Topological Landau Theory

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Theory Canada 17

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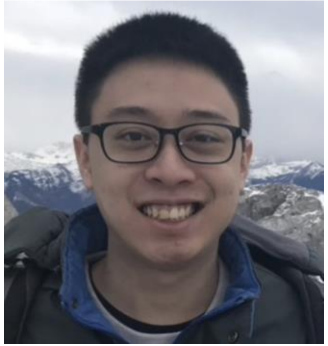
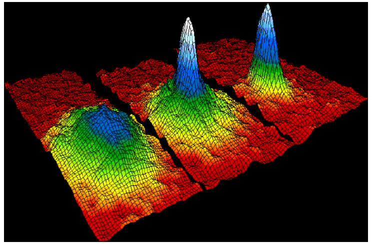


Canon Sun
(Alberta)

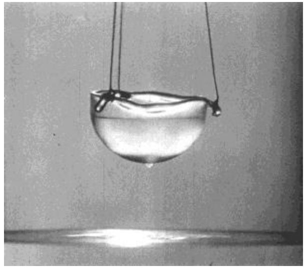
C. Sun & JM, arXiv:2412.15103
(PRL, in press)

Phases of matter

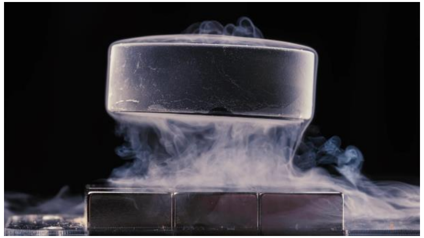
Ferromagnet



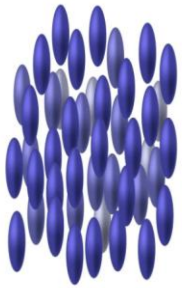
Bose-Einstein condensate



Superfluid



Superconductor



Liquid crystal

and more…

Landau theory

- Framework to understand phase transitions
- Also used to understand the phases themselves
- General—applies to a wide variety of systems
- Phenomenological—no need for microscopic theory

Overview

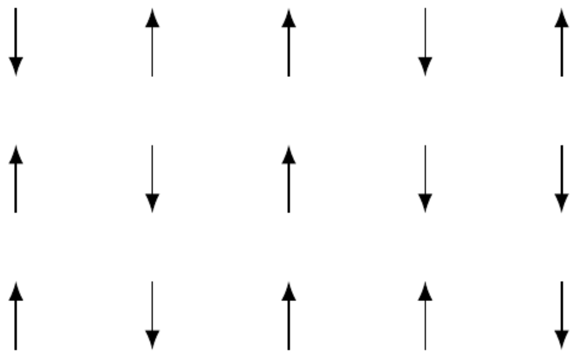
1. Landau theory
    - Ising ferromagnet
    - Superconductor

2. Topological Landau theory
    - Berry phase in quantum mechanics
    - Berry phase in statistical mechanics

3. Microscopic model
    - Topology of the superconducting order parameter ($\neq$ topological superconductivity!)
    - Measurement of Berry phase in Josephson junctions

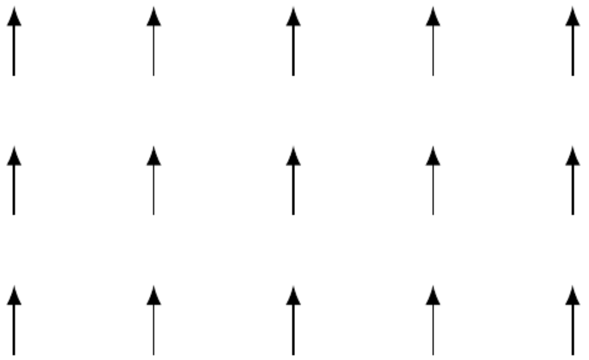
Ising ferromagnet

Paramagnet



$$m = 0$$

Ferromagnet



$$m \neq 0$$

Landau free energy

- Free energy depends on order parameter (magnetization)

- Order parameter is small near continuous phase transition

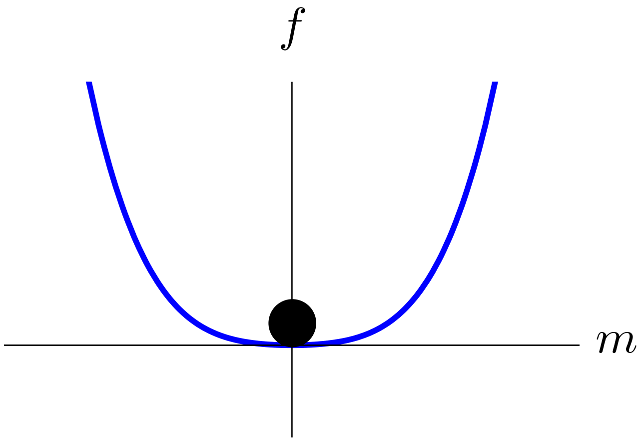
  => expand free energy as power series in m

- Free energy must be compatible with symmetry $m \mapsto -m$

$$f = f_0 + am^2 + bm^4 + \dots$$

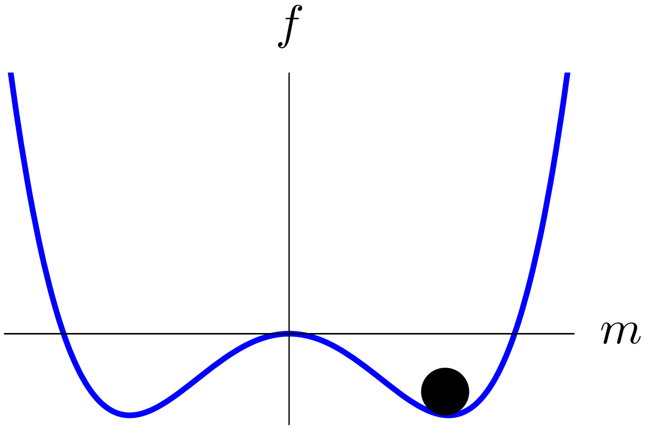
Spontaneous symmetry breaking

Free energy: $f = am^2 + bm^4 + \ldots$



$a > 0$

Paramagnet $m_{\text{min}} = 0$



$a < 0$

Ferromagnet $m_{\text{min}} \neq 0$

General prescription

1. Identify order parameter
   - Magnetization m

2. Construct free energy
   - Free energy must be invariant under symmetry operations
   - Spin flip $m \mapsto -m$

$$f = am^2 + bm^4 + \dots$$

3. Minimize free energy
   - Quadratic coefficient determines critical temperature

$$m_{\mathrm{min}} = \begin{cases} 0 & a > 0 \\ \sqrt{-\frac{a}{2b}} & a < 0 \end{cases}$$

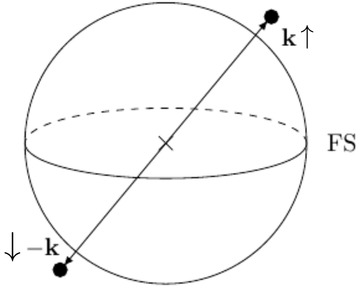
Superconductivity

Phenomenology

- Zero resistivity
- Meissner effect (flux expulsion)

Cooper pair

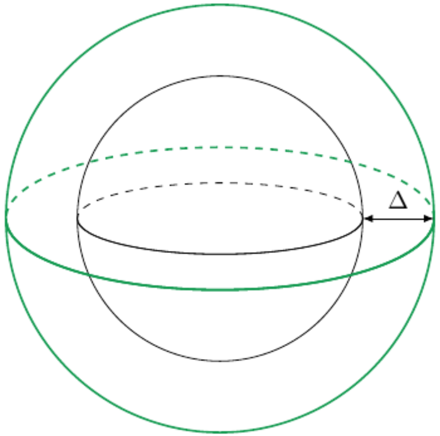
- Electrons pair to form a Cooper pair
- Cooper pairs condense



Order parameter

Pair amplitude

$$\langle \hat{c}_{\mathbf{k}\uparrow}\hat{c}_{-\mathbf{k}\downarrow}\rangle \begin{cases} = 0 & \text{metallic} \\ \neq 0 & \text{superconducting} \end{cases}$$



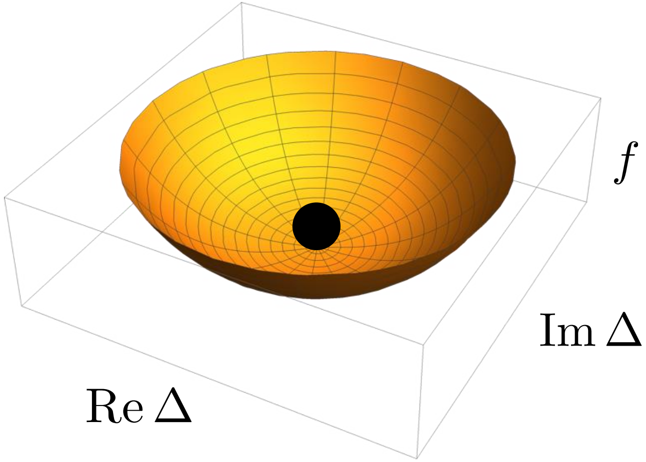
Gap function

$$\Delta = -\frac{V}{\mathcal{V}}\sum_{\mathbf{k}}\langle \hat{c}_{\mathbf{k}\uparrow}\hat{c}_{-\mathbf{k}\downarrow}\rangle \in \mathbb{C}$$

Free energy should be invariant under a change of phase

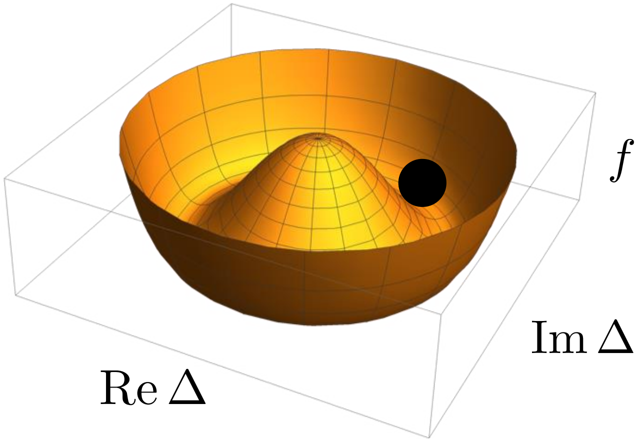
Spontaneous symmetry breaking

Free energy: $$f = a|\Delta|^2 + b|\Delta|^4 + \ldots$$



$a > 0$

Metal $\Delta_{\text{min}} = 0$



$a < 0$

Superconductor $\Delta_{\text{min}} \neq 0$

Gauge symmetry

Wavefunction

- Complex function

- Wavefunctions that differ by a phase correspond to the same quantum state

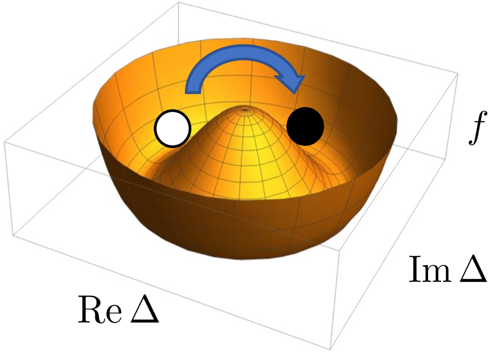
$$|\psi\rangle \rightarrow e^{i\varphi}|\psi\rangle$$

Gap function

- Complex function

- Gap functions that differ by a phase correspond to the same thermodynamic ground state

$$\Delta \rightarrow e^{i\varphi}\Delta$$



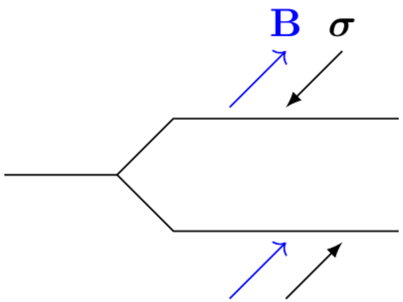
Berry phase in QM

Spin-1/2 in a magnetic field

Hamiltonian

$$H = -\mathbf{B}\cdot\boldsymbol{\sigma} = -|B|\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$$

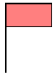
Spectrum

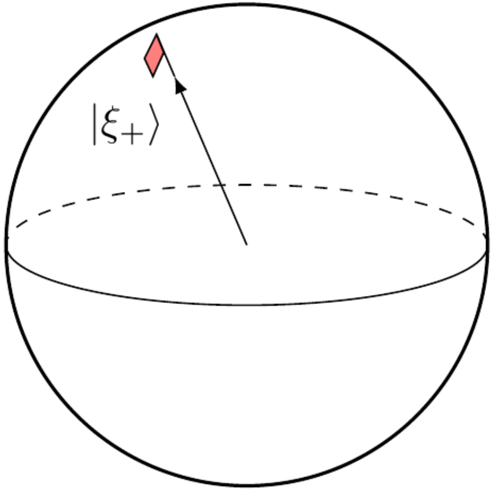


$$|\xi_-\rangle = \left(\sin\tfrac{\theta}{2}e^{-i\phi}, -\cos\tfrac{\theta}{2}\right)$$

$$|\xi_+\rangle = \left(\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}e^{i\phi}\right)$$

Bloch sphere

 $=$ phase of $|\xi_+\rangle$



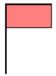
Berry phase in QM

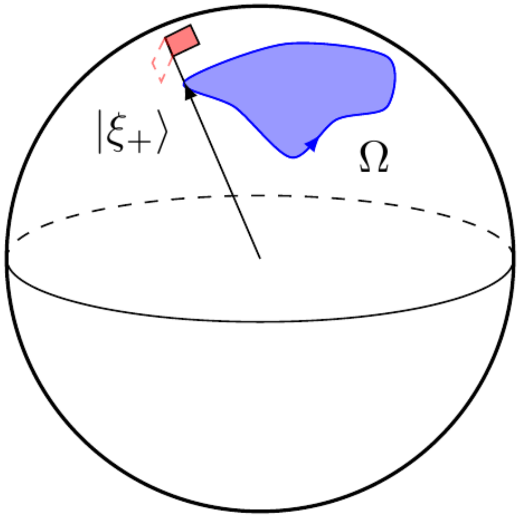
Adiabatic theorem

If the parameters of the Hamiltonian are varied slowly, then the system will remain in the same eigenstate.

Returning to the initial parameters brings the system back to its starting state, <u>up to a phase</u>

Bloch sphere

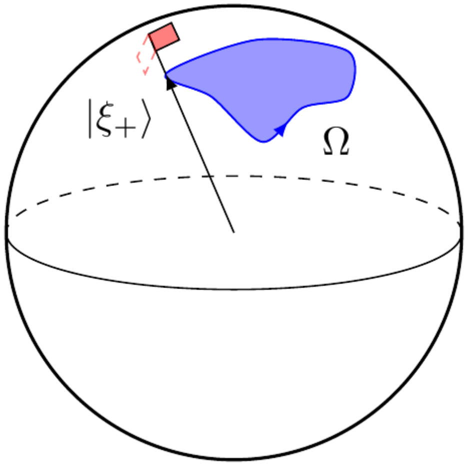

= phase of $|\xi_+\rangle$



$$|\xi_+\rangle \to e^{iq\Omega}|\xi_+\rangle \qquad q = -1/2$$

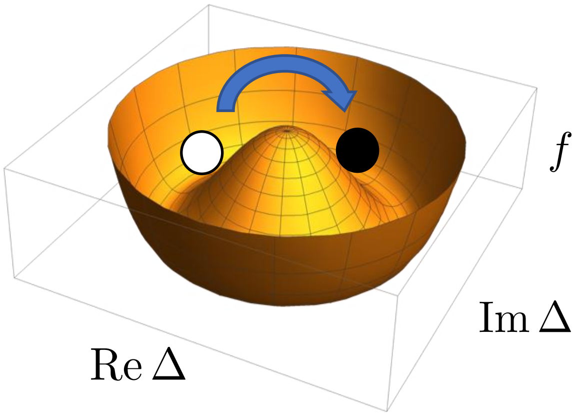
Berry phase

Quantum mechanics



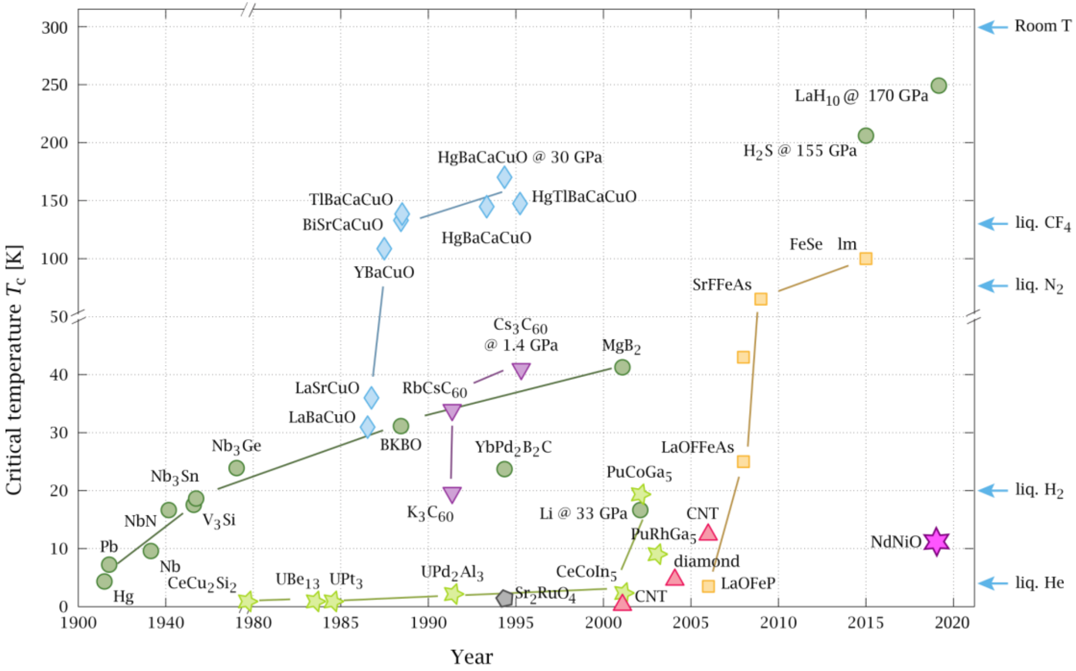
$$|\xi_+\rangle \to e^{iq\Omega}|\xi_+\rangle$$

Statistical mechanics



$$\Delta \to e^{i\varphi}\Delta \quad ?$$

Cuprate superconductors



Order parameter: $\Delta_{\mathbf{k}} = \Delta \left(k_x^2 - k_y^2\right) / k_F^2$

Competition between phases

Order parameter

$$\Delta_{\mathbf{k}} = \Delta_1 + \Delta_2 (k_x^2 - k_y^2)/k_F^2$$

Quadratic term

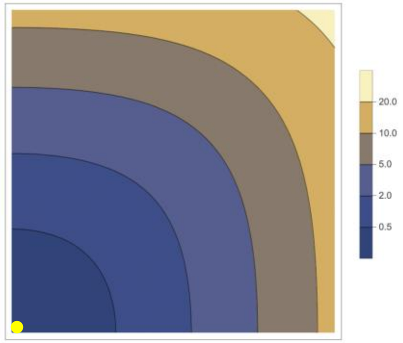
$$f_2 = a_1 |\Delta_1|^2 + a_2 |\Delta_2|^2$$

Quadratic term determines:

1. Critical temperature
2. The order parameter

Can we have a $\Delta_1^* \Delta_2$ term?

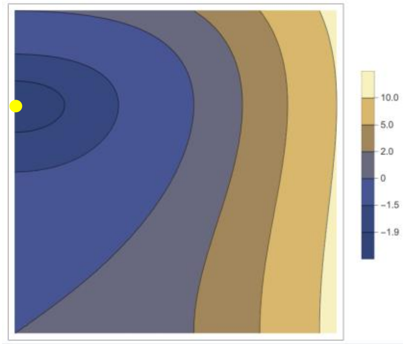
$|\Delta_2|$



$|\Delta_1|$

$a_1, a_2 > 0$

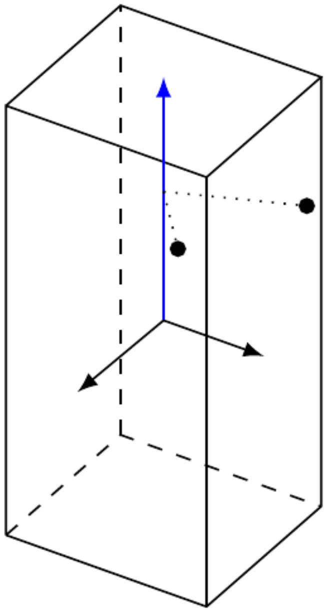
$|\Delta_2|$



$|\Delta_1|$

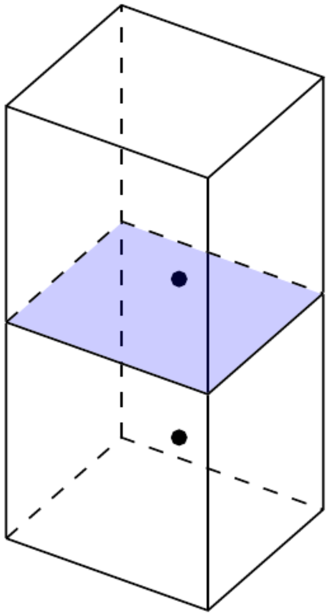
$a_1 > 0, a_2 < 0$

D_{4h} point group



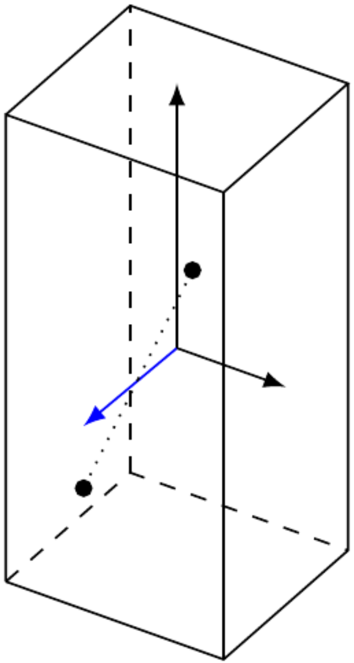
C_4 rotation about z-axis

$$(k_x, k_y, k_z) \mapsto (-k_y, k_x, k_z)$$



Mirror parallel to x-y plane

$$(k_x, k_y, k_z) \mapsto (k_x, k_y, -k_z)$$



C_2 rotation about x-axis

$$(k_x, k_y, k_z) \mapsto (k_x, -k_y, -k_z)$$

Symmetry of order parameter

Order parameter $\Delta_{\mathbf{k}} = \Delta_1 + \Delta_2(k_x^2 - k_y^2)/k_F^2$

	C_4 rotation z-axis	x-y mirror	C_2 rotation x-axis
	$(k_x, k_y, k_z) \mapsto (-k_y, k_x, k_z)$	$(k_x, k_y, k_z) \mapsto (k_x, k_y, -k_z)$	$(k_x, k_y, k_z) \mapsto (k_x, -k_y, -k_z)$
1	1	1	1
$k_x^2 - k_y^2$	$-(k_x^2 - k_y^2)$	$k_x^2 - k_y^2$	$k_x^2 - k_y^2$

$\Delta_1^* \Delta_2$ not invariant under C_4 rotation about z-axis

Symmetry of order parameter

Order parameter $\Delta_{\mathbf{k}} = \Delta_1 + \Delta_2(k_x^2 + k_y^2)/k_F^2$

	C_4 rotation z-axis	x-y mirror	C_2 rotation x-axis
	$(k_x, k_y, k_z) \mapsto (-k_y, k_x, k_z)$	$(k_x, k_y, k_z) \mapsto (k_x, k_y, -k_z)$	$(k_x, k_y, k_z) \mapsto (k_x, -k_y, -k_z)$
1	1	1	1
$k_x^2 - k_y^2$	$-(k_x^2 - k_y^2)$	$k_x^2 - k_y^2$	$k_x^2 - k_y^2$
$k_x^2 + k_y^2$	$k_x^2 + k_y^2$	$k_x^2 + k_y^2$	$k_x^2 + k_y^2$

$\Delta_1^* \Delta_2$ invariant under all symmetry operations!

Matrix quadratic term

Order parameter

$$\Delta_{\mathbf{k}} = \Delta_1\phi_1(\mathbf{k}) + \Delta_2\phi_2(\mathbf{k})$$

with

$$\phi_1(\mathbf{k}) = 1$$ "s-wave"

$$\phi_2(\mathbf{k}) = \sqrt{5}\left(1 - \frac{3}{2}\frac{k_x^2 + k_y^2}{k_F^2}\right)$$ "extended s-wave"

Orthonormal basis functions that transform the **same** way under symmetry operations

Quadratic term

Most general term compatible with symmetry

$$f_2 = \begin{pmatrix}\Delta_1^* & \Delta_2^*\end{pmatrix}\begin{pmatrix}A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\Delta_1 \\ \Delta_2\end{pmatrix}$$

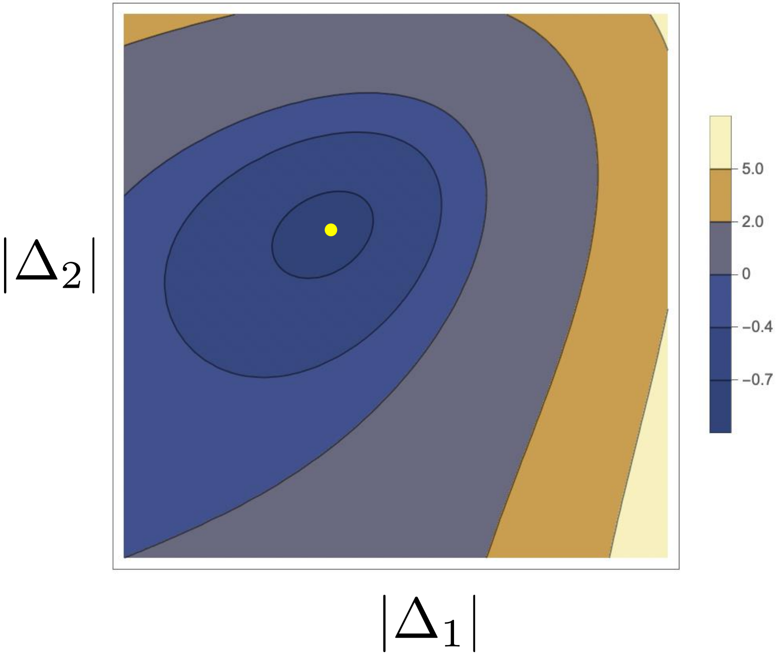
where A is a Hermitian matrix

Berry phase of order parameter

Free energy

$$f_2 = \begin{pmatrix} \Delta_1^* & \Delta_2^* \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}$$

- Critical temperature = temperature when smallest eigenvalue changes sign
- Order parameter = eigenvector of smallest eigenvalue
- The matrix A depends on microscopic parameters
    - Order parameter changes as parameters are varied
    - Berry phase



Model Hamiltonian

Free Hamiltonian

$$\hat{H}_0 = \sum_{\mathbf{k}\sigma} \hat{c}^{\dagger}_{\mathbf{k}\sigma} \xi_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma}$$

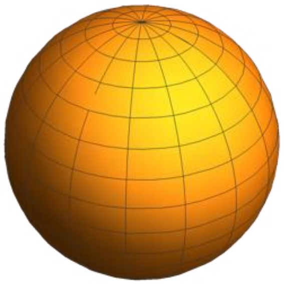
Interaction Hamiltonian

$$\hat{H}_{\text{int}} = -\frac{1}{\mathcal{V}} \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \hat{c}_{-\mathbf{k}'\downarrow} \hat{c}_{\mathbf{k}'\uparrow}$$

Scattering matrix element

$$V_{\mathbf{k}\mathbf{k}'} = \sum_{\alpha,\beta=1}^{2} V_{\alpha\beta} \phi_\alpha(\mathbf{k}) \phi_\beta(\mathbf{k}')$$

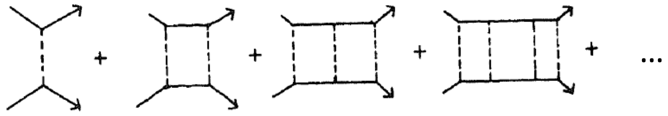
D_{4h} orbitals



$$\phi_1(\mathbf{k}) = 1$$

$$\phi_2(\mathbf{k}) = \sqrt{5}\left(1 - \frac{3\sin^2\theta_{\mathbf{k}}}{2}\right)$$

Free energy



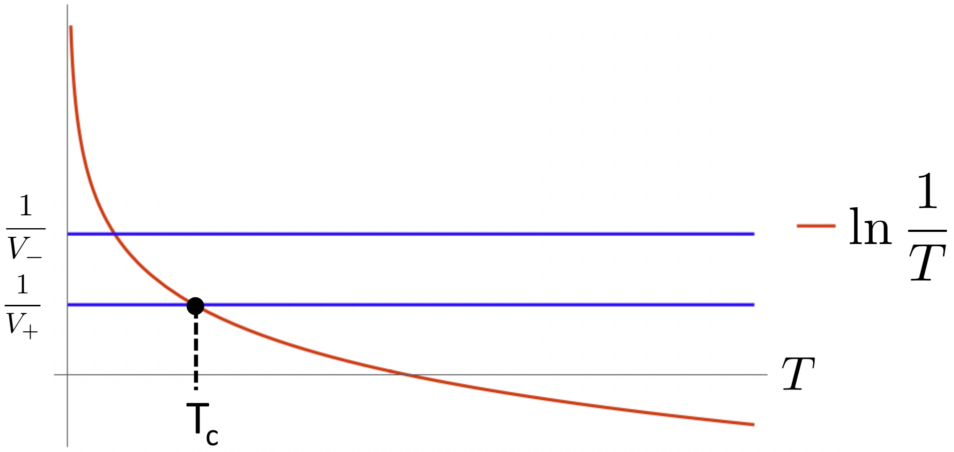
Quadratic term

$$f_2 = \sum_{\alpha\beta} \Delta^*_\alpha A_{\alpha\beta} \Delta_\beta$$

where

$$A_{\alpha\beta} = V^{-1}_{\alpha\beta} - N(0) \ln\left(\frac{2e^\gamma}{\pi} \frac{\Lambda}{k_B T}\right) \delta_{\alpha\beta}$$

- Logarithmic divergence as $T \to 0$
- There is a phase transition at sufficiently low temperature
- Order parameter $\propto$ eigenvector $\hat{\Delta}_+$ of V with largest eigenvalue



Adiabatic order parameter dynamics

Time-dependent Ginzburg-Landau equation

$$-\hbar\eta\frac{\mathrm{d}\Delta_\alpha}{\mathrm{d}t} = \frac{\partial f}{\partial \Delta_\alpha^*}, \qquad \eta \;=\; \pi N(0)/(8k_B T)$$

Landau & Khalatnikov 1954
Gor'kov & Eliashberg 1968

- Vary interaction parameters $\boldsymbol{\lambda}(t)$ in V "slowly", on characteristic time scale τ
- Slightly below T_c, critical (non-critical) eigenvector relaxes on time scale τ_{crit} ($\tau_{\mathrm{non\text{-}crit}}$) with $\tau_{\mathrm{non\text{-}crit}} \ll \tau_{\mathrm{crit}}$
- Provided $\tau \gg \tau_{\mathrm{crit}}$, order parameter stays in the "instantaneous ground state" $\hat{\Delta}_+(\boldsymbol{\lambda}(t))$ but acquires Berry phase φ :

$$\varphi = \oint_C d\boldsymbol{\lambda}\cdot\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}) \qquad\qquad \mathcal{A}_j(\boldsymbol{\lambda}) = i\hat{\Delta}_+^\dagger(\boldsymbol{\lambda})\partial_j\hat{\Delta}_+(\boldsymbol{\lambda})$$

Berry connection

Thermodynamic 3D Weyl point

Most general scattering matrix element

(time-reversal symmetry breaking)

$$V = V_0 I_2 + |V| \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

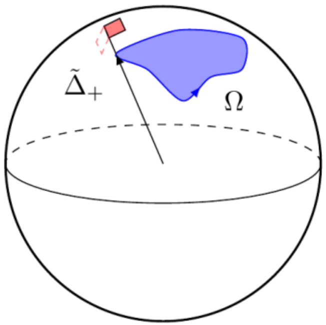
Eigenvalues

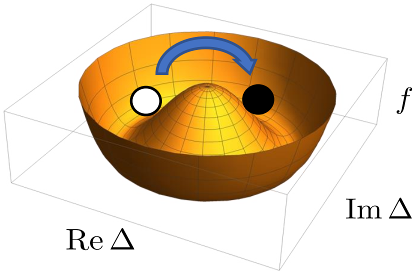
$$V_\pm = V_0 \pm |V|$$

Eigenvectors

$$\tilde{\Delta}_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix} \qquad \tilde{\Delta}_- = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

U(1) Berry phase





$$\tilde{\Delta}_+ \to e^{iq\Omega} \tilde{\Delta}_+ \qquad q = -1/2$$

“monopole bundle”

Thermodynamic 2D Dirac point

Most general *time-reversal symmetric* scattering matrix element

$$V = V_0 I_2 + |V| \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$$

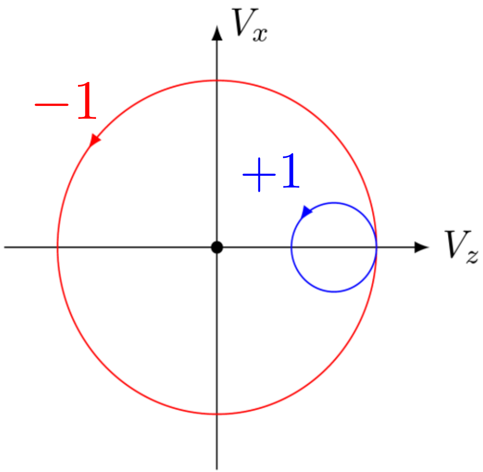
Eigenvalues

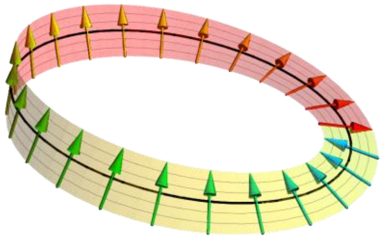
$$V_\pm = V_0 \pm |V|$$

Eigenvectors

$$\tilde{\Delta}_+ = \begin{pmatrix} \cos\frac{\phi}{2} \\ \sin\frac{\phi}{2} \end{pmatrix} \qquad \tilde{\Delta}_- = \begin{pmatrix} -\sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix}$$

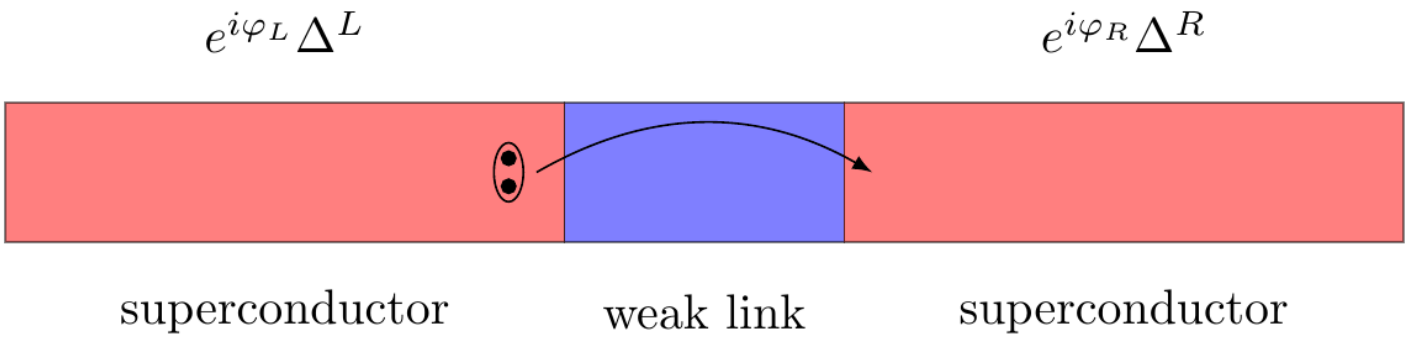
Z_2 Berry phase





“Möbius bundle”

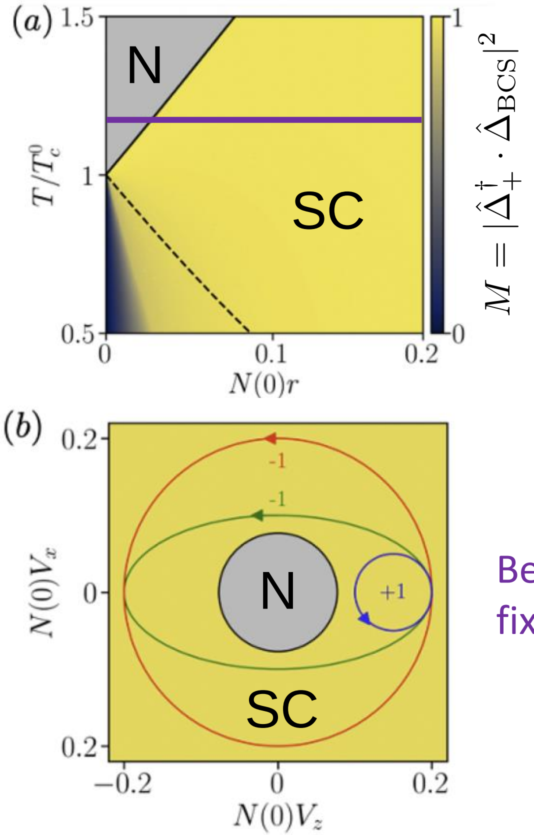
Josephson effect



Cooper pairs tunneling through a weak link leads to a current

$$\boxed{I \propto \sin(\varphi_L - \varphi_R)}$$

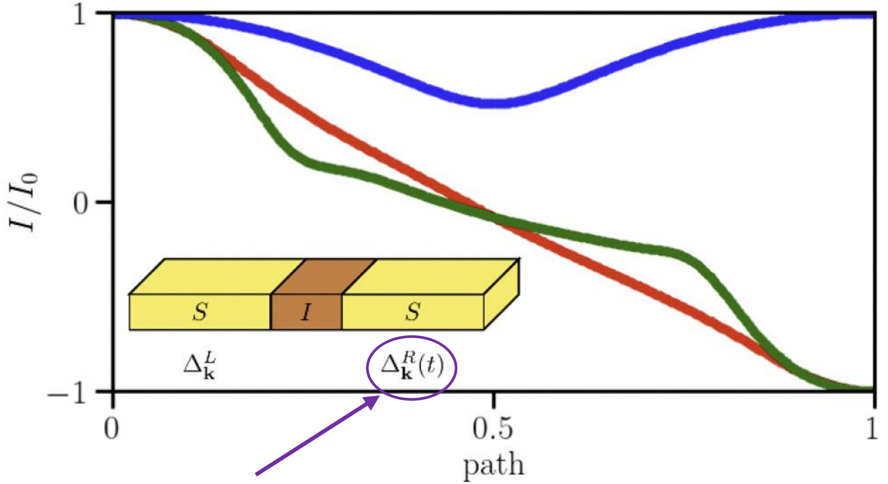
Topological Josephson effect



“thermodynamic Dirac cone”

Berry phase at fixed T

Josephson current changes sign due to π Berry phase of Dirac cone



vary interactions adiabatically

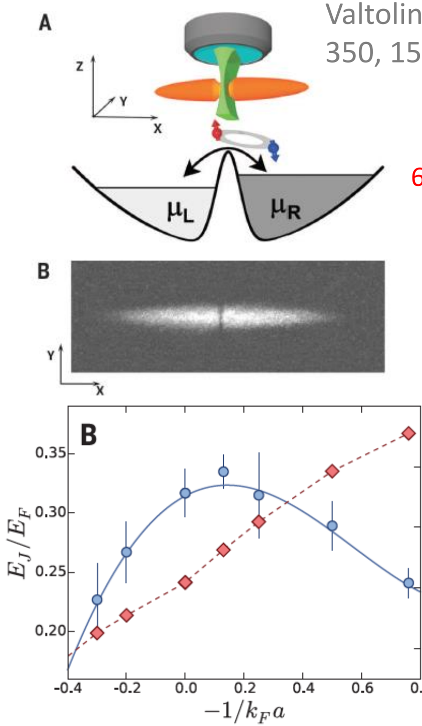
Tunable atomic junctions?

a = scattering length (Feshbach resonance)

QUANTUM SIMULATION

Josephson effect in fermionic superfluids across the BEC-BCS crossover

Valtolina et al., Science 350, 1505 (2015)

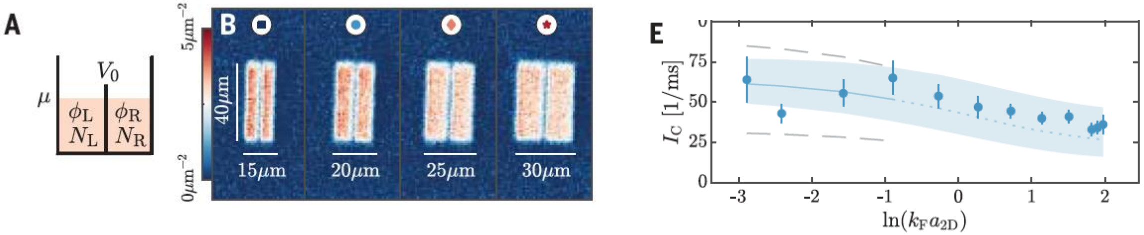


^{6}Li

QUANTUM GASES

An ideal Josephson junction in an ultracold two-dimensional Fermi gas

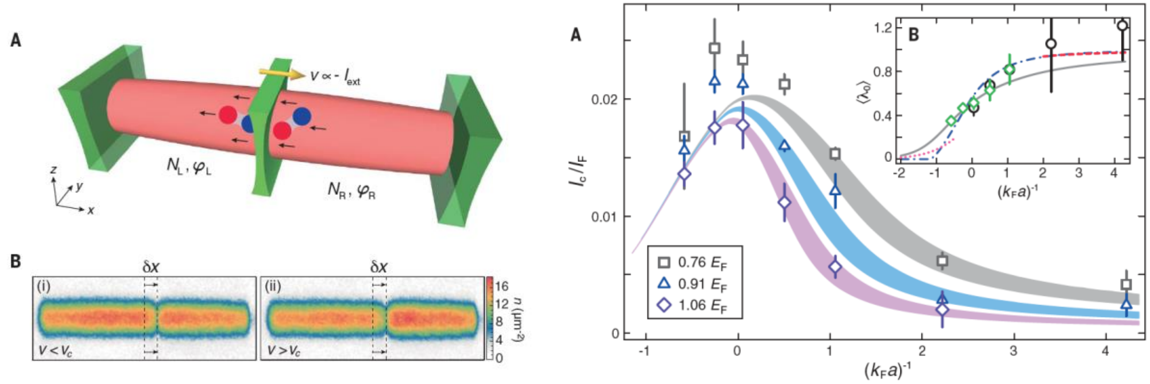
Luick et al., Science 369, 89 (2020)



QUANTUM GASES

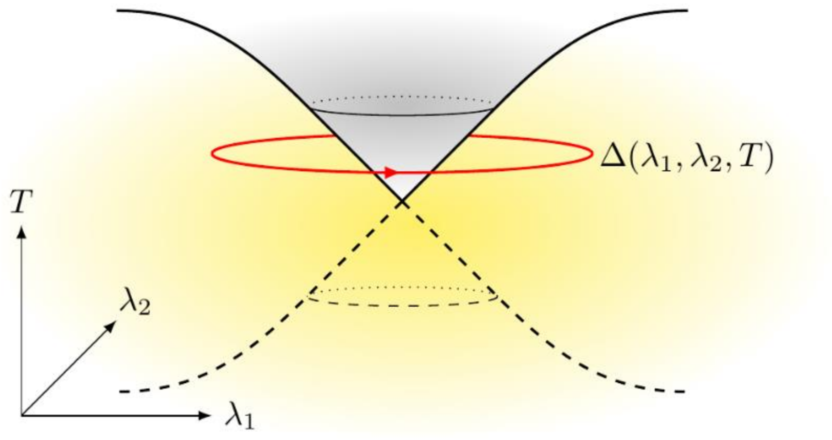
Strongly correlated superfluid order parameters from dc Josephson supercurrents

Kwon et al., Science 369, 84 (2020)



Summary

- ”Hidden” topology in Landau theory
- Berry phase of order parameter from thermodynamic analogs of Weyl & Dirac points
- Realization in microscopic model of superconductor/superfluid with two “competing” order parameters
- Berry phase can be measured in a Josephson junction
- Potential realization in ultracold atomic Josephson junctions (?)



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C. Sun and JM, arXiv:2412.15103

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Some formal aspects

Symmetry group G

- e.g. $\mathrm{SO}(3), D_{4h}$

Order parameter

- Order parameter lives in some Hilbert space
- e.g. $\mathcal{H} = L^2(S^2)$

Hilbert space can be decomposed into invariant subspaces:

$$\mathcal{H} = \bigoplus_{\mu,\alpha} \mathcal{H}_\alpha^{(\mu)}$$

- μ labels the irrep
- α labels the multiplicity

Order parameter can be expanded as

$$\Delta(\mathbf{k}) = \sum_{\mu,\alpha,m} \Delta_{\alpha m}^{(\mu)} \phi_{\alpha m}^{(\mu)}(\mathbf{k})$$

- m labels vector within irrep
- e.g. $\phi_1(\mathbf{k}) = 1$ and $\phi_2(\mathbf{k}) = k_x^2 + k_y^2$ are different functions that belong to the same irrep

Quadratic term

Most general quadratic term compatible with symmetry

$$f_2 = \sum_{\mu,\alpha,\beta,m} \Delta_{\alpha m}^{(\mu)*} A_{\alpha\beta}^{(\mu)} \Delta_{\beta m}^{(\mu)}$$

Parallels with topological band theory

Quadratic term

- Hermitian matrix A

- Depends on parameters $A(\boldsymbol{\lambda})$
  - Microscopic, e.g. interaction strength
  - Macroscopic, e.g. temperature

- Phase transition happens when eigenvalue crosses zero

- Eigenvector below zero = relevant order parameter

Bloch Hamiltonian

- Hermitian matrix H

- Depends on parameters $H(\mathbf{k})$

- Metal-insulator transition when eigenvalue crosses zero

- Eigenvector below zero = filled Bloch state