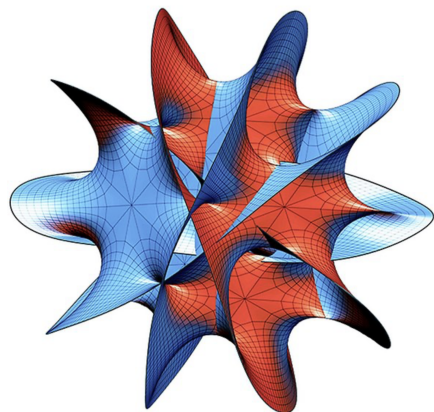
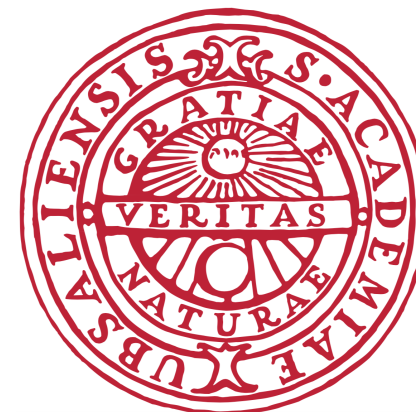




# Elliptics 2026

## Les Diablerets

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## Higher-genus polylogarithms: overview lecture

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Oliver Schlotterer (Uppsala University  
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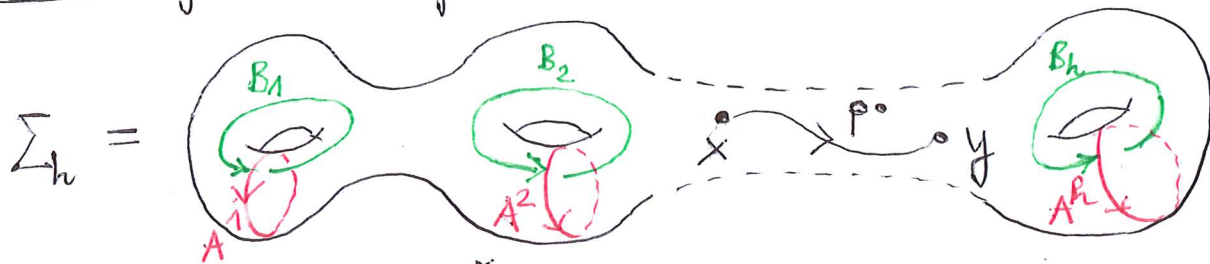


September 9<sup>th</sup> 2026



# Higher-genus polylogarithms

Goal: function space that closes under integration on



↳ iterated int's  $\int_y^x \bar{w}_1 \bar{w}_2 \dots \bar{w}_r$  of "suitable" kernels  $\bar{w}_j$

• genus  $h=0$ :  $\bar{w}_j \in \frac{dz}{z-p}$  ,  $p \in \mathbb{C}$

• genus  $h=1$ : merom'  $g^{(k)}(z-p, \tau)$  or single-valued  $f^{(k)}(z-p, \tau)$   
with  $k \in \mathbb{N}$  & modulus  $\tau \in \mathbb{H}$

$$\frac{\theta_1'(0, \tau) \theta_1(z + \eta, \tau)}{\theta_1(z, \tau) \theta_1(\eta, \tau)} \times \begin{cases} 1 \\ \exp(2\pi i \eta \frac{\text{Im } z}{\text{Im } \tau}) \end{cases} = \frac{1}{\eta} + \sum_{k=1}^{\infty} \eta^{k-1} \times \begin{cases} g^{(k)}(z-p, \tau) \\ f^{(k)}(z-p, \tau) \end{cases}$$

• higher  $h \geq 1$ ,  $g^{(k)} \rightarrow$  Enriques kernels  $g^{I_1 \dots I_k}_J(z, p)$   
 $f^{(k)} \rightarrow$  DHS kernels  $f^{I_1 \dots I_k}_J(z, p)$

indices  $I_1, \dots, I_k, J = 1, 2, \dots, h$

• advocate to study identities / closure under  $\int$  /  
moduli dependence via flat connections

$$dJ = J \wedge J$$

↑ initially  $d_z$ , later  $\sum_{i=1}^n dx_i$ , towards the end  
 $\sum_{i=1}^n dx_i + \sum_{\alpha=1}^{3h-3} \partial_{\tau_\alpha}$

# Outline:

I) Enriques kernels

II) DHS kernels

III) Polylogs

IV) Gauge equivalence

V) Single-valued polylogs

VI) Multi-variable connections

VII) Moduli-variations

non-sep  
deg. co-  
vered  
in Mi-  
chele's

I) Enriques kernels [1112.0864 & 2502.14769]

• start from 3rd kind diff. with poles  $\sim (\frac{1}{z-p} - \frac{1}{z-q})$  talk

built from prime form [e.g. Pokraka's talk]

$$\partial_z \log \frac{E(z,p)}{E(z,q)} = \partial_z \log \left( \frac{\theta[v](\int_p^z \omega)}{\theta[v](\int_q^z \omega)} \right)$$

any odd spin structure

Abelian diff's,  $\int_{A^I} \omega_j = \delta_j^I$  and  $\int_{B^I} \omega_j = B_{IJ}$  period matrix

• rk 2 kernel with pole  $\sim \delta_j^I / (z-p)$  from convolution

$$g_{IJ}^I(z,p) = \int_{A^I} dt \omega_j(t) \partial_z \log \frac{E(z,p)}{E(z,t)} - i\pi \delta_j^I \omega_j(z)$$

• higher-rk kernels from iterative convolution

$$g_{IK_1 \dots K_S}^I(z,p) = - \int_{A^I} dt g_{IJ}^L(z,t) g_{K_1 \dots K_S}^{K_1 \dots K_S}(t,p) \\ - \omega_j(z) (-2\pi i)^{S+1} \frac{B_{S+1}}{S!} \delta_{IJ}^{IK_1 \dots K_S} - \sum_{r=1}^S (-2\pi i)^r \frac{B_r}{r!} \delta_{IJ}^{K_1 \dots K_r} g_{IK_{r+1} \dots K_S}^{IK_{r+1} \dots K_S}(z,p)$$

• originally defined via zero' flat connection on  $\sum_h$

$$K_E(z,p) = dz \sum_{r=0}^{\infty} g_{I_1 \dots I_r}^{I_1 \dots I_r}(z,p) B_{I_1} \dots B_{I_r} a^J$$

in free Lie  $[a^J, b_I]$  with  $B_I X = [b_I, X]$  s. that

$$(i) \operatorname{Res}_{z=p} K_E(z, p) = \sum_{I=1}^h [b_I, a^I]$$

(ii) only  $B_j$ -monodromies

$$K_E(A^I \cdot z, p) = K_E(z, p) = e^{2\pi i B_j} K_E(B_j \cdot x, p)$$

notational confessions:

$$\bullet w_I(z) = g^{\otimes I}_I(z, p) = f^{\otimes I}_I(z, p)$$

$$\bullet Y_{Ij} = \operatorname{Im} \Omega_{Ij} \text{ \& inverse } Y^{Ij} = [(\operatorname{Im} \Omega)^{-1}]^{Ij}$$

used to raise / lower indices, e.g.  $w^I = Y^{Ij} w_j$

## II) DHS kernels [2306.08644]

$$\bullet \text{render } \partial_z \log \frac{E(z, p)}{E(z, q)} \text{ single-valued in } p, q \text{ via Green fct}$$

$$G(z, p) = -\log |E(z, p)|^2 + 2\pi i \left( \operatorname{Im} \int_p^z w_I \right) \left( \operatorname{Im} \int_p^z w^I \right)$$

$\bullet$  [rk 2 kernel with pole  $\sim \delta_j^I / (z-p)$  from convolution]

$$f^I_j(z, p) = \int_{\Sigma_h} d^2t \bar{w}^I(t) w_j(t) \partial_z (G(z, t) - G(z, p))$$

$\bullet$  [higher-rk kernels from iterative convolution]

$$f^{IK_1 \dots K_S}_j(z, p) = - \int_{\Sigma_h} d^2t f^L_j(z, t) \bar{w}^I(t) f^{K_1 \dots K_S}_L(t, p)$$

$\bullet$  [originally defined via  $\underline{\text{sv}}$  flat connection on  $\Sigma_h$ ]

$$J_{\text{DHS}}(z, p) = dz \sum_{r=0}^{\infty} f^{I_1 \dots I_r}_j(z, p) B_{I_1} \dots B_{I_r} a^J - \pi d\bar{z} b_j \bar{w}^J(z)$$

$$\text{still: } \operatorname{Res}_{z=p} J_{\text{DHS}}(z, p) = [b_I, a^I]$$

compensating  
 $\partial_{\bar{z}} f^{I_1 \dots I_r}_j(z, p)$

### III ) Polylogs

- both connections  $J \in \{K_E, J_{DHS}\}$  yield homotopy-invariant

$$\prod_{E \text{ or } DHS} (x, y | p) = \text{Perp} \left( \int_y^x dz J(z, p) \right) = \sum_{W \in \{a^1, \dots, a^h, b^1, \dots, b_h\}^X} W \prod_{E \text{ or } DHS} (W | x, y | p)$$

polylogs on  $\Sigma_h \setminus \{p\}$

- $\Gamma_E(\dots)$  expressible via

$$\tilde{\Gamma} \left( \begin{matrix} \vec{I}_1 & \vec{I}_2 & \dots & \vec{I}_r \\ J_1 & J_2 & \dots & J_r \\ p_1 & p_2 & \dots & p_r \end{matrix} \middle| x, y \right) = \int_y^x dz g_{J_1}^{\vec{I}_1}(z, p_1) \tilde{\Gamma} \left( \begin{matrix} \vec{I}_2 & \dots & \vec{I}_r \\ J_2 & \dots & J_r \\ p_2 & \dots & p_r \end{matrix} \middle| z, y \right)$$

- numerical evaluation using Schottky-rep of  $\tilde{\Gamma}$

[2406.10051]; also : Crowdy-Marshall method

[Andrzej's talk & 2509.01598]

- $\prod_{DHS}$  is modular invariant upon prescribing  $Sp(2h, \mathbb{Z})$

$$\exists \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ action } \begin{cases} a^J \rightarrow (C\Omega + D)^J{}_K a^K \\ b_I \rightarrow [C\Omega + D]^{-1}{}_I^K b_K \end{cases}$$

## I. Enriquez kernels

Introduced meromorphic but multi-valued  $g^{I_1 \cdots I_r}_J(z, p)$  on blackboard

[Enriquez 1112.0864; D'Hoker, OS 2502.14769]

## II. DHS kernels

Introduced single-valued but non-mero'  $f^{I_1 \cdots I_r}_J(z, p)$  on blackboard

[D'Hoker, Hidding, OS 2306.08644]

## III. Polylogs

Constructed polylogs from path-ordered exponentials of

Enriquez connection  $\mathcal{K}_E$  & DHS connection  $\mathcal{J}_{\text{DHS}}$

## IV. Gauge equivalence DHS vs. Enriquez

Connections  $\mathcal{K}_E$  &  $\mathcal{J}_{\text{DHS}}$  related by gauge transformation  $\mathcal{U}$  together with Lie-algebra automorphism  $(a^J, b_I) \rightarrow (\hat{a}^J, \hat{b}_I) = \text{Lie series in } (a^K, b_L)$

$$\left( d - \mathcal{K}_E(x, y; a, b) \right) = \mathcal{U}(x, p)^{-1} \left( d - \mathcal{J}_{\text{DHS}}(x, y; \hat{a}, \hat{b}) \right) \mathcal{U}(x, p)$$

[D'Hoker, Enriquez, OS, Zerbini 2501.07640]

## IV. Gauge equivalence DHS vs. Enriquez

Connections  $\mathcal{K}_E$  &  $\mathcal{J}_{\text{DHS}}$  related by gauge transformation  $\mathcal{U}$  together with

Lie-algebra automorphism  $(a^J, b_I) \rightarrow (\hat{a}^J, \hat{b}_I) = \text{Lie series in } (a^K, b_L)$

$$\left( d - \mathcal{K}_E(x, y; a, b) \right) = \underbrace{\mathcal{U}(x, p)^{-1}}_{\substack{\text{series in } \mathcal{J}_{\text{DHS}} \\ \text{polylogs and } b_I}} \left( d - \underbrace{\mathcal{J}_{\text{DHS}}(x, y; \hat{a}, \hat{b})}_{\substack{\text{series in } \mathcal{J}_{\text{DHS}} \text{ polylogs} \\ \& \text{ ad}_{b_K} \text{ acting on } a^J, b_I}} \right) \mathcal{U}(x, p)$$

[D'Hoker, Enriquez, OS, Zerbini 2501.07640]

- expansion of  $\mathcal{U}(x, p)$ ,  $\hat{a}^J$ ,  $\hat{b}_I$  computable order by order in  $b_I$  from monodromies / poles of  $\mathcal{K}_E$  &  $\mathcal{J}_{\text{DHS}}$  (no explicit closed formula yet)
- at genus  $h = 1$ , simplify to  $\mathcal{U}(x, p) \rightarrow e^{2\pi i b \frac{\text{Im}(x-p)}{\text{Im}(\tau)}}$  &  $(\hat{a}, \hat{b}) \rightarrow (a + \frac{\pi b}{\text{Im} \tau}, b)$
- corollary: Enriquez / DHS polylog series at any  $h \geq 1$  related by

$$\mathbb{F}_E(x, y; p; a, b) = \mathcal{U}(x, p)^{-1} \mathbb{F}_{\text{DHS}}(x, y; p; \hat{a}, \hat{b}) \mathcal{U}(y, p)$$

## IV. Gauge equivalence DHS vs. Enriquez

Given that the polylog series obey

$$\mathbb{F}_{\text{DHS}}(x, y; p; \hat{a}, \hat{b}) = \mathcal{U}(x, p) \mathbb{F}_{\text{E}}(x, y; p; a, b) \mathcal{U}(y, p)^{-1}$$

the associated spaces  $\mathcal{H}(\bullet)$  of polylogs are related by

$$\mathcal{H}(\mathcal{J}_{\text{DHS}}) = \mathcal{H}(b_I \bar{\omega}^I) \cdot \mathcal{H}(\mathcal{K}_{\text{E}}) \longleftarrow \begin{array}{l} \text{antiholo' iterated} \\ \text{int's of } \bar{\omega}^{I_1} \dots \bar{\omega}^{I_r} \end{array} \left. \begin{array}{l} \text{meromorphic \& in fact smallest fct.} \\ \text{space that closes under integration} \end{array} \right\} \begin{array}{l} \text{[D'Hoker, Enriquez, OS, Zerbini 2501.07640]} \end{array}$$

... where  $\mathbb{C} \ni$  coeff's of  $\int_y^x \bar{\omega}^{I_1} \dots \bar{\omega}^{I_r}$  may depend on  $p$  & moduli of  $\Sigma_h$

$\exists$  numerous other ways of spanning same polylog space, e.g. using kernels with higher poles which are both single-valued and meromorphic

[Enriquez, Zerbini 2110.09341, 2212.03119]

## V. Single-valued polylogarithms at arbitrary genus

- also for single-valued kernels  $f^{\vec{I}}_J$ , polylogs of  $\mathbb{F}_{\text{DHS}}$  exhibit ( $p$ - and moduli-dependent) **monodromies** under  $\int^x \rightarrow \int^{\mathfrak{A}^I \cdot x}$  or  $\int^{\mathfrak{B}^I \cdot x}$  or  $\odot_p$
- lesson from single-valued polylogs at genus 0 [**Brown 2004**] and genus 1:  
**[Baune, Broedel, Moeckli 2511.15240; OS, Sohnle, Tao 2511.15883]**
  - \* compose polylog series  $\mathbb{F}_{\bullet}(x; e_i)^{-1}$  with cplx. conjugate  $\overline{\mathbb{F}_{\bullet}(x; e_i)}$
  - \* need **Lie-algebra automorphism**  $e_i \rightarrow \sigma(e_i)$  “change of alphabet”

## V. Single-valued polylogarithms at arbitrary genus

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- single-valued polylogs  $\forall h \geq 1$  via **[Baune, Broedel, Moeckli 2606.17911]**

$$\overline{\mathbb{F}_{\text{DHS}}(z, p; \sigma(a), \sigma(b))} \mathbb{F}_{\text{DHS}}(z, p; a, b)^{-1}$$

getting Lie series  $\sigma(a^J) = a^J + \dots$  and  $\sigma(b_I) = -b_I - Y_{IJ}a^J/\pi + \dots$

order by order in  $a^J, b_I$  by imposing  $\mathfrak{A}^I, \mathfrak{B}_J$  **monodromy cancellation**.

## VI. Multi-variable flat connections

So far: polylogs on punctured surfaced  $\Sigma_h \setminus \{p\}$ ,

—→ very hard to check closure under  $\int dp$ .

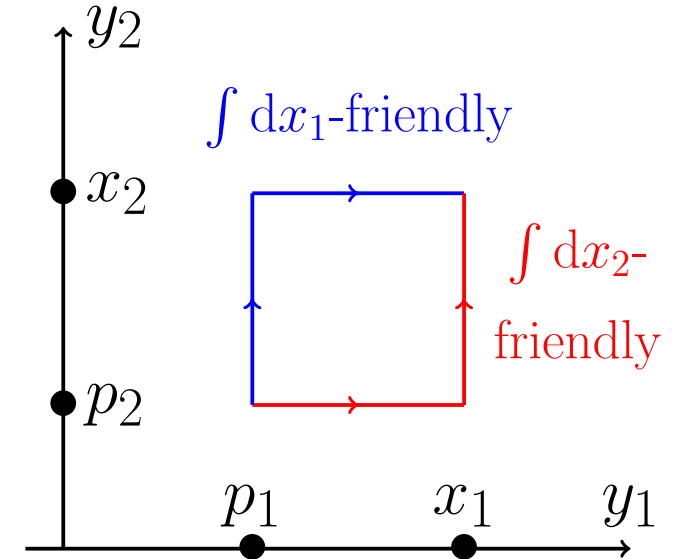
- to be handled by multi-variable connections – either  $\mathcal{J}_{\text{DHS}}(\vec{x})$  or  $\mathcal{K}_{\text{E}}(\vec{x})$   
on config. spaces  $\text{Cf}_n(\Sigma_h)$  or  $\text{Cf}_n(\tilde{\Sigma}_h)$  of  $\vec{x} = (x_1, \dots, x_{n \geq 2})$  @  $x_i \neq x_j$
- both take values in Lie algebra  $\mathfrak{t}_{h,n}$  [Bezrukavnikov 1994; see Axel's talk] with  
generators  $a_i^J, b_{iJ}$  @  $i = 1, \dots, n$  and  $t_{ij} = t_{ji}$  @  $1 \leq i < j \leq n$   
subject to relations (for all pairwise distinct  $i, j, k = 1, 2, \dots, n$ )

$$\begin{aligned} [a_i^I, a_j^J] &= [b_{iI}, b_{jJ}] = 0 & [b_{iI}, a_j^J] &= \delta_I^J t_{ij} \\ [a_i^I, t_{jk}] &= [b_{iI}, t_{jk}] = 0 & [b_{iI}, a_i^I] &= - \sum_{j \neq i} t_{ij} \end{aligned}$$

## VI. Multi-variable flat connections

### Key motivation:

by flatness of  $\mathcal{K}_E(\vec{x})$ ,  $\mathcal{J}_{\text{DHS}}(\vec{x})$ , can change  
 fibration bases for polylogs in their Pexp,  
 tailored to integrate over any  $x_{j=1,\dots,n}$



- components  $K_i(\vec{x})$ ,  $J_i(\vec{x})$  below are  $(1, 0)$ -forms in  $x_i$ , scalars in  $x_{j \neq i}$

$$\mathcal{K}_E(\vec{x}) = \sum_{i=1}^n dx_i K_i(\vec{x}) \quad \text{mero' with } K_i(\mathfrak{B}_{kJ} \cdot \vec{x}) = e^{-2\pi i B_{kJ}} K_i(\vec{x})$$

[Enriquez 1112.0864]

$$\mathcal{J}_{\text{DHS}}(\vec{x}) = \sum_{i=1}^n \left\{ dx_i J_i(\vec{x}) - \pi d\bar{x}_i b_{iJ} \bar{\omega}^J(x_i) \right\} \quad \text{single-valued in } \vec{x}$$

[D'Hoker, OS 2602.01461]

- earlier  $g^{\vec{I}}_J(z, p)$ ,  $f^{\vec{I}}_J(z, p)$  kernels suffice to express  $K_i(\vec{x})$ ,  $J_i(\vec{x})$ !

## VI. 1 Explicit form of components

- components  $K_i(\vec{x})$ ,  $J_i(\vec{x})$  below are  $(1, 0)$ -forms in  $x_i$ , scalars in  $x_{j \neq i}$

$$\mathcal{K}_E(\vec{x}) = \sum_{i=1}^n dx_i \textcolor{teal}{K}_i(\vec{x}) \quad \text{mero' with } K_i(\mathfrak{B}_{kJ} \cdot \vec{x}) = e^{-2\pi i B_{kJ}} K_i(\vec{x})$$

$$\mathcal{J}_{\text{DHS}}(\vec{x}) = \sum_{i=1}^n \left\{ dx_i \textcolor{teal}{J}_i(\vec{x}) - \pi d\bar{x}_i b_{iJ} \bar{\omega}^J(x_i) \right\} \quad \text{single-valued in } \vec{x}$$

- decompose kernels into  $p$ -indep. tracless parts /  $p$ -dep. trace

$$f^{\vec{I}K}{}_J(x, p) = \partial_x \textcolor{blue}{\Phi}^{\vec{I}K}{}_J(x) - \textcolor{red}{\delta}_J^K \partial_x \textcolor{red}{\mathcal{G}}^{\vec{I}}(x, p)$$

- with  $\omega_J(x) := \partial_x \Phi^\emptyset_J(x)$  & shorthand  $B_{i\vec{I}} = B_{iI_1} \cdots B_{iI_r}$  for  $\vec{I} = I_1 \cdots I_r$ ,

$$J_i(\vec{x}) = \sum_{\vec{I}} B_{i\vec{I}} \left\{ \partial_{x_i} \textcolor{blue}{\Phi}^{\vec{I}}{}_J(x_i) a_i^J + \sum_{k \neq i} \textcolor{red}{\partial}_{x_i} \textcolor{red}{\mathcal{G}}^{\vec{I}}(x_i, x_k) t_{ik} \right\}$$

and similarly for  $K_i(\vec{x})$  upon  $f^{\vec{I}}{}_J(x, p) \rightarrow g^{\vec{I}}{}_J(x, p)$  in  $J_i(\vec{x})$

## VI. 2 Fay / interchange identities from flatness

Flatness conditions on  $\mathcal{J}_{\text{DHS}}(\vec{x}) \ni$  vanishing of  $[J_i(\vec{x}), J_k(\vec{x})] \forall i \neq k$

$\implies$  all bilinear identities among  $f^{\vec{I}}_J(x, p)$  [D'Hoker, OS 2602.09108]

- interchange identities among 2 points  $x, y \in \Sigma_h$  including

$$\omega_I(x) f^I_J(y, x) + \omega_I(y) f^I_J(x, y) = 0$$

[D'Hoker, Hidding, OS 2306.08644; D'Hoker, OS 2407.11476]

- Fay identities among 3 points  $x, y, z \in \Sigma_h$  including

$$\begin{aligned} f^I_J(y, z) f^J_K(z, x) &= f^I_J(y, x) f^J_K(z, x) - f^I_J(z, y) f^J_K(y, x) \\ &\quad - \omega_J(y) f^{IJ}_K(z, y) - \omega_J(y) f^{JI}_K(z, x) - \omega_J(z) f^{JI}_K(y, x) \end{aligned}$$

[D'Hoker, OS 2407.11476]

Same identities with  $f^{\vec{I}}_J(x, p) \rightarrow g^{\vec{I}}_J(x, p)$  from  $[K_i(\vec{x}), K_j(\vec{x})] = 0$ .

[Baune, Broedel, Im, Lisitsyn, Moeckli 2409.08208]

[D'Hoker, OS 2602.09108]

## VII. Moduli variations

Motivation: elliptic polylogs at rational pts  $r_i \in \mathbb{Q} + \tau\mathbb{Q} \forall i = 0, 1, \dots, s$ ,

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_s \\ r_1 & \cdots & r_s \end{smallmatrix} \middle| r_0, \tau\right) \longrightarrow \text{iterated } \int d\tau \text{ of modular forms}$$

- if  $r_i \in \frac{\mathbb{Z} + \tau\mathbb{Z}}{N}$ , modular forms of congruence subgroup  $\Gamma(N)$  of  $\mathrm{SL}_2(\mathbb{Z})$   
[Broedel, Duhr, Dulat, Penante, Tancredi 1803.10256]
- if  $r_0 = 1$  and  $r_i \in \mathbb{Z}$ , get iterated Eisenstein integrals of  $\mathrm{SL}_2(\mathbb{Z})$   
[Enriquez 1301.3042; Broedel, Matthes, OS 1507.02254]

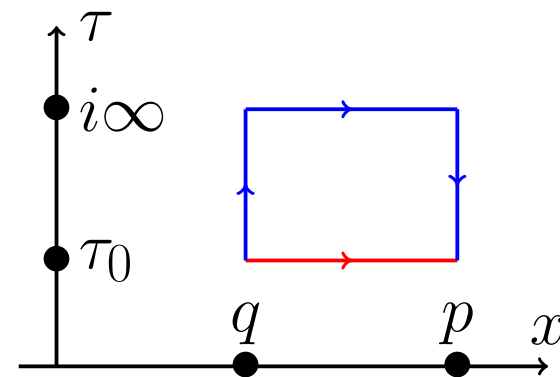
For  $\infty$  family of iterated integrals from algebra-valued connection  $K(x, \tau)$ ,

$$\mathrm{Pexp}\left(\int_q^p dx K(x, \tau)\right) = 1 + \int_q^p dx K(x, \tau) + \int_q^p dx_1 K(x_1, \tau) \int_q^{x_1} dx_2 K(x_2, \tau) + \cdots$$

need flat connection  $\mathcal{J} = dx K(x, \tau) + d\tau L(x, \tau)$

$$d_{x,\tau} \mathcal{J} = \mathcal{J} \wedge \mathcal{J} \implies \text{path indep. } \mathrm{Pexp} \int \mathcal{J}$$

$dx K(x, \tau_0)$  integrals from those of  $d\tau L(\bullet, \tau)$



## VII. 1 Genus $h = 1$ : Calaque-Enriquez-Etingof connection

In elliptic polylog case, use flat  $\mathcal{J}_{\text{CEE}} \in (\text{free Lie}[a, b] \rtimes \text{derivations on Lie}[a, b])$   
 [Calaque, Enriquez, Etingof math/0703237]

$$\mathcal{J}_{\text{CEE}}(x, \tau) = a \, dx + \sum_{k=1}^{\infty} \text{ad}_b^k(a) \left( dx \, g^{(k)}(x, \tau) + \frac{k \, d\tau}{2\pi i} g^{(k+1)}(x, \tau) \right) + \frac{d\tau}{2\pi i} L_0(\tau)$$

... here specialized to 2 pts  $x_1 = x$  and  $x_2 = 0$  (otherwise:  $\mathcal{J}_{\text{CEE}} \in \mathfrak{t}_{1,n}$ )

## VII. 1 Genus $h = 1$ : Calaque-Enriquez-Etingof connection

In elliptic polylog case, use flat  $\mathcal{J}_{\text{CEE}} \in (\text{free Lie}[a, b] \rtimes \text{derivations on Lie}[a, b])$   
[Calaque, Enriquez, Etingof math/0703237]

$$\mathcal{J}_{\text{CEE}}(x, \tau) = \textcolor{red}{a} \, dx + \sum_{k=1}^{\infty} \textcolor{red}{ad}_b^k(\textcolor{red}{a}) \left( dx \, g^{(k)}(x, \tau) + \frac{k \, d\tau}{2\pi i} g^{(k+1)}(x, \tau) \right) + \frac{d\tau}{2\pi i} L_0(\tau)$$

Easily check to be closed,  $d_{x, \tau} \mathcal{J}_{\text{CEE}}(x, \tau) = 0$ , but leftover flatness condi-

tion  $\mathcal{J}_{\text{CEE}} \wedge \mathcal{J}_{\text{CEE}} = 0$  requires  $\textcolor{blue}{der Lie}[a, b]$ -valued,  $x$ -independent  $L_0(\tau)$ ,

$$L_0(\tau) = \epsilon_0 + \sum_{k=2}^{\infty} (k-1) G_k(\tau) \delta_k$$

with holo' Eisenstein  $G_k(\tau)$  and  $\textcolor{blue}{derivations} \, \epsilon_0, \delta_k$  determined by

$$\epsilon_0(a) = 0, \quad \delta_k(a) = \sum_{p=1}^{k/2-1} (-1)^p [\textcolor{blue}{ad}_b^p(a), \textcolor{blue}{ad}_b^{k-p-1}(a)]$$

$$\epsilon_0(b) = \textcolor{blue}{a}, \quad \delta_k(b) = 0$$

## VII. 2 Genus $h = 1$ : Tsunogai derivations

For the  $d\tau$  component of  $\mathcal{J}_{\text{CEE}}(x, \tau) = dx K(x, \tau) + d\tau L(x, \tau)$ ,

$$L(x, \tau) = \frac{1}{2\pi i} \left\{ \sum_{k=1}^{\infty} \text{ad}_b^k(a) k g^{(k+1)}(x, \tau) + L_0(\tau) \right\}$$

evaluation at  $x = 0$  or  $1$  where  $g^{(\ell \geq 2)}(x, \tau) \rightarrow -G_\ell(\tau)$

combines  $\delta_k \in \mathfrak{der} \text{Lie}[a, b]$  with inner derivation  $-[\text{ad}_b^{k-1}(a), \bullet]$

$$L(x = 0 \text{ or } 1, \tau) = \epsilon_0 + \sum_{k=2}^{\infty} (k-1) G_k(\tau) \epsilon_k$$

Introduces famous Tsunogai derivations  $\epsilon_k(X) = \delta_k(X) - [\text{ad}_b^{k-1}(a), X]$

$$\epsilon_0(a) = 0, \quad \epsilon_k(a) = \sum_{p=0}^{k/2-1} (-1)^p [\text{ad}_b^p(a), \text{ad}_b^{k-p-1}(a)]$$

$$\epsilon_0(b) = a, \quad \epsilon_k(b) = \text{ad}_b^k(a)$$

[Tsunogai 1995;  
MANY others]

## VII. 3 Higher genus $h \geq 2$ : extending DHS connection

Now extend  $n$ -var.  $\mathcal{J}_{\text{DHS}}(\vec{x})$  by holo' moduli components  $L_\alpha(\vec{x}) \in \mathfrak{der} \mathfrak{t}_{h,n}$

$$\mathcal{J}(\vec{x}) = \mathcal{J}_{\text{DHS}}(\vec{x}) + \sum_{\alpha=1}^{3h-3} d\tau_\alpha L_\alpha(\vec{x})$$

Flat only in absence of antiholo'  $d\bar{\tau}_\alpha$ , i.e. only w.r.t.  $dx + \sum_{\alpha=1}^{3h-3} d\tau_\alpha \partial_{\tau_\alpha}!$

[D'Hoker, Enriquez, OS, Zerbini 2608.29169]

Map  $(1,0)$ -forms  $L_\alpha$  in moduli  $\tau_\alpha$  to  $(2,0)$ -forms  $L_{ww}$  in point  $w \in \Sigma_h$

via integration against Beltrami  $(-1,1)$ -form differentials  $\mu_\alpha(w)$ ,

$$L_\alpha(\vec{x}) = \frac{1}{\pi} \int_{\Sigma_h} d^2w \mu_\alpha(w) L_{ww}(\vec{x}), \quad \alpha = 1, \dots, 3h-3$$

Implements variation of complex structure of  $\Sigma_h$  (preserving its square  $-1$ )

which maps the metric on  $\Sigma_h$  from  $\sim |dz|^2$  to  $\sim |dz - \mu(z)d\bar{z}|^2$

[Verlinde, Verlinde '87; D'Hoker, Phong '88]

## VII. 4 Variational derivatives

Study flatness conditions for  $(2, 0)$ -form  $L_{ww}(\vec{x})$  pointwise in  $w \in \Sigma_h$ ,

$$\mathcal{J}(\vec{x}) = \mathcal{J}_{\text{DHS}}(\vec{x}) + \frac{1}{\pi} \sum_{\alpha=1}^{3h-3} d\tau_\alpha \int_{\Sigma_h} d^2w \mu_\alpha(w) L_{ww}(\vec{x})$$

Assuming  $L_{ww}$  to have  $a$ -degree one just like  $(1, 0)$ -form part of  $\mathcal{J}_{\text{DHS}}(\vec{x})$

$$\text{flatness of } \mathcal{J} \quad \Rightarrow \quad \partial_x L_{ww}(\vec{x}) = \delta_{ww} \mathcal{J}_{\text{DHS}}^{(1,0)}(\vec{x})$$

On RHS: variational derivative  $\delta_{ww} \cong (2, 0)_w$ -form counterpart of  $\partial_{\tau_\alpha}$ :

$$\partial_{\tau_\alpha} F(\vec{x}, \vec{\tau}) = \frac{1}{\pi} \int_{\Sigma_h} d^2w \mu_\alpha(w) \delta_{ww} F(\vec{x}, \vec{\tau})$$

If  $F(\vec{x}, \vec{\tau})$  is CFT correlator on  $\Sigma_h$ , then  $\delta_{ww} \cong$  inserting stress-tensor  $T(w)$

$$F(\vec{x}, \vec{\tau}) = \left\langle \prod_{j=1}^n \phi(x_j) \right\rangle_{\Sigma_h} \quad \Rightarrow \quad \delta_{ww} F(\vec{x}, \vec{\tau}) = \left\langle T(w) \prod_{j=1}^n \phi(x_j) \right\rangle_{\Sigma_h}$$

## VII. 4 Variational derivatives

Study flatness conditions for  $(2, 0)$ -form  $L_{ww}(\vec{x})$  pointwise in  $w \in \Sigma_h$ ,

$$\mathcal{J}(\vec{x}) = \mathcal{J}_{\text{DHS}}(\vec{x}) + \frac{1}{\pi} \sum_{\alpha=1}^{3h-3} d\tau_{\alpha} \int_{\Sigma_h} d^2w \mu_{\alpha}(w) L_{ww}(\vec{x})$$

Assuming  $L_{ww}$  to have  $a$ -degree one just like  $(1, 0)$ -form part of  $\mathcal{J}_{\text{DHS}}(\vec{x})$

$$\text{flatness of } \mathcal{J} \Rightarrow \partial_x L_{ww}(\vec{x}) = \delta_{ww} \mathcal{J}_{\text{DHS}}^{(1,0)}(\vec{x})$$

Evaluate RHS via variational der's  $\delta_{ww} \omega^I(x) = -\omega^I(w) \partial_w \partial_x \mathcal{G}(w, x)$  and

$$\delta_{ww} f_{I_1 \dots I_r}^J(x, p) = \sum_{k=0}^r \partial_w f_{I_1 \dots I_k} L^J(x, w) f_{I_{k+1} \dots I_r}^L(w, p)$$

while considering ...

$$\dots \quad a_{iJ} = Y_{JK} a_i^K, \quad b_i^J = Y^{JK} b_{iK} \quad \text{as } \tau_{\alpha}\text{-independent}$$

## VII. 4 Variational derivatives

Study flatness conditions for  $(2, 0)$ -form  $L_{ww}(\vec{x})$  pointwise in  $w \in \Sigma_h$ ,

$$\mathcal{J}(\vec{x}) = \mathcal{J}_{\text{DHS}}(\vec{x}) + \frac{1}{\pi} \sum_{\alpha=1}^{3h-3} d\tau_{\alpha} \int_{\Sigma_h} d^2w \mu_{\alpha}(w) L_{ww}(\vec{x})$$

Assuming  $L_{ww}$  to have  $a$ -degree one just like  $(1, 0)$ -form part of  $\mathcal{J}_{\text{DHS}}(\vec{x})$

$$\text{flatness of } \mathcal{J} \Rightarrow \partial_x L_{ww}(\vec{x}) = \delta_{ww} \mathcal{J}_{\text{DHS}}^{(1,0)}(\vec{x})$$

Determines desired  $d\tau_{\alpha}$  extension up to  $\vec{x}$ -indep.,  $\mathfrak{der} \mathfrak{t}_{h,n}$ -valued  $\tilde{L}_{ww}^0$

$$\begin{aligned} L_{ww}(\vec{x}) = \tilde{L}_{ww}^0 - \sum_{\vec{I}, \vec{J}} \sum_{k=1}^n \left\{ \partial_w \mathcal{G}_{\vec{I}}(w, x_k) \partial_w \Phi_{\vec{J}}^M(w) B_k^{\theta(\vec{I})} B_k^{\vec{J}} a_{kM} \right. \\ \left. + \sum_{\ell > k}^n \partial_w \mathcal{G}_{\vec{I}}(w, x_k) \partial_w \mathcal{G}_{\vec{J}}(w, x_{\ell}) B_k^{\theta(\vec{I})} B_{\ell}^{\theta(\vec{J})} t_{k\ell} \right\} \end{aligned}$$

with antipode  $B_k^{\theta(I_1 I_2 \dots I_r)} = (-1)^r B_k^{I_r} \dots B_k^{I_2} B_k^{I_1}$  & all  $w, \vec{x}$  in DHS kernels.

## VII. 5 Derivations from flatness conditions

Besides  $\partial_x L_{ww}(\vec{x}) = \delta_{ww} \mathcal{J}_{\text{DHS}}^{(1,0)}(\vec{x})$ , flatness conditions include

$$[\mathcal{J}_{\text{DHS}}^{(1,0)}(\vec{x}), L_{ww}(\vec{x})] = 0, \quad \partial_{\bar{x}_i} L_{ww}(\vec{x}) + \pi \bar{\omega}_J(x_i) [b_i^J, L_{ww}(\vec{x})] = 0$$

Fixes action of  $\vec{x}$ -independent part  $\tilde{L}_{ww}^0$  of  $L_{ww}(\vec{x})$  on  $\mathfrak{t}_{h,n}$  generators

$$[L_{ww}^0, b_i^I] = \omega^I(w) \omega^K(w) a_{iK} + \sum_{\vec{P} \neq \emptyset} \delta_{ww} \mathcal{A}_{\vec{P}}^I{}^K B_i^{\vec{P}} a_{iK}$$

$$[L_{ww}^0, a_{iI}] = \frac{1}{2} \sum_{\vec{P}, \vec{Q}} \left\{ \delta_{ww} \mathcal{A}_{\vec{P}I\vec{Q}}^M{}^N [B_i^{\theta(\vec{P})} a_{iM}, B_i^{\vec{Q}} a_{iN}] \right. \\ \left. - \delta_{ww} \mathcal{D}_{\vec{P}I\vec{Q}} \sum_{k \neq i} [B_i^{\theta(\vec{P})} t_{ik}, B_i^{\vec{Q}} t_{ik}] \right\}$$

... after  $\vec{x}$ -indep. redefinition to render  $L_{ww}^0$  holomorphic in  $w$ ,

$$L_{ww}^0 = \tilde{L}_{ww}^0 - \omega^J(w) \sum_{k=1}^n \left\{ \frac{1}{h-1} \partial_w \Phi_J^M(w) a_{kM} + \frac{1}{h} \sum_{\vec{I} \neq \emptyset} \partial_w \Phi_{J\vec{I}}^M(w) B_k^{\vec{I}} a_{kM} \right\}$$

## VII. 5 Derivations from flatness conditions

Action of  $\vec{x}$ -independent part  $L_{ww}^0$  on  $\mathfrak{t}_{h,n}$  generators

$$\begin{aligned}
[L_{ww}^0, b_i^I] &= \omega^I(w) \omega^K(w) a_{iK} + \sum_{\vec{P} \neq \emptyset} \delta_{ww} \mathcal{A}_{\vec{P}}^I{}^K B_i^{\vec{P}} a_{iK} \\
[L_{ww}^0, a_{iI}] &= \frac{1}{2} \sum_{\vec{P}, \vec{Q}} \left\{ \delta_{ww} \mathcal{A}_{\vec{P}I\vec{Q}}^M{}^N [B_i^{\theta(\vec{P})} a_{iM}, B_i^{\vec{Q}} a_{iN}] \right. \\
&\quad \left. - \delta_{ww} \mathcal{D}_{\vec{P}I\vec{Q}} \sum_{k \neq i} [B_i^{\theta(\vec{P})} t_{ik}, B_i^{\vec{Q}} t_{ik}] \right\}
\end{aligned}$$

where coefficients of the  $\mathfrak{t}_{h,n}$ -elements are  $\delta_{ww}$  variations of

$$\begin{aligned}
\mathcal{A}_{K\vec{P}L}^I{}^J &= \int_{\Sigma_h} d^2x \omega^I(x) \bar{\omega}_K(x) \int_{\Sigma_h} d^2y \mathcal{G}_{\vec{P}}(x, y) \bar{\omega}_L(y) \omega^J(y) \\
\mathcal{D}_{\vec{P}} &= \frac{1}{h} \int_{\Sigma_h} d^2x \omega^I(x) \bar{\omega}_I(x) \mathcal{G}_{\vec{P}}(x, x), \quad \vec{P} \neq \emptyset
\end{aligned}$$

## VII. 5 Derivations from flatness conditions

Action of  $\vec{x}$ -independent part  $L_{ww}^0$  on  $\mathfrak{t}_{h,n}$  generators

$$\begin{aligned}
[L_{ww}^0, b_i^I] &= \omega^I(w) \omega^K(w) a_{iK} + \sum_{\vec{P} \neq \emptyset} \delta_{ww} \mathcal{A}_{\vec{P}}^I{}^K B_i^{\vec{P}} a_{iK} \\
[L_{ww}^0, a_{iI}] &= \frac{1}{2} \sum_{\vec{P}, \vec{Q}} \left\{ \delta_{ww} \mathcal{A}_{\vec{P}I\vec{Q}}^M{}^N [B_i^{\theta(\vec{P})} a_{iM}, B_i^{\vec{Q}} a_{iN}] \right. \\
&\quad \left. - \delta_{ww} \mathcal{D}_{\vec{P}I\vec{Q}} \sum_{k \neq i} [B_i^{\theta(\vec{P})} t_{ik}, B_i^{\vec{Q}} t_{ik}] \right\}
\end{aligned}$$

where coefficients of the  $\mathfrak{t}_{h,n}$ -elements are  $\delta_{ww}$  variations of

modular tensors  $\mathcal{A}_{K\vec{P}L}^I{}^J$  and  $\mathcal{D}_{\vec{P}}$  except for the rk-3 case (1-letter  $\vec{P}$ )

$$\delta_{ww} \mathcal{A}_{P}^I{}^J = \omega^I(w) \partial_w \Phi_P^J(w) + \frac{1}{h-1} \omega^L(w) \partial_w \Phi_L^I(w) \delta_P^J - (I \leftrightarrow J)$$

## VII. 6 Higher-genus analogues of Tsunogai derivations

Similar to  $\epsilon_k(X) = \delta_k(X) - [\text{ad}_b^{k-1}(a), X]$  at genus  $h = 1$ , shift  $[L_{ww}^0, \bullet]$  to

$$M_{ww}^0 = L_{ww}^0 - \sum_{\vec{P} \neq \emptyset} \delta_{ww} \mathcal{D}_{\vec{P}} \sum_{1 \leq k < \ell} B_k^{\vec{P}} t_{k\ell}$$

Simplifies earlier formulae: decouple  $(a_{iI}, b_i^I)$  at different  $i \in \{1, \dots, n\}$

$$[M_{ww}^0, b_i^I] = \sum_{\vec{P}} \delta_{ww} \mathcal{E}_{\vec{P}}^I{}^K B_i^{\vec{P}} a_{iK}$$

$$[M_{ww}^0, a_{iI}] = \frac{1}{2} \sum_{\vec{P}, \vec{Q}} \delta_{ww} \mathcal{E}_{\vec{P}I\vec{Q}}^M{}^N [B_i^{\theta(\vec{P})} a_{iM}, B_i^{\vec{Q}} a_{iN}]$$

$$[M_{ww}^0, t_{ij}] = 0$$

where  $\mathcal{E}^{MN} = -Y^{MN}/\pi$  and  $\mathcal{E}_{J\vec{P}L}^I{}^K = \mathcal{A}_{J\vec{P}L}^I{}^K + \delta_J^I \mathcal{D}_{\vec{P}} \delta_L^K$

## VII. 6 Higher-genus analogues of Tsunogai derivations

The  $\delta_{ww}$  variations in  $L_{ww}$  directly translate into  $\partial_{\tau_\alpha}$  in  $L_\alpha = \mathcal{J}|_{d\tau_\alpha}$ ,

$$\begin{aligned} [M_\alpha^0, b_i^I] &= \sum_{\vec{P}} \partial_{\tau_\alpha} \mathcal{E}^I_{\vec{P}}{}^K B_i^{\vec{P}} a_{iK} \\ [M_\alpha^0, a_{iI}] &= \frac{1}{2} \sum_{\vec{P}, \vec{Q}} \partial_{\tau_\alpha} \mathcal{E}^M_{\vec{P}I\vec{Q}}{}^N [B_i^{\theta(\vec{P})} a_{iM}, B_i^{\vec{Q}} a_{iN}] \end{aligned}$$

At genus  $h = 1$ , recover Eisenstein from  $\int$  representation ( $\vec{P} = P_1 \cdots P_r$ )

$$\partial_\tau \mathcal{E}^I_{J\vec{P}L}{}^K \big|_{h=1} = \delta_J^I \delta_L^K \partial_\tau \int_{\Sigma_1} d^2x \mathcal{G}_{\vec{P}}(x, x) \big|_{h=1} = (\text{Im } \tau)^r (r+1) G_{r+2}(\tau)$$

Exactly reproduce series in Tsunogai derivations of  $[L(x=0, \tau), \bullet]$  on  $a$  &  $b$

$$[M_\alpha^0, \bullet] \big|_{h=1} = \epsilon_0 + \sum_{k=2}^{\infty} (k-1) G_k(\tau) \epsilon_k = [L(x=0, \tau), \bullet]$$

## Conclusion & Outlook

- many recent developments on higher-genus polylogs thanks to symbiosis with conformal field theory, algebraic geometry and topology
- QFT / gravity applications: hope for higher-genus polylogs to facilitate Feynman-integral evaluations / offer prototypes for higher-dim. varieties
- string-theory applications: tensor generalizations of modular graph forms & constructing chiral amplitudes from Enriquez kernels / supergravity
- future links to number theory via single-valued polylogs / zeta generators

**Thank you for your attention !**