The Wightman Axioms for Hopf-Frobenius Modules

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Definition of Frobenius Algebras

A Frobenius algebras is a unital associative k-algebra with a pairing $\langle\cdot,\cdot\rangle : F \otimes F \to k$, such that

- Non-degenerate: If $\langle x, y\rangle = 0$ for all y, then $x = 0$. Similarly, $\langle x, y\rangle = 0$ for all x implies $y = 0$.
- Associativity: $\langle xy, z\rangle = \langle x, yz\rangle$, or in the complex case, $\langle xy, z\rangle = \langle x, y^* z\rangle$.

Equivalently: A linear functional $\varepsilon : F \to k$, that satisfies the non-degeneracy condition

$$\langle x, y\rangle = \varepsilon(xy) \quad \langle x, y\rangle = \varepsilon(x^* y)$$

Frobenius Algebras

Examples

Space	μ	$\langle \cdot, \cdot \rangle$
$C_c^\infty(X)$	Pointwise	$\langle f, g \rangle = \int f^* g$
Schwartz functions $\mathcal{S}$	Pointwise	$\langle f, g \rangle = \int f^* g$
$\mathbb{R}^n$ w/ basis $\{e_i\}$	$e_i e_j = \delta_{ij} e_i$	$\langle e_i, e_j \rangle = \delta_{ij}$
Hilbert space w/ basis $\{e_i\}$	$e_i e_j = \delta_{ij} e_i$	$\langle e_i, e_j \rangle = \langle e_i, e_j \rangle_H$
kG (finite group)	Group mult.	$\langle g, h \rangle = \delta_{gh,1}$

$$\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \sup_x |x^\alpha D^\beta f(x)| < \infty, \forall \alpha, \beta \}$$

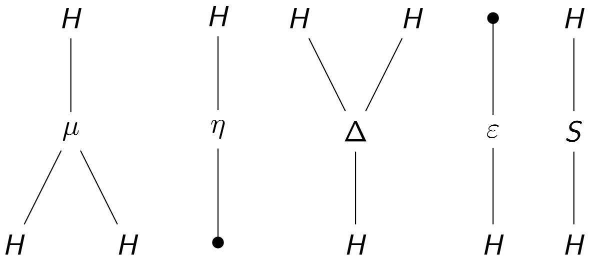
Hopf Algebras

Definition

A Hopf algebra is a unital associative k-algebra, H, with maps

- Comultiplication: $\Delta : H \to H \otimes H$
- Counit: $\varepsilon : H \to k$
- Antipode: An involution $S : H \to H$

which satisfy appropriate compatibility conditions.



Hopf Algebras

Examples

Space	Δ	ε
kG	$\Delta g = g \otimes g$	$\varepsilon(g) = 1$
Tensor alg. $T(X)$	$\Delta x = x \otimes 1 + 1 \otimes x$	$\varepsilon(x) = 0$
Symmetric alg. $S(X)$	$\Delta x = x \otimes 1 + 1 \otimes x$	$\varepsilon(x) = 0$
Univ. Env. Alg. $U(\mathfrak{g})$	$\Delta x = x \otimes 1 + 1 \otimes x$	$\varepsilon(x) = 0$
$\Lambda(X)$	$\Delta x = x \otimes 1 - 1 \otimes x$	$\varepsilon(x) = 0$

Sweedler Notation: Denote the comultiplication by

$$\Delta x = \sum x_{(1)} \otimes x_{(2)}.$$

Hopf-Frobenius Modules

Definition

A left Hopf-Frobenius module is a pair (H, F) where H is a Hopf algebra, F is a Frobenius algebra, such that the following conditions are satisfied:

- Module: F is a left H-module.
- Weak Cartan: $\sum \left\langle g_{(1)} \cdot x, g_{(2)} \cdot y \right\rangle = \varepsilon(g) \langle x, y \rangle$.
- Antipode Condition: $\langle S(g) \cdot x, y \rangle = \langle x, g \cdot y \rangle$.

Hopf-Frobenius Modules

Example 1 — Derivatives of Functions

Take our Hopf algebra to be $k[\partial]$ and the Frobenius algebra to be $\mathcal{S}$, the space of Schwartz functions. This is an HF-module with

- Action: $\partial \cdot f = f'$, the derivative of f
- Weak Cartan / Antipode: Both of these conditions reduce to

  $$\int (\partial \cdot f) g = -\int f(\partial \cdot g)$$

  The **weak derivative** condition.

Hopf-Frobenius Modules

Example 2 — Operators on a Hilbert Space

Let F be a separable Hilbert space, and let $\{A_i\}$ be a family of compact mutually-commuting *skew-Hermitian* operators on F. Then we can find an orthonormal basis of simultaneous eigenvectors, $\{e_k\}$.
Take $(F, \{e_k\})$ as our Frobenius algebra, and let $H = U(\{A_i\})$ be the universal enveloping algebra of the abelian Lie algebra generated by the $\{A_i\}$. This is a HF-module where

- Action: $A_i \cdot e_k = A_i e_k = \alpha_i^k e_k$, the natural action of the Hermitian operators on the eigenvector basis.
- Weak Cartan / Antipode: Both of these conditions reduce to

$$\langle Hx, y \rangle = - \langle x, Hy \rangle$$

The **skew-Hermitian** condition.

Hopf-Frobenius Modules

Example 3 — Lie Groups

Consider the Lie group $U(n)$, and let H be its group algebra over $\mathbb{C}$. Take $\mathbb{C}^n$ with basis $\{e_i\}$ as our Frobenius algebra.

- Action: Left multiplication.
- Weak Cartan / Antipode: Both of these conditions reduce to

$$\langle g \cdot x, g \cdot y \rangle = \langle x, y \rangle$$

The unitarity condition.

Hopf-Frobenius Modules

Frobenius		Hopf	
States	**Hil**	Operators	$B(\mathbf{Hil})$
Functions	$\mathcal{S}$	Derivatives	$\mathbb{C}[\partial_1, \ldots, \partial_n]$
Space	$\mathbb{R}^{1,3}$	Transformations	Λ

Wightman Axioms

Relativistic Quantum Mechanics

1. **Hilbert Space:** There is a separable Hilbert space of states X.
2. **Evolution:** There is a unitary representation of $SL(2, \mathbb{C})$ on X, which we denote $U(L, a)$
3. **Spectral Condition:** We can write $U(1, a) = \exp(iP^\mu a_\mu)$ and the eigenvalues of P satisfy $p_0^2 - \sum_{i=1}^3 p_i^2 > 0$ and $p^0 > 0$.
4. **Vacuum:** There is a unique vector $|0\rangle \in X$ invariant under $U(L, a)$.

Wightman Axioms

Field Conditions

1. **Dense Subset:** There is a dense subset $D \subseteq X$ which contains $|0\rangle$ and is mapped to itself by $U(L, a)$.
2. **Fields:** Quantum fields are operator-valued distributions on a Schwartz space, $\phi_i : \mathcal{S} \to B(D)$.
3. **Transformations:** There is an action of $SL(2)$ on the fields, $S(L)$ such that $U(L, a)\phi(x)U(L, a)^{-1} = S(L)\phi(L^{-1}(x - a))$
4. **Cyclic Vacuum:** The image of polynomials in $\phi_i(f)$ on $|0\rangle$ is dense in X.

Wightman Axioms

Microcausality Condition

1. If $f, g \in \mathcal{S}$ have spacelike supports ($f(x)g(y) = 0$ for all x, y with $d(x, y) \geq 0$) then $[\phi_i(f), \phi_j(g)]_\pm(v) = 0$ for all $v \in D$.

Hopf-Frobenius Wightman Axioms

A (weak) Hopf-Frobenius-Wightman QFT is a collection of Hopf algebras $(B(F_{\text{state}}), kSL(2))$ and Frobenius algebras $(F_{\text{state}}, \mathcal{S}, \mathbb{R}^{1,3})$ such that

1. States and Evolution: $(kSL(2), F_{\text{state}})$ forms a Hopf-Frobenius module.
2. Vacuum: There is a unit $1 \in F_{\text{state}}$, which is the unique element preserved by $kSL(2)$.
3. Fields: Quantum fields are linear maps $\mathcal{S} \to B(F_{\text{state}})$ and form a Frobenius algebra F_{field}.
4. Field Transformations: $(kSL(2), F_{\text{field}})$ forms a Hopf-Frobenius module.

Hopf-Frobenius Wightman Axioms

Other Conditions

1. **Dense Subset:** We can view F_{state} as all of D.
2. **Cyclic Vacuum:** The image of polynomials in $\phi_i(f)$ on $|0\rangle$ is dense in X.
3. **Spectral Condition:** We can write $U(1,a) = \exp(iP^\mu a_\mu)$ and the eigenvalues of P satisfy $p_0^2 - \sum_{i=1}^3 p_i^2 > 0$ and $p^0 > 0$.
4. **Causality:** If $f, g \in \mathcal{S}$ have spacelike supports ($f(x)g(y) = 0$ for all x, y with $d(x,y) \geq 0$) then $[\phi_i(f), \phi_j(g)]_\pm(v) = 0$ for all $v \in D$.