How the Schwarzschild-de Sitter horizons remain in thermal equilibrium at vastly different temperatures

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Outline

1. Can the Schwarschild and de Sitter horizons in SdS be in thermal equilibrium at $T_{black\ hole} \gg T_{cosmological}$?
2. Tolman-Ehrenfest criterion of thermal equilibrium
3. Generalization to “stationary heat conduction” between S and dS horizons
4. Temperature regularization at the horizons
5. Conclusions

Schwarzschild-de Sitter space

- A Schwarzschild black hole horizon radiates at the Hawking temperature $\mathcal{T}_H = \frac{1}{8\pi m}$ (units $\hbar = c = G = 1$)
- A de Sitter horizon radiates at the Gibbons-Hawking temperature $\mathcal{T}_{GH} = \frac{H}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}$
- A Schwarschild-de Sitter/Kottler spacetime

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{1 - 2m/r - \Lambda r^2/3} + r^2 d\Omega^2_{(2)}$$

has two horizons. If the black hole has size $2m \ll H^{-1}$ = Hubble radius, then

$$\boxed{\mathcal{T}_H \gg \mathcal{T}_{GH}}$$

Can these two horizons be at thermal equilibrium at vastly different temperatures? The standard lore is **NO** because $\mathcal{T}_H \neq \mathcal{T}_{GH}$, but **think again.**

Tolman-Ehrenfest: thermal equilibrium in spacetime

Usually “thermal equilibrium” means:

$$\frac{\partial \mathcal{T}}{\partial t} = 0 \; + \; \text{“heat flux} = 0\text{”}$$

The Tolman-Ehrenfest criterion (1928-30) applies:

- to a test fluid
- in a static spacetime
- the fluid is at rest in coordinates adapted to the time symmetry

$$\boxed{\mathcal{T}\sqrt{-g_{00}} = \text{const.}}$$

T cannot be constant in thermal equilibrium in curved spacetime (unless $g_{00} \neq$ const.) Meaning:
heat = energy = mass → sinks in a gravitational field and a temperature gradient is necessary for equilibrium.

Here we apply TE to the Hawking+Gibbons-Hawking radiation fluid in the static SdS spacetime (already done in the literature):

- it is a test fluid (neglect backreaction);
- SdS is static between the horizons;
- heat flux $\neq 0$ → need to generalize TE criterion.

There is a chance for the Schwarzschild and de Sitter horizons to be in thermal equilibrium.
Analogy: stationary heat conduction in a rod with ends in contact with thermostats. Heat flux density $q \neq 0$ but time-independent.

Generalized Tolman-Ehrenfest criterion

The stress-energy tensor of a dissipative relativistic fluid is

$$T_{ab} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a u_b + q_b u_a$$

where $u^a =$ 4-velocity, $h_{ab} \equiv g_{ab} + u_a u_b$, $\rho =$ energy density, $P =$ isotropic pressure, $\pi_{ab} =$ anisotropic (trace-free) stresses, $q^a =$ heat flux density.

Project the eq. of motion $\nabla^b T_{ab} = 0$ onto time and 3-space, assume steady flow $\dot{q}_a = 0$:

$$\cancel{\dot{\rho}} + (P+\rho)\cancel{\Theta} + \cancel{\pi_{ab}}\,\sigma^{ab} + \nabla_a q^a + q_a \dot{u}^a = 0$$

$$(P+\rho)\,\dot{u}_a + h_a{}^c\left(\cancel{\dot{q}_c} + \nabla_c P + \nabla^b \cancel{\pi_{bc}}\right) + \left(\cancel{\omega_a}{}^b + \cancel{\sigma_a}{}^b\right) q_b + \frac{4\cancel{\Theta}}{3}\, q_a = 0$$

Use the Buchdahl relation (1949) valid in static spacetimes

$$\dot{u}_a = \nabla_a \ln\left(\sqrt{-g_{00}}\right)$$

and

$$\nabla_a q^a = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\, q^a\right) = -q^a \underbrace{\nabla_a \ln\sqrt{-g_{00}}}_{\dot{u}_a} = -\frac{q^j \nabla_j P}{P+\rho}$$

For a static, spherical geometry $q^a = (0, q^r, 0, 0)$ and $P = w\rho$,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 q^r\right) = \frac{1}{P+\rho}\frac{dP}{dr} = \frac{w}{(w+1)}\frac{d\rho}{dr}$$

Integrate

$$q^r(r) = \frac{C}{\sqrt{-g/\sin^2\vartheta}}\,\rho^{\frac{w}{w+1}}$$

then

$$\nabla_a(\ln\sqrt{-g_{00}}) + \frac{w}{w+1}\,\frac{d\rho}{dr} = 0\,,$$

which integrates to

$$\rho(r) = \rho_0\left(\frac{1}{\sqrt{-g_{00}}}\right)^{\frac{w+1}{w}}$$

$$q^r(r) = \frac{q_0}{\sqrt{g_{00}/\sin^2\vartheta}}$$

In particular, for the SdS geometry

$$q^r(r) = \frac{q_0}{r^2\sqrt{-g_{00}}} = \frac{q_0}{r^2\sqrt{-2m/r - \Lambda r^2/3}}$$

Impose Eckart's constitutive relation (1940)

$$q_a = -\mathcal{K} h_{ab} \left(\nabla^b \mathcal{T} + \mathcal{T} \dot{u}^b \right)$$

→

$$q^r = \mathcal{K} g_{00} \left[\frac{d\mathcal{T}}{dr} + \mathcal{T} \frac{d}{dr} \left(\ln \sqrt{-g_{00}} \right) \right]$$

and finally, assuming $\mathcal{K} = \text{const.}$,

$$\mathcal{T} \sqrt{-g_{00}} = \mathcal{T}_0 - \frac{q_0}{\mathcal{K}} \int \frac{dr}{r^2 \left(1 - 2m/r - \Lambda r^2/3 \right)}$$

(reduces to the standard TE criterion for $q_0 = 0$).

Compute integral explicitly →

$$\begin{aligned}\mathcal{T}(r) &= \frac{1}{\sqrt{1-2m/r-\Lambda r^2/3}}\Bigg[\mathcal{T}_0 \\ &\quad + \frac{q_0}{2m\mathcal{K}}\left(\ln x - \sum_i \frac{\left(\Lambda R_i^2-3\right)\ln\left(x-x_i\right)}{3\left(\Lambda R_i^2-1\right)}\right)\Bigg]\end{aligned}$$

where

- $x \equiv r/r_0$
- R_i are the roots of $f(r) \equiv 1-2m/r-\Lambda r^2/3 = 0$
- $x_i \equiv R_i/r_0$

$\mathcal{T}(r)$ interpolates between Hawking and Gibbons-Hawking temperatures at the horizons, generates a static radial heat flux, and the analog of stationary heat conduction (rod with ends at thermostats) is established.

SPECIAL CASES:

- **Schwarzschild:**

$$\mathcal{T}(r) = \frac{1}{\sqrt{1-2m/r}} \left[\mathcal{T}_0 - \frac{q_0}{2m\mathcal{K}} \ln\left(1 - \frac{2m}{r}\right) \right]$$

  reduces to $\mathcal{T}_H$ as $r \to +\infty$ if $\mathcal{T}_0 = 1/(8\pi m)$ (diverges on the horizon).

- **de Sitter:** one must set $q_0 = 0$ because the inward-directed radiation travels to $r = 0$ and beyond, exiting the horizon → no net flux:

$$\mathcal{T}(r) = \frac{\mathcal{T}_0}{\sqrt{1-H^2r^2}}$$

  (diverges on the de Sitter horizon).

TEMPERATURE REGULARIZATION AT THE HORIZONS

Divergence of $\mathcal{T}$ well-known in horizon thermodynamics (Birrell & Davies 1984; Wald 199; Giddings 2016); related to the vacuum state of the quantum field used to compute $\mathcal{T}$ @ horizons (e.g., the Boulware vacuum is singular on the S. horizon).

Hawking temperature $\mathcal{T}_H = \frac{\kappa}{2\pi}$ seen by a static observer at $r = +\infty$, where $\kappa =$ surface gravity.

To remain at a fixed radius, a detector must be accelerated with uniform acceleration a and, in the inertial vacuum state, it detects Unruh radiation at $\mathcal{T}_U = a/(2\pi)$.

$$\mathcal{T} = \frac{\kappa}{2\pi\sqrt{-K^c K_c}}\,, \qquad K^c = \text{Killing vector}$$

As the Schwarzschild horizon is approached, $r \to 2m$, one obtains the surface gravity

$$\kappa = \lim_{r\to 2m} \sqrt{-K^c K_c}$$

and

$$\mathcal{T} = \frac{a}{2\pi} \to \frac{\kappa}{2\pi\sqrt{-K^c K_c}} \, .$$

In coordinates adapted to the local symmetry, $K^c K_c = g_{00}$ and

$$\mathcal{T}\sqrt{-g_{00}} \simeq \frac{a}{2\pi}\sqrt{-g_{00}} \simeq \frac{\kappa}{2\pi}$$

for the Unruh observer, so

local Hawking temperature $=$ Tolman-Ehrenfest temperature

Applying the TE criterion to the radiation fluid as we did is not far-fetched, not new.

Second, the divergence of $\mathcal{T}$ at the horizon is illusory: there is now (some) consensus that the Hawking radiation dos not arise at the horizon, but in a *macroscopic* quantum atmosphere around it, with thickness $\gg$ Planck size (Giddings 16; Dey + 17, 19; Ong & Good 20; Nambu & Soda 22; Kaczmarek & Szcześniak 24).It extends to

$$r \simeq \frac{3\sqrt{3}}{4}(2m) \simeq 1.3r_S$$

Likewise, the wavelength of the typical Hawking quanta is not r_s but

$$\lambda \simeq 79r_S$$

(Giddings '16; Birrell & Davies '84)

Another way to regularize the temperature is by including the trace anomaly for the radiation (Eune + 2017, Eur. Phys. J. C 77, 244)

CONCLUSIONS

- For thermal equilibrium, temperature cannot be constant!
- Similarity SdS-stationary heat conduction in a rod with thermostats; heat flows radially from S to SdS horizon.
- $\mathcal{T}(r)$ and $q^a(r)$ computed
- So, thermal equilibrium is possible but:
  - Eckart's thermodynamics is non-causal
  - is subject to instabilities

  Should be revisited with Israel-Stewart's or other causal thermodynamics formalism.

THANK YOU