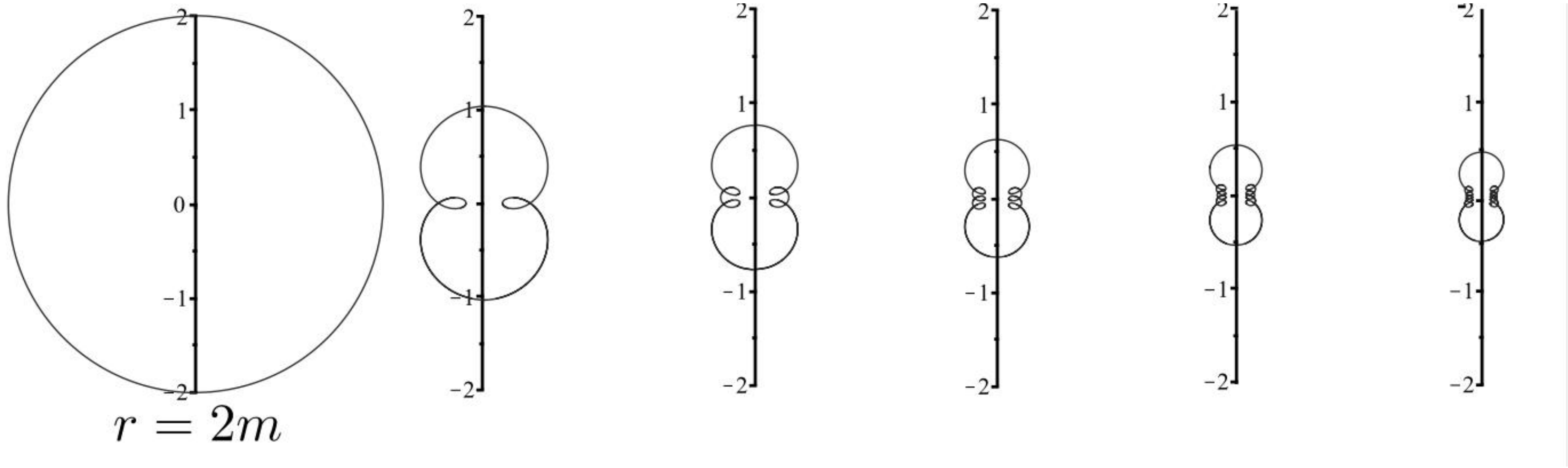
$$-\Delta u = \lambda u$$

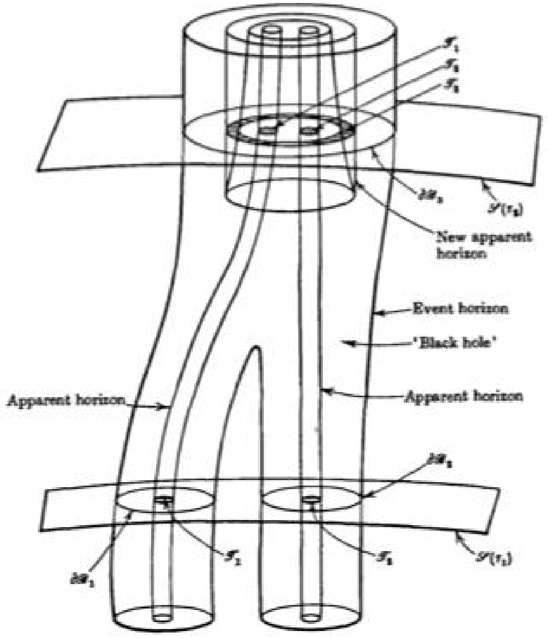
- On the Page metric, utilizing $SU(2) \times U(1)$ isometry reduces this to a Sturm-Liouville problem

$$-\frac{1}{SB(x)}\partial_x[A(x)B(x)\partial_x u(x)] = \left[\lambda - \frac{n^2}{4\alpha^2 SA} - \frac{\mu}{4SB}\right] u(x)$$

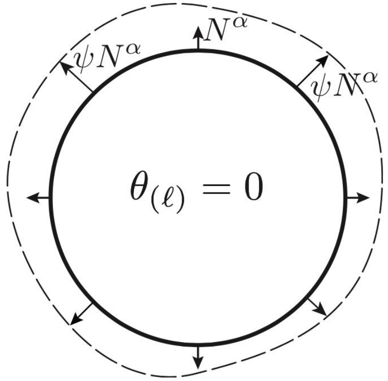
n and μ are integers

Pseudo-spectral method





$$L_\Sigma \psi = -\Delta\psi + \left(\frac{1}{2}\mathcal{R} - 2||\sigma_{(\ell)}||^2 - 2G_{++} - G_{+-}\right)\psi$$



- Booth, Hennigar, Mondal, 2020, PRD 102, 044031
- Pook-Kolb, Booth, Hennigar, 2021. PRD 104, 084084
- Hawking & Ellis 1973

Pseudo-spectral method

$$-\Delta u = \lambda u$$

- Discretize $x = x_i$ and $u(x) \approx \sum_{j}^{\mathcal{N}} c_j T_j(x)$ $\quad 0 \leq i, j \leq \mathcal{N}$

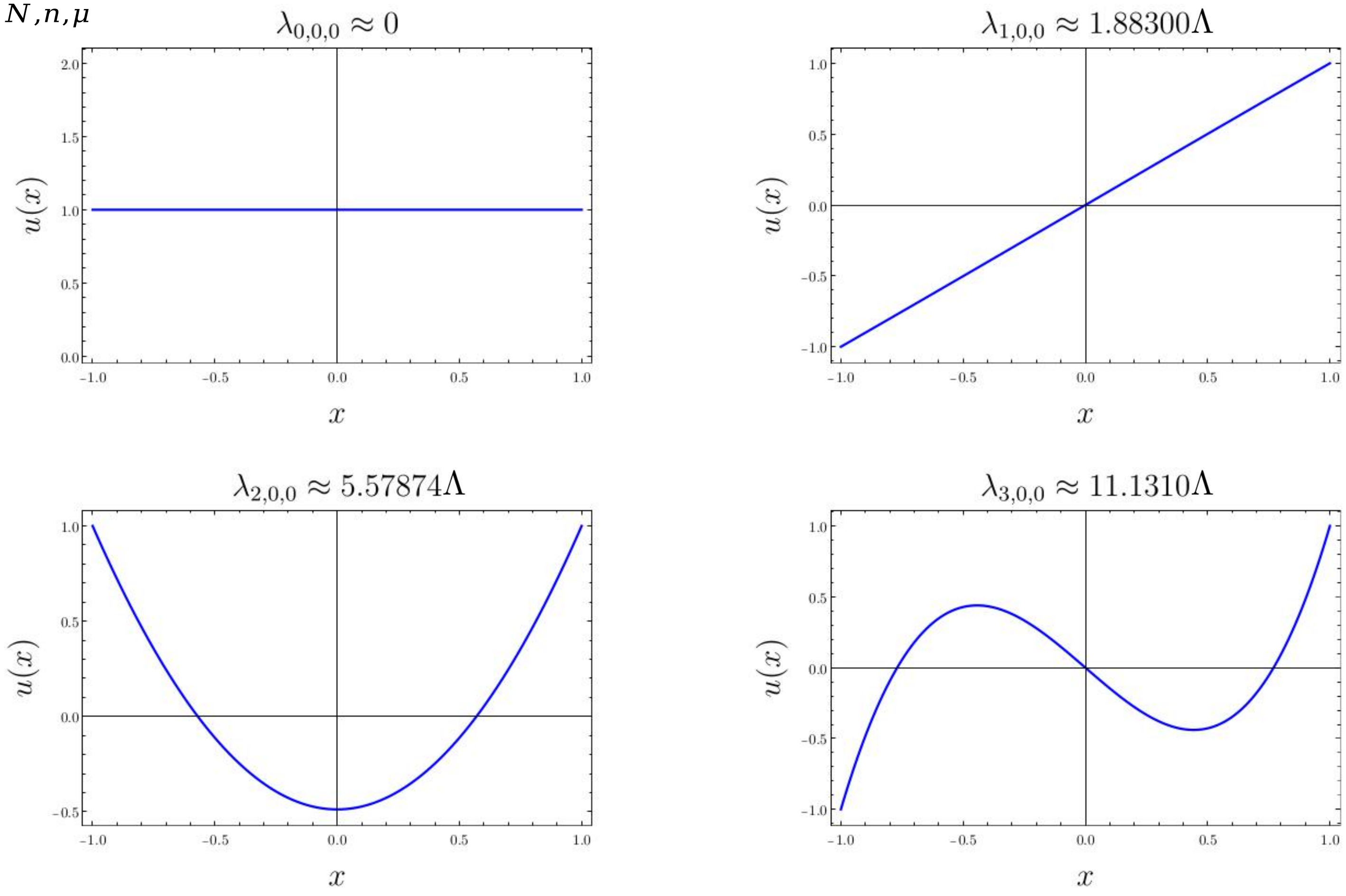
$$-\Delta u = \lambda u \;\rightarrow\; -\Delta_{ij} c_j = \lambda \Phi_{ij} c_j$$

$$-\Delta_{ij} = \Delta T_j(x_i) \qquad \Phi_{ij} = T_j(x_i)$$

- Eigenvalues and coefficients are found by solving char. eqn of $\Phi^{-1}\Delta$

Numerical results

$\lambda_{N,n,\mu}$



✔ Lichnerowicz-Obata bound

$$\lambda_1 \geq \frac{4\Lambda}{3}$$

✔ Hall-Murphy upper bound

$$\lambda_1 < 1.89\Lambda$$

- Besse, Springer 2018
- Hall, Murphy, 2014. Ann. Global Anal. Geom. 87-101

Lichnerowicz Op on the Page Metric

$$g_{\alpha\beta} \mapsto g_{\alpha\beta} + h_{\alpha\beta}$$
$$\mathrm{Ric}(g+h) = \Lambda \cdot (g+h)$$

$$\Delta_L(h)_{ab} = -(\nabla_b \nabla^b h_{ab} + 2R_{acbd}h^{cd})$$

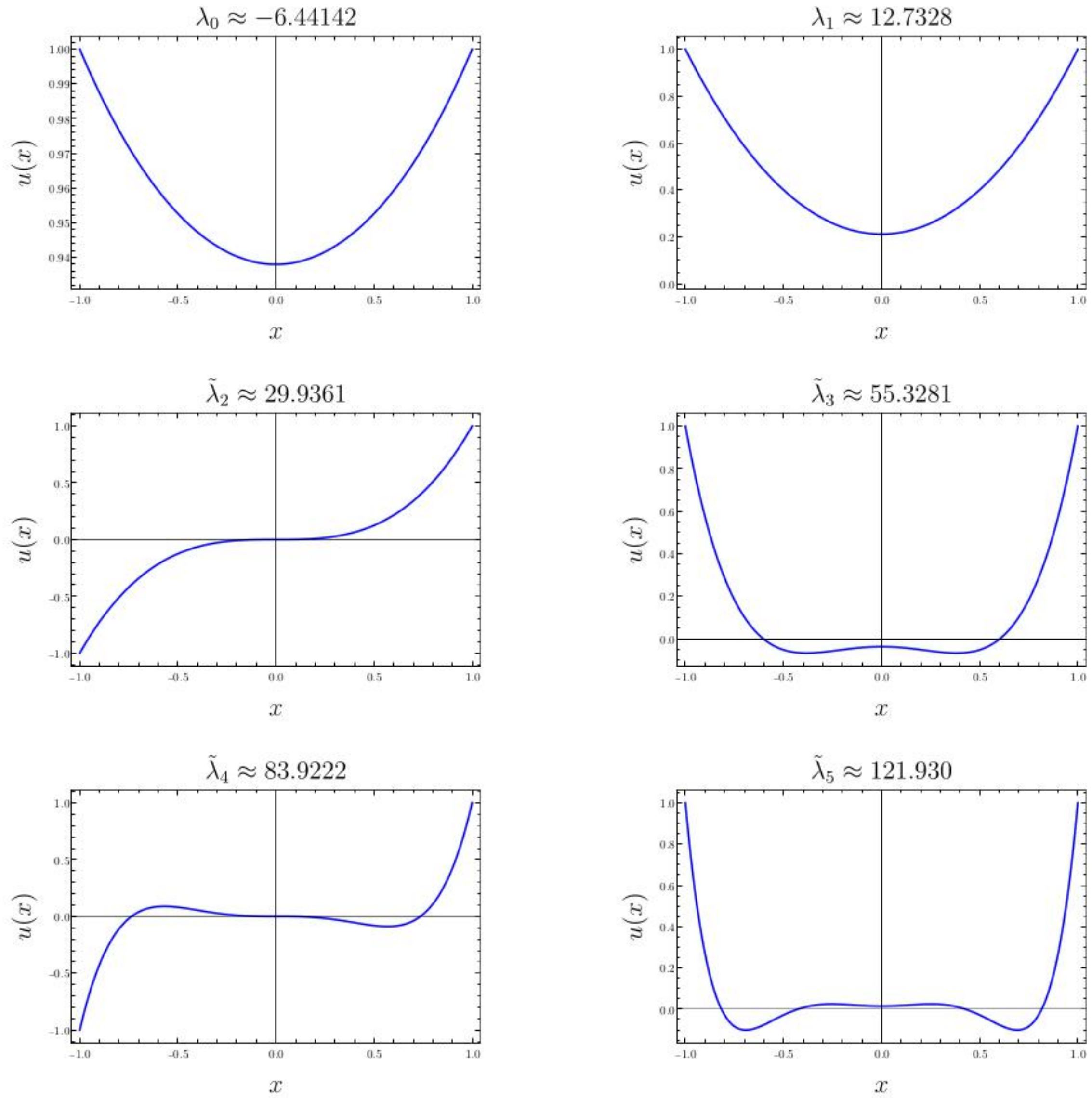
$$\Delta_L(h)_{ab} = \lambda h_{ab}$$

- Difficult coupled 2nd-order system
- Young reduces it to a master ODE for $h_{00} = h_{00}(x)$

$$h_{00}'' + C(x)h_{00}' + D(x)h_{00} = -\frac{\tilde{\lambda}}{A(x)}h_{00}$$

- Young, 1983. PRD 28, 2436-2438

Numerical results



✔ Young suggests there to be one negative eigenvalue

Young's negative eigenvalue was instead claimed to be

$$-5.75 < \tilde{\lambda}_0 < -5.74$$

- Young, 1983. PRD 28, 2436-2438

Summary

- Numerically approximated eigenvalues and eigenfunctions of studied operators on the Page metric.
- The results align with claims made in both the scalar and tensor mode cases.
- Can be repeated on spacetimes with problems that may be reduced similarly.