

The Variation of Calabi-Yau Hodge Structures and Feynman Integrals

Elliptics and Beyond, Les Diablerets

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SCATTERING AMPLITUDES

Formal definition of a Calabi-Yau manifold

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The $c_1(T_M) = 0$ condition makes it easy to construct many Calabi-Yau families by simple combinatorics.

Families of Calabi-Yau n -fold hypersurfaces in projective space

Let $M \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d , with normal bundle $\mathcal{N} = \mathcal{O}_M(dH)$, H hyperplane class. The exact sequence

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$$c(T_M) = \frac{(1 + H)^{n+2}}{1 + dH} = 1 + \underbrace{[(n+2) - d]H}_{c_1(T_M)} + \underbrace{[(1-d)^2 + \frac{1}{2}n(n+3-2d)]H^2}_{c_2(T_M)} + \dots$$

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$n = 1$: cubic $d = 3$ in $\mathbb{P}^2$ $wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$ is a complex family of elliptic curves $n = \dim_{\mathbb{C}} = 1$, with complex modulus z .

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$n = 3$: quintic in $\mathbb{P}^4$ is a CY 3-fold with **101 complex moduli**.

$\vdots$

Topological data of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

$$n = 1: (x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1): 10-9=1,$$

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$$\chi = \int_{M_n} c_n = \int_{\mathbb{P}^*} c_n \mathcal{N} = c_n d,$$

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$$n = 2: \text{Def. arithmetic genus } \chi_k(X) := \sum_{i=0}^n (-1)^i h^{ki}. \text{ Hirzebruch Riemann-Roch for surfaces } \chi_0(S) = \frac{1}{12} \int_S (c_1^2 + c_2). \text{ We have } h^{00} = 1 \text{ and } SU(2) \text{ holonomy implies further } h^{02} = 1 \text{ and } h^{01} = 0, \text{ i.e. } \chi_0(M_2) = 2 \text{ and } c_1 = 0 \Rightarrow \chi(M_2) = 24. \text{ Only one topological type of CY 2-folds, the } K3.$$

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$$n = 3: \text{Hirzebruch Riemann Roch trivial } \chi_0 = \frac{1}{24} \int_{M_3} c_1 c_2 = 1 - 0 + 0 - 1 \checkmark, \\ \chi_1 = -h^{11} + h^{21} = \frac{1}{24} \int_{M_3} c_1 c_2 - 12 c_3 \Rightarrow \chi = 2(h^{11} - h^{21}) \checkmark. \text{ E.g.} \\ c_3(TM) = -40H^3 \Rightarrow \chi = -200 \text{ and } h^{11} = 1, h^{21} = 101 \text{ for the quintic.}$$

Topological data of Calabi-Yau n -folds

Conjecture: For $n \geq 3$ there exists for each n a huge, but finite number of topological types of Calabi-Yau n -folds

Theorem C. T. C. Wall¹:

The topological type of a Calabi-Yau 3-fold M is fixed by their integer Hodge structure $(h^{2,1}, h^{1,1})$, their triple intersection $D_i \cdot D_j \cdot D_k \in \mathbb{N}$ and $c_2(TM) \cdot D_k, D_k \in H_4(M, \mathbb{Z})$.

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While the topological classification of Calabi-Yau n -folds for $n \geq 3$ is an **open problem** it is relatively easy to find **huge classes of constructions**

¹We assume M to be oriented, simply connected and torsion-free.

Classes of Calabi-Yau varieties: Branched covers

Branched covers: Let $\mathbb{P}$ be a n -dim Fano variety with $K(\mathbb{P}) = c_1(\mathbb{P}) > 0$ then a d -fold cover

$$y^d = p(\underline{x}), \quad \text{branched at} \quad [p(\underline{x})] = \frac{d}{d-1} K(\mathbb{P})$$

over $\mathbb{P}$ is a not necessarily smooth CY n -fold.

Example elliptic curve: $\mathbb{P} = \mathbb{P}^1$ with $d = 2$

$$y^2 = p(\underline{x}) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) .$$

$K(\mathbb{P}^1) = 2H$, $H = [x]$, $[p(\underline{x})] = 2K(\mathbb{P}^1) = 4H$ ✓ The invariant complex modulus is $z = (a_1 - a_2)(a_3 - a_4)/((a_1 - a_3)(a_2 - a_4))$.

$\mathbb{P} = (\mathbb{P}^1)^\ell$ and $d = 2, 3$ describe generically singular CY ℓ -varieties for two dimensional ℓ -loop fishnets amplitudes, with 4, 3 lattice valence, C. Duhr,

A. Klemm, F. Loebbert, C. Nega and F. Porkert, Phys. Rev. Lett. **130**, no.4, 4 (2023) .

Classes of Calabi-Yau varieties: Complete Intersection

Complete intersections: The vanishing locus of r polynomials $P_k(\underline{x}_1, \dots, \underline{x}_m) = 0$, $k = 1, \dots, r$ in $\mathbb{P} = \prod_{j=1}^m \mathbb{P}_j^{n_j}$ defines a CY $\left(\sum_{j=1}^m n_j - r\right)$ -fold if

$$\sum_{k=1}^r d_{kj} = n_j + 1 \quad , \forall j = 1, \dots, m .$$

Here d_{kj} are degrees of the k th polynomial in the variables of the j th $\mathbb{P}^{n_j}$ factor.

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The complete intersection CY $(\ell - 1)$ -folds with $m = \ell + 1$, $r = 2$, $d_{kj} = 1$ and $n_j = 1 \ \forall k, j$, describe the ℓ loop banana integral with $(\ell + 1)$ mass parameters m_i^2/p^2 in two dimensions. K. Bönisch, F. Fischbach, A. Klemm, C. Nega and R. Safari: JHEP 05 (2021) 066

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The complete intersection CY 3-fold with $m = 1$, $r = 2$, $n_1 = 7$, $d_{k1} = 2$ contributes to the 5PM, 1SF sector of black-hole scattering M. Driesse, G. U. Jakobsen, A. Klemm, G. Mogull, C. Nega, J. Plefka, B. Sauer and J. Usovitsch, Nature 641 (2025) 603-607

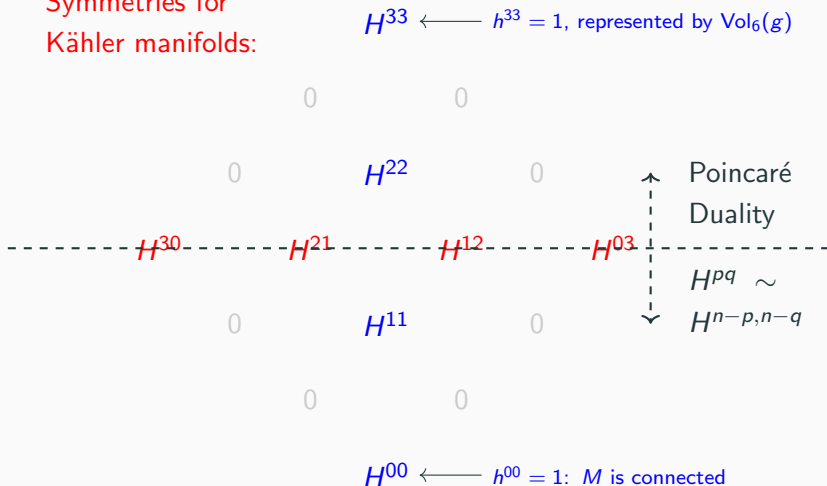
Cohomology and deformations of 3-folds

The cohomology
groups of a CY-
3fold:

$$\begin{array}{ccccccc} & & & & H^{33} & & \\ & & & 0 & & 0 & \\ & & 0 & & H^{22} & & 0 \\ & H^{30} & & H^{21} & & H^{12} & & H^{03} \\ & & 0 & & H^{11} & & 0 \\ & & & 0 & & 0 & \\ & & & & H^{00} & & \end{array}$$

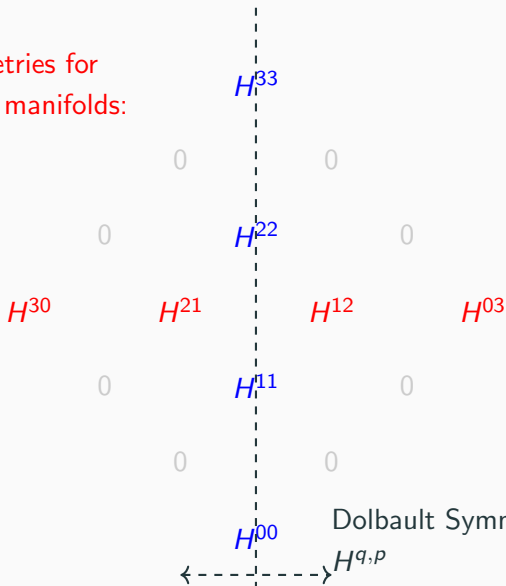
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Symmetries for
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Special Properties
of CY-manifolds:

$$H^{33}$$

$$0$$

$$0$$

$$0$$

$$H^{22}$$

$$0$$

$$H^{30}$$

$$H^{21}$$

$$H^{12}$$

$$H^{03}$$

$h^{30} = 1$: $dx^1 \wedge dx^2 \wedge dx^3$
represents the unique
(3,0)-form Ω

$$0$$

$$H^{11}$$

$$0$$

$$0$$

$$0$$

vanish since dx^i and
 $dx^i \wedge dx^j$ are not $SU(3)$ -
invariant

$$H^{00}$$

Cohomology and deformations of 3-folds

$$H^{2,1}(M) \xrightarrow[\lrcorner \Omega]{\sim}$$

$H^1(M, TM)$ Kodaira:

$H^1(M, TM)$ describes

first order
complex

structure

deforma-
tions

Tian–Todorov:

They are globally
unobstructed, i.e.

$$\dim_{\mathbb{C}}(\mathcal{M}_{cs}) = h^{2,1}$$

H^{33}

0

0

0

H^{22}

0

H^{30}

H^{21}

H^{12}

H^{03}

0

H^{11}

0

0

0

H^{00}

CY 3-folds as

complex defor-

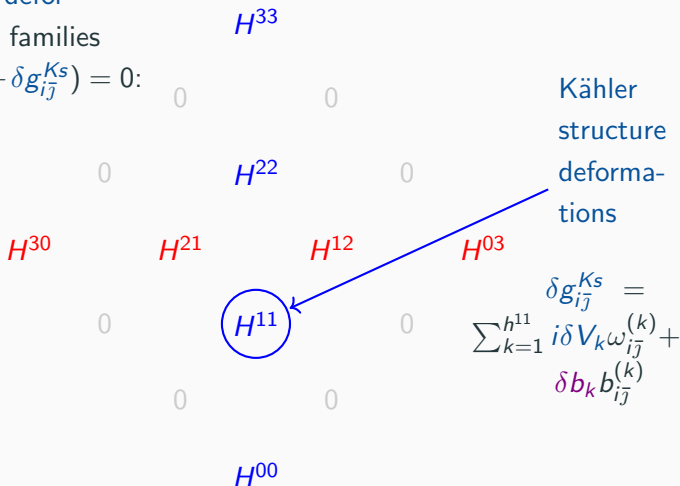
mation families

$$R_{i\bar{j}}(g + \delta g_{ij}^{cs}) = 0:$$

Cohomology and deformations of 3-folds

CY 3-folds as
Kähler deformation families

$$R_{i\bar{j}}(g + \delta g_{i\bar{j}}^{Ks}) = 0:$$

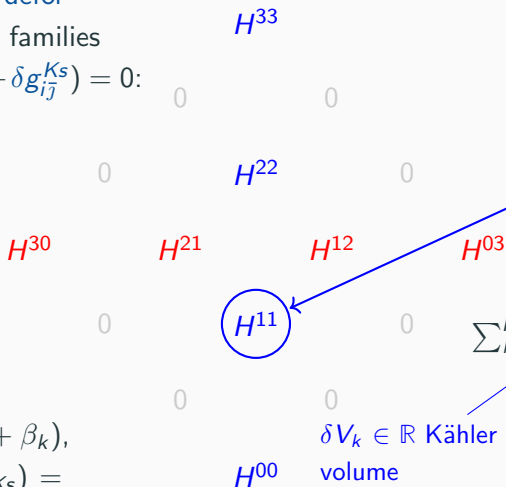


Cohomology and deformations of 3-folds

CY 3-folds as
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$$R_{i\bar{j}}(g + \delta g_{i\bar{j}}^{Ks}) = 0:$$

Kähler
structure
deforma-
tions



$$\delta g_{i\bar{j}}^{Ks} = \sum_{k=1}^{h^{11}} i\delta V_k \omega_{i\bar{j}}^{(k)} + \delta b_k b_{i\bar{j}}^{(k)}$$

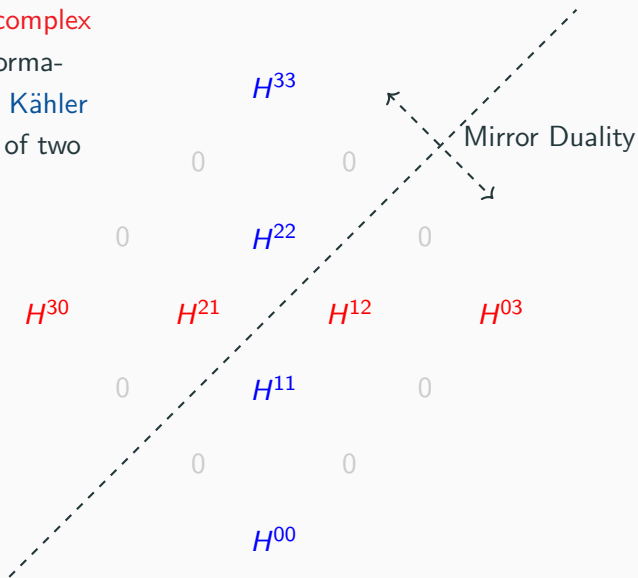
$$t_k = 2\pi i(iV_k + \beta_k),$$

i.e. $\dim_{\mathbb{C}}(\mathcal{M}_{cKs}) = h^{11}$

$\delta V_k \in \mathbb{R}$ Kähler volume
 $\delta \beta_k \in \mathbb{R}$ value of N-S b-field (stringy)

Cohomology and deformations of 3-folds

Mirror Symmetry exchanges the **complex structure** deformations and the **Kähler deformations** of two CY (M, W)



Construction of mirror pairs

M generic quintic $[\hat{p}_5(\underline{x}) = 0] = [5H] \subset \mathbb{P}^4$, $h_{21} = 101$, $h_{11} = 1$.

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Generalisation (Batyrev): $(\Delta, \hat{\Delta})$ is a dual pair of reflexive lattice polyhedra,

$\mathbb{P}_\Delta$ the associated toric space and $[H_i]$ its divisors.

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(M, W) mirror
pairs

$$W = \{[p_\Delta = 0] = [\sum_i \hat{H}_i] \subset \mathbb{P}_{\hat{\Delta}}\}$$

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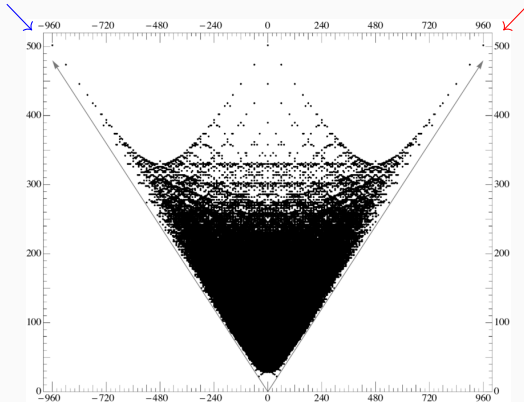
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(M, W) mirror pairs

$$W = \{[p_\Delta = 0] = [\sum_i \hat{H}_i] \subset \mathbb{P}_{\hat{\Delta}}\}$$

$$(h^{11}, h^{2,1}) = (11, 491)$$

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Construction of mirror pairs

M generic quintic $[\hat{p}_5(\underline{x}) = 0] = [5H] \subset \mathbb{P}^4$, $h_{21} = 101$, $h_{11} = 1$.

W special quintic $[p_5 = \sum_{i=0}^4 x_i^5 - z \prod_{i=0}^4 x_i = 0] \subset \widehat{\mathbb{P}^4/\mathbb{Z}_5^3}$, $h_{21} = 1$, $h_{1,1} = 101$

Generalisation (Batyrev): $(\Delta, \hat{\Delta})$ is a dual pair of reflexive lattice polyhedra, $\mathbb{P}_\Delta$ the associated toric space and $[H_i]$ its divisors.

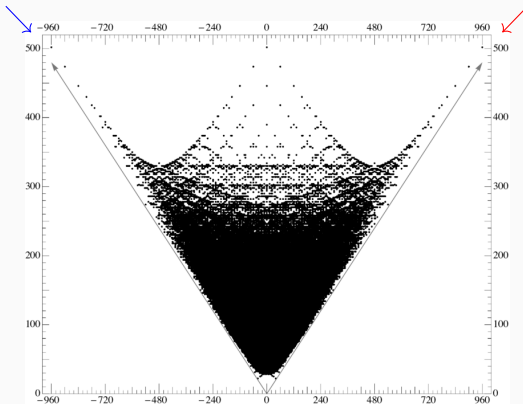
Geom.	$\#(\Delta_k, \hat{\Delta}_k)$
2 pts	1
ell crv	16
K3	4319 ¹
CY 3-flds	473800776 ¹
CY 4-flds	$\mathcal{O}(10^{22})?$
$\vdots$	$\vdots$

¹ Kreuzer and

Skarke '02, $k = 3, 4$

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Periods on Calabi-Yau n-folds

Period integrals

$$\Pi_{ij}(z) = \int_{\Gamma_i} \gamma^j(z)$$

define a non-degenerate pairing between (middle) homology and (middle) cohomology, well defined by Stokes' theorem:

$$\Pi : H_n(M_n, \mathbb{Z}) \times H^n(M_n, \mathbb{Q}) \rightarrow \mathbb{C}.$$

It is possible and often natural to consider extensions of $\mathbb{Z}$. There is an intersection pairing

$$\Sigma : H_n(M_n, \mathbb{Z}) \times H_n(M_n, \mathbb{Z}) \rightarrow \mathbb{Z},$$

that can be made in particular integral. If n is odd Σ is antisymmetric and can be made symplectic. If n is even Σ is symmetric on the even self-dual lattice $H_n(M_n, \mathbb{Z})$. E.g. for $K3$ $b_2 = 22$ and

$$\sigma = b_2^+ - b_2^- = \frac{1}{3} \int_{M_2} c_1^2 - 2c_2 = -16 \text{ hence } b_2 \text{ has signature } (3, 19) \text{ and is } E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}.$$

If n is odd we fix an integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \dots, r$ with $\text{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

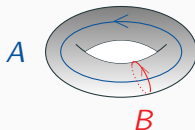
It is clearly defined only up to an $\text{Sp}(b_n(M), \mathbb{Z})$ choice.

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Exp: Calabi-Yau 1-fold: $P = wy^2 - x(x - w)(x - wz) = 0 \subset \mathbb{P}^2$

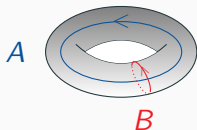


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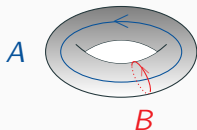
$$\Omega(z) = \oint_{\gamma} \frac{2dx \wedge dy}{P} = \frac{dx}{y}, \quad \partial_z \Omega(z) \sim \frac{xdx}{y}$$

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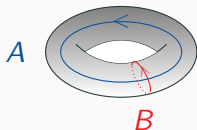
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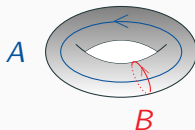
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$$\mathcal{L} \int_{\Gamma} \Omega = \left[z(1-z) \partial_z^2 + (1-2z) \partial_z - \frac{1}{4} \right] \int_{\Gamma} \Omega = 0.$$

Rational cohomology after Griffiths

Hypersurfaces and complete intersection:

$$M = \{\cap_{i=1}^r P_i(\underline{x}, \underline{z}) = 0 \subset \mathbb{P} | \underline{x} \text{ hom. coord of ambient space } \mathbb{P}\}$$

Define measure on the ambient space (w_i weights if $\mathbb{P}$ general toric)

$$\mu_{n+r} = \sum_{k=1}^{n+r+1} (-1)^k w_k x_k \, d x_1 \wedge \dots \wedge \widehat{d x_k} \wedge \dots \wedge d x_{n+r+1} .$$

For $r = 1$ and $\gamma \sim S^1$ encircles $P_1 = P = 0$ in $\mathbb{P}$

$$\Omega_n = \oint_{\gamma} \frac{a_0 \mu_{n+1}}{P(\underline{x}, \underline{z})},$$

is the Griffiths residuum form of the $(n, 0)$ form. Note that it is well defined under coord rescaling due to the $c_1 = 0$ condition $d = n + 2$. a_0 makes Ω_n better behaved under $\text{Aut}(\mathcal{M}_{cs})$.

Rational cohomology after Griffiths

For the branched covers, say $y^2 = p_{\deg(2K)}(\underline{x})$, we can integrate over y using the residuum formula and get

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in particular the familiar holomorphic differential for the elliptic curve.

The Griffith residuum form has a straightforward generalisation to complete intersections

$$\Omega_n = \oint_{\gamma_1} \dots \oint_{\gamma_r} \frac{(\prod_{i=1}^r a_{0,i}) \mu_{n+r}}{P_1 \dots P_r},$$

Deriving the Picard-Fuchs differential ideal

The periods fulfil Fuchsian type holonomic differential equations that together with boundary conditions determine them.

Poor man's derivation of Picard Fuchs equations: Pick a cycle Γ of $T^n = (S^1)^n$ topology perform all residua and find differential operators that annihilate the result.

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Poor man's derivation of Picard Fuchs equations: Pick a cycle Γ of $T^n = (S^1)^n$ topology perform all residua and find differential operators that annihilate the result. E.g. let $m = n + 2$ and $\{P = \sum_{i=1}^m x_i^m - ma \prod_{i=1}^m x_i = 0 \subset \mathbb{P}^{n+1}\}$ CY n -folds. Then

$$\oint_{T^n} \Omega \sim \oint_{T^{n+1}} \frac{\mu_{n+1}}{\prod_i x_i (1 - \sum_i x_i^m / ma \prod_i x_i)} = \sum_{k=0}^{\infty} \frac{(mk)!}{(k!)^m} z^k$$

with $z = 1/(ma)^m$ is annihilated by $(n + 1)$ -order PF operator

$$\left(\theta^{n+1} - mz \prod_{k=1}^{n+1} (m\theta + k) \right) \int_{\Gamma} \Omega = 0 ,$$

where $\theta = z \frac{d}{dz}$ is the logarithmic derivative.

Deriving the Picard-Fuchs differential ideal

Griffiths derivation of Picard Fuchs equations: Consider derivatives of $\partial^k \Omega \sim \oint_{\gamma} \frac{Q_{kd}(\underline{x}, \underline{z}) \mu}{P^{k+1}}$ w.r.t. the moduli. Griffiths proved for the **pure variation of Hodge structures**

$$\partial^k \Omega \in \oplus_{q=0}^k \mathcal{H}^{n-q,q} =: \mathcal{F}^{n-k}$$

are independent meromorphic forms with vanishing residuum in $\mathcal{F}^0$. $\mathcal{H}^{n-q,q} \sim H^{n-q,q}$ by Hyper cohomology and since $\text{rank}(\mathcal{F}^0) = b_n$ there must be a $\mathbb{Q}(z)$ relation between the derivatives for $k > b_n$ modulo exact form, i.e. under the integrals. This is the Picard-Fuchs operator.

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$$\int \frac{\partial}{\partial x_s} \left(\frac{Q(\underline{x}, \underline{z}) \mu}{P^k} \right) = 0 \rightarrow \int \frac{k Q(\underline{x}, \underline{z}) \partial_s P \mu}{P^{k+1}} = \int \frac{\partial_s Q(\underline{x}, \underline{z}) \mu}{P^k}.$$

allows to reduce the $1/P^{b_n+1}$ degree using a Gröbner-basis algorithm.

Deriving the Picard-Fuchs differential ideal

Gelfand's derivation of Picard–Fuchs equations: First over-parametrize the problem. $P = \sum_{i=1}^m a_i x_i^m - a_0 \prod_{i=1}^m x_i$. Then the equation

$$(\theta_0^m - \prod_i \theta_i) \frac{\mu}{P} = 0$$

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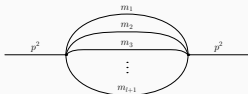
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with $\theta_i = a_i \frac{d}{da_i}$ just reflects a linear relation between points in the Newton polytope of P . The automorphism of the representation of $P = 0$ in the ambient space implies that the problem can depend only on $z = (-1)^m \prod_{i=1}^m a_i / a_0^m$. Eliminate the a_i in terms of z . Then one gets from each linear relation a resonant GKZ differential operator. Here after some trivial manipulation

$$\mathcal{L}^{n+1} = \theta_z^{n+1} - mz \prod_{k=1}^{n+1} (m\theta_z + k) .$$

Application Banana Integrals in 2d:

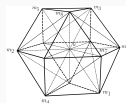
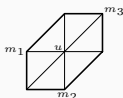
The maximal cut Banana integrals with $t = p^2/\mu^2$, $\xi_i = m_i/\mu$ or $z_i = \xi_i^2/t$.



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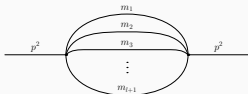
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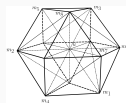
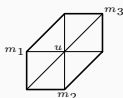
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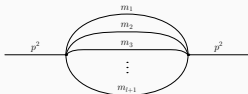
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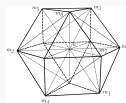
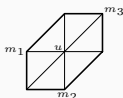
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$$I(z_i) = \int_{\Gamma} \oint_{S^1} \frac{\mu_{\ell}}{\left(t - \left(\sum_{i=1}^{\ell+1} \xi_i^2 x_i\right) \left(\sum_{i=1}^{\ell+1} x_i^{-1}\right)\right) \prod_i^{\ell+1} x_i} = \int_{\Gamma} \Omega =: \Pi_{\Gamma}$$

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Better representation as complete intersection

In K. Boehnisch, F. Fischbach, C.Nega, AK 2020 a representation without redundant deformations as complete intersection of two degree $(1, \dots, 1)$ polynomials $P_1 = P_2 = 0 \subset \mathbb{P}_{l+1} := \bigotimes_{i=1}^{l+1} \mathbb{P}_{(i)}^1$ was identified.

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According to [S. Hosono, AK, S. Theisen and S.-T. Yau 1994](#) the periods of this geometry fulfil a simple GKZ system with $2(\ell + 2)$ redundant deformation parameters and linear relations

$$\begin{aligned} l^{(1)} &= (-1, -1; 1, 1, 0, 0, \dots, 0, 0, 0, 0) \\ &\vdots \\ l^{(\ell+1)} &= (-1, -1; 0, 0, 0, 0, \dots, 0, 0, 1, 1) \end{aligned}$$

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and Frobenius solution

$$\omega_0(\underline{z}; \underline{\epsilon}) = \sum_{n_1, \dots, n_{l+1} \geq 0} c(\underline{n} + \underline{\epsilon}) \underline{z}^{n+\epsilon}, \quad c(\underline{n}) = \frac{\prod_{j=1}^2 \left(-\sum_{k=1}^{l+1} l_{0j}^{(k)} n_k \right)!}{\prod_{i=1}^{2l+2} \left(\sum_{k=1}^{l+1} l_i^{(k)} n_k \right)!}.$$

in the $\ell + 1$ invariant MUM coordinates $z_k = m_i^2 / p^2 = \prod_{i=1}^{2(\ell+4)} a_i^{l_i^{(k)}}$.

Advantages for the geometric formulation: Banana Integrals

At the point of maximal unipotent monodromy mirror symmetry predicts via the $\hat{\Gamma}$ -class formalism an integral monodromy basis of the periods.

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An extended $\hat{\Gamma}$ -class for the Fano variety $P_2 = 0 \subset \bigotimes_{i=1}^{l+1} \mathbb{P}_{(i)}^1$

$$\hat{\Gamma}_F(TF) = \frac{\hat{A}(TF)}{\hat{\Gamma}^2(TF)} = \frac{\Gamma(1 - c_1)}{\Gamma(1 + c_1)} \cos(\pi c_1) .$$

fixes also an integral basis of the solutions to the inhomogeneous Picard fuchs differential ideal including the full Banana integral. This basis choice was confirmed by [Hiroschi Iritani 2021](#).

Complex moduli spaces of Calabi-Yau n -fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n -fold M is parametrized by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ unobstructed complex deformation parameters z , i.e $\mathcal{M}_{cs}(M)$ is a manifold of complex dimension $h^{n-1,1} =: r$.

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Global Torelli Theorems hold for elliptic curves and K3 surfaces. In this case $\mathcal{M}_{cs}(M)$ is a symmetric domain which maps to the period domain. E.g. for $n=1$ $\mathcal{M}_{cs}(M) \simeq SL(2, \mathbb{Z}) \backslash \mathbb{H}$ with $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\} \simeq SL(2, \mathbb{R})/SO(2)$. This allows a modular parametrisation of the amplitudes, which is in general not available for $n \geq 3$

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are **the Riemann bilinear** relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \quad (1)$$

defining the real positive exponential of the **Kähler potential** $K(z)$ for the **Weil–Petersson metric** $G_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_j} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

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$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \quad (1)$$

defining the real positive exponential of the **Kähler potential** $K(z)$ for the **Weil–Petersson metric** $G_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_j} K(z)$ on $\mathcal{M}_{cs}(M_n)$. As well as from relations on holomorphic bilinears and their derivatives that follow from **Griffiths transversality**¹

$$\int_{M_n} \Omega \wedge \underline{\partial}_{l_k}^k \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{l_n}(\underline{z}) \in \mathbb{Q}(\underline{z}) & \text{if } k = n . \end{cases} \quad (2)$$

Here $\underline{\partial}_{l_k}^k \Omega = \partial_{z_{l_1}} \dots \partial_{z_{l_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^k H^{n-p,p}$ are sections of holomorphic sub bundles $F^n \subset F^{n-k} \subset \dots \subset F^1 \subset F^0$ determining the **Hodge filtration** of the Hodge bundle F^0 .¹ R. Bryant and P. Griffiths, Prog.

Simple consequences of Griffith transversality

Consider the Picard-Fuchs operator

$$\mathcal{L}^{(n+1)}(z) = \sum_{i=0}^{n+1} a_i(z) \partial_z^i \quad \text{with} \quad a_{n+1} = 1$$

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Here $C_{\underline{z}}(z) = C_{\underbrace{z \dots z}_n}(z)$ from (2) fulfils the differential equation

$$C'_{\underline{z}} = -\frac{2}{n+1} a_n C_{\underline{z}} \tag{3}$$

Periods, Monodromy and special Geometry

Local $\rightarrow$ global: How to find the periods over cycles in $H_3(W, \mathbb{Z})$?
Find the basis in which monodromies $\Pi \mapsto T_* \Pi$ around the singular points $*$ are in $\mathrm{Sp}(4, \mathbb{Z})$

Periods, Monodromy and special Geometry

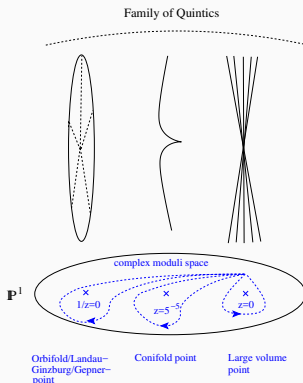
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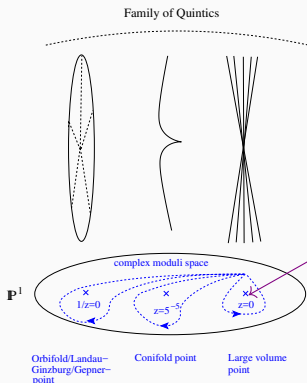
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Identifies also the expansion point for the mirror map as point of maximal unipotent monodromy

Landmans monodromy theorem

The theorem of Landman [AMS 1973](#) states that all possible monodromy matrices on an algebraic n -fold have to obey the relation

$$(\mathbf{T}^k - \mathbf{1})^{n+1} = 0 . \quad (4)$$

Here $k \in \mathbb{N}_0$, implying that the indicial α has to be a rational number. $\mathbf{T}$ can be unipotent of lower order $m < n$, i.e., $(\mathbf{T}^k - \mathbf{1})^{m+1} = 0$, where m is the size of the biggest Jordan block in $\mathbf{T}$. The maximal n that can appear is $n = \dim(M)$.

Unipotency of order $m \leq n$ implies that a period on an n -fold cannot degenerate worse than with a logarithmic singularity of type $\log(\Delta)^n$. Suppose that a maximal cut of a Feynman integral degenerates in the physical parameter Δ as $\log(\Delta)^m$. Then Landman's theorem implies that the geometry associated to this integral cannot be an algebraic manifold of dimension less than m .

Degenerations of one parameter mixed Hodge structures

Limiting mixed Hodge diamond $H_{\text{lim}}^{p,q}$ for the four different degenerations $F-$, $C-$, $K-$, $M-$ points of one parameter variations of CY mixed Hodge structures:

$$F : \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$$C : \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$$K : \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 1 & & 0 & & 1 & \\ & & 0 & & 0 & & 0 \\ 0 & & 1 & & 0 & & 1 \\ & & & 0 & & 0 & \\ & & & 0 & & & \\ & & & 0 & & & \end{array}$$

$$M : \begin{array}{ccccccc} & & & & & 1 & \\ & & & 0 & & 0 & \\ & & 0 & & 1 & & 0 \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 1 & \\ & & 0 & & 1 & & 0 \\ & & & 0 & & 0 & \\ & & & & & 1 & \end{array}$$

Special Geometry on CY 3-folds

Special geometry **Bryant and Griffiths '83** implies that the periods can be expressed by a prepotential $\mathcal{F}$

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- ← double logarithmic solution
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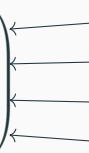
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and **Candelas et al '91** identified near the MUM point $z = 0$

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Hosono et. al '93 generalised to multiparameter CY and related the classical terms to the CTC Wall data $\kappa = D^3$, $\sigma = (\kappa \bmod 2)/2$ in

$$\mathcal{F} = -\frac{\kappa}{6}t^3 + \frac{\sigma}{2}t^2 + \frac{c_2 \cdot D}{24}t + \frac{\chi(M)}{2} \frac{\zeta(3)}{(2\pi i)^3} - \frac{1}{(2\pi i)^3} \sum_{\substack{\beta \in H_2(M, \mathbb{Z}) \\ \beta \neq 0}} n_0^\beta \text{Li}_3(Q^\beta).$$

Special Geometry for CY 3-folds and beyond

Special Geometry allows to chose a basis of cosmology in the the t coordinates

$$\begin{aligned}\Omega_0 &= \alpha_0 + t^i \alpha_i - \partial_i \mathcal{F} \beta^i - (2\mathcal{F} - t^i \partial_i \mathcal{F}) \beta^0, \\ \chi_i &= \partial_i \Omega_0 = \alpha_i - \partial_i \partial_j \mathcal{F} \beta^j - (\partial_i \mathcal{F} - t^j \partial_i \partial_j \mathcal{F}) \beta^0, \\ \chi^i &= -\beta^i + t^i \beta^0, \\ \Omega^0 &= \beta^0.\end{aligned}$$

Using $C_{ijk} = -\partial_i \partial_j \partial_k \mathcal{F}$ it follows immediately that the Gauss–Manin connection acts on this basis as

$$\partial_i \begin{pmatrix} \Omega_0 \\ \chi_j \\ \chi^j \\ \Omega^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_i^k & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Omega_0 \\ \chi_k \\ \chi^k \\ \Omega^0 \end{pmatrix}.$$

The splitting of the period matrix into semi-simple and nilpotent part is discussed in [J.Dücker, J.Piribauer, AK JHEP 07 \(2025\) 225](#) using special geometry and the Frobenius structure of the periods of n -folds $n > 3$.

Griffith transversality and iterated integrals

Let denote ϖ_k $k = 0, \dots, n$ some basis in the kernel of $\mathcal{L}^{(n+1)}$ and the $(n+1) \times (n+1)$ Wronskian

$$W = ||\partial_z^i \varpi_k||$$

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Then Griffiths-transversality (2) implies that the inverse of the Wronskian is up rational factors Z linear in (derivatives) of periods

$$W^{-1} = \Sigma W^T Z^{-1} \quad (5)$$

By the variation of constant methods, the particular solution is **an iterated integral** of the form

$$\varpi^{(p)}(z) = \sum_k \varpi_k \int^z d\tilde{z} W_k W^{-1} \quad (6)$$

where by (5) W^{-1} is linear $\partial_z^i \varpi_l$.

Summary of the Geometric reformulation

The fact $\vec{\Pi}(\underline{z})$ fulfil the Picard-Fuchs differential $\{\mathcal{L}_s^{(\ell)}\}\Pi_{\Gamma_k} = 0$ is equivalent to the fact that they are **flat** sections of the Hodge bundle:

$$\nabla_k \vec{\Pi} = (\partial_{z_k} - A_k(\underline{z}))\vec{\Pi} = 0, \quad k = 1, \dots, h^{n-1,1}(M),$$

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- Flat means they are **(locally) trivial**. Complexity results entirely from their **monodromy** around regular singularities at $\Delta_i = 0$.
- The **right section** –Feynman integral– is found by integral global monodromy structure $\mathfrak{M}_i \in \begin{cases} \mathrm{Sp}(h_n, \mathbb{Z}) & n \text{ odd} \\ \mathrm{O}(h_n, \mathbb{Z}) & n \text{ even} \end{cases}$ and its singular behaviour at **ultraviolet, on-shell, and infrared limits**, which correspond to **degenerations of the Hodge structure** of the CY: **Maximal unipotent points, conifolds etc** .

Solution to the full non-maximal cut Feynman integral

We saw that maximal cut Feynman integral is identified with the periods over closed cycles $\Gamma_k \in H_n(X, \mathbb{Z})$. The actual Feynman Integral $I_\ell^{full}(z)$ is rather a chain integral of Ω over open cycles in relative cohomology, so called Griffiths normal functions. They fulfil the same Picard-Fuchs equation with **an inhomogeneity**, which has a rather straightforward physical interpretation:

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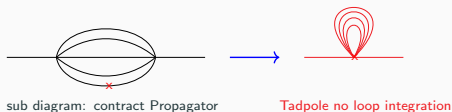
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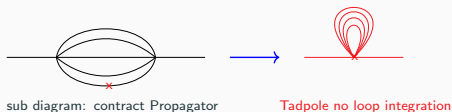
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Leading to an inhomogeneous differential equation for the Banana integral

$$\mathcal{L}^{(\ell)} I_\ell^{full}(z) = -(\ell + 1)!z$$

Summary: Dictionary Feynman integrals and geometry

Perturbative QFT	Geometry X	Differential eq.
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ -deformed) ⌚ Monodromy group $\in \Gamma(\mathbb{Z})$; irreducible ?	Homogeneous Gauss Manin ($d - A(z))\underline{\Pi} = 0$
actual Feynman integral	Chain integral (ϵ -deformed)	Inhomogeneous Gauss Manin connection ($d - A(z))\underline{\Pi} = B(z)$

This picture emerged due to work of I. Gel'fand, D. Broadhurst, D. Kreimer, S. Bloch, P.

Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. McLeod, M. Hoppel, M. Wilhelm,

J. Broedel, L. Tancredi, S. Müller-Stach, ... + at least 248 cits. in K. Bönisch, C. Duhr, F. Fischbach, C. Nega, AK

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The key way to systematize this and obtain results in any dimension is to introduce so called **master integrals** for ℓ -loop **Feynman integrals** in general dimensions $D \in \mathbb{R}_+$ Johannes Henn (13)

$$I_{\underline{\nu}}(\underline{z}, D) := \int \prod_{r=1}^{\ell} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^p \frac{1}{D_j^{\nu_j}} \quad (7)$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \dots, p$ are the propagators, q_j is the j^{th} momentum through D_j , $m_j^2 \in \mathbb{R}_+$ are masses. Subject to momentum conservation the q_j are linear in the external momenta and the internal loop momenta k_r . We define $\epsilon := \frac{D_{\text{cr}} - D}{2}$.

Master Integrals and integration by parts relations

Note that the propagator exponents in the **master integrals** are arbitrary non-negative integers and together with $D_{cr} \in \mathbb{Z}$ span a **non-negative cone in a lattice** $(\underline{\nu}, D_{cr}) \in \mathbb{Z}_{\geq 0}^{p+1}$.

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relate the master integrals with different exponents $\underline{\nu}$.

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$$\int \frac{\partial}{\partial x_k} \left(\frac{Q(x, z)\mu}{\mathcal{F}^r} \right) = 0 \rightarrow \int \frac{\mu r Q(x, a) \partial_k \mathcal{F}}{\mathcal{F}^{r+1}} = \int \frac{\mu \partial_k Q(x, a)}{\mathcal{F}^r} .$$

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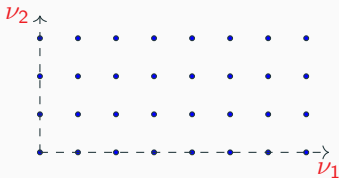
The master integrands define a **finite** cohomology, which sometimes extends the cohomology $H^{\ell}(M_{\ell}, \mathbb{Z})$ of a CY M_{ℓ} .

Master Integrals and integration by parts relations

In particular there is a **finite region in $\mathbb{Z}_{\geq 0}^{p+1}$** Smirnov and Petukhov LMP 2011 that contains all relevant master integrals. I.e. in those master integrals one can express all derivatives w.r.t. the z_k as a linear combination with **rational coefficients** by the IBP relations.

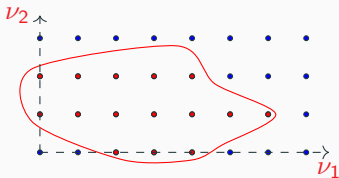
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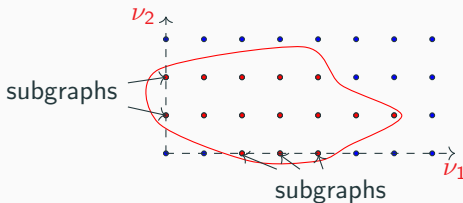
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A complete set of IBP relations corresponds to the complete Picard Fuchs ideal or Gauss-Manin connection for the period integrals.

IBP relation summary:

Hence the IBP relations characterise a suitable finite set of master integrals

$$I(\underline{x}, D) := \int \prod_{r=1}^l \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^p \frac{1}{D_j^{\nu_j}},$$

in a finite region in $\mathbb{Z}_{\geq 0}^{p+1}$, by a generalised first order Gauss Manin connection

$$dI(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon) I(\underline{x}, \epsilon)$$

$$\epsilon = (D_{cr} - D)/2.$$

Basis change on master integrals ϵ -factorized form

$$\underline{I}(\underline{x}, \epsilon) \rightarrow \underline{I}^{better}(\underline{z}(x); \epsilon) = R_0(\underline{z}(x); \epsilon) \underline{I}(\underline{z}(x); \epsilon)$$

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ϵ factorized Gauss-Manin connection for the 5PM 1SF [8][9]

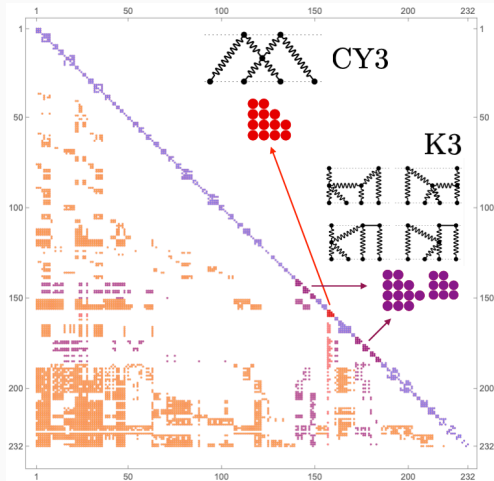


Figure 1: Non-zero entries of the 232×232 extended Gauss-Manin connection matrix $A^{best}(x)$. The not-magnified diagonal sectors give rise to polylogarithms.

The use of the ϵ factorized form

$$\nabla_{\epsilon} I(z, \epsilon) = (d_z - \epsilon \mathbf{A}(z)) I(z, \epsilon) = 0$$

Suppose we have boundary conditions $I(z_0, 0) = I_0$ then we can write the solution vector $I(z, \epsilon)$ as path ordered exponential

$$I(z, \epsilon) = \mathbb{P} \exp\left[\epsilon \int_{z_0}^z \mathbf{A}(z) dz\right] I_0$$

in terms of iterated integrals in the form

$$I(z, \epsilon) = \left[\mathbf{1} + \epsilon \int_{z_0}^z \mathbf{A}(z') dz' + \epsilon^2 \int_{z_0}^z \int_{z_0}^{z'} \mathbf{A}(z') dz' \mathbf{A}(z'') dz'' + \dots \right] I_0$$

For example for the 1 loop diagrams this formalism gives rise to the *function class* of multiple poly logarithms. For higher loop amplitudes this class generalizes to iterated Calabi-Yau periods.

What are families on Calabi-Yau varieties?

Reviews on Calabi-Yau varieties

First: T. Hubsch, "Calabi-Yau manifolds: A Bestiary for physicists," World Scientific, 1994.

Short: S. Hosono, A. Klemm and S. Theisen, "Lectures on mirror symmetry," Lect. Notes Phys. **436**, 235-280 (1994) [arXiv:hep-th/9403096]

Longer: A. Klemm, "The B-model approach to topological string theory on Calabi-Yau n-folds", p. 79-397, Birkhäuser/Springer, 2018.

Math: D. Cox and S. Katz, "Mirror symmetry and algebraic geometry," AMS, pages xii+469 1999

Math: M. Gross, D. Huybrechts, D. Joyce, "Calabi-Yau manifolds and related geometries," Universitext, 2001.

Toric (ambient spaces) : D. Cox, J. Little and H. Schenck, "Toric varieties," AMS, Providence, RI (2011).

What are families on Calabi-Yau varieties?

Databases and software tools for Calabi-Yau varieties:

4,319 and 473,800,776 reflexive polyhedra in dimensions three and four, respectively; in particular, 30,108 distinct Hodge pairs of CY threefold families: Kreuzer and Skarke <https://hep.itp.tuwien.ac.at/~kreuzer/CY/>

Software tools for calculating their triangulations, intersection numbers, etc.: Liam McAllister Group <https://github.com/LiamMcAllisterGroup/cytools>

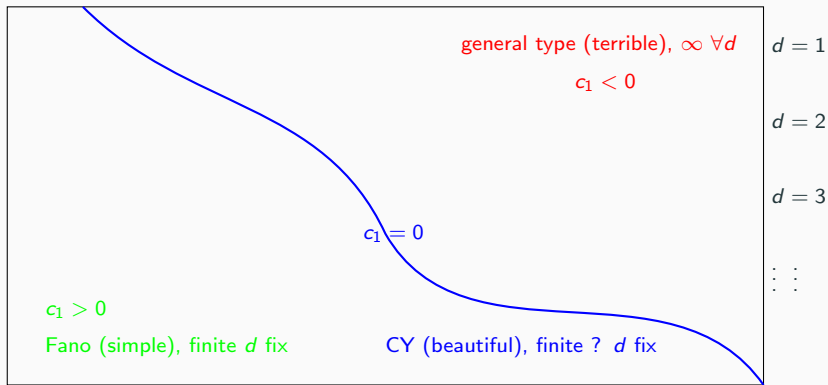
Updated one-parameter list of 624 CY operators by Almkvist, van Enckevort, van Straten and Zudilin: AESZ list

<https://cyclcluster.mpim-bonn.mpg.de/aesz.html>

Software for analytic continuation of solutions, monodromies, etc.:

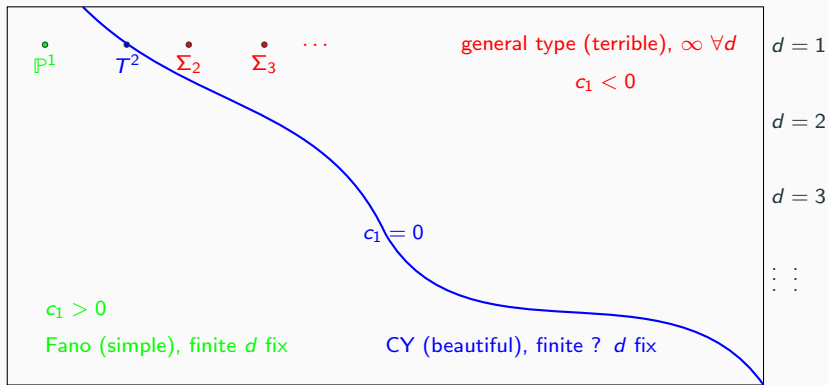
ore_algebra https://www.algebra.uni-linz.ac.at/people/mkauers/ore_algebra/generated/ore_algebra.analytic.html

Additional Philosophical Remarks: Kodaira map of algebraic varieties

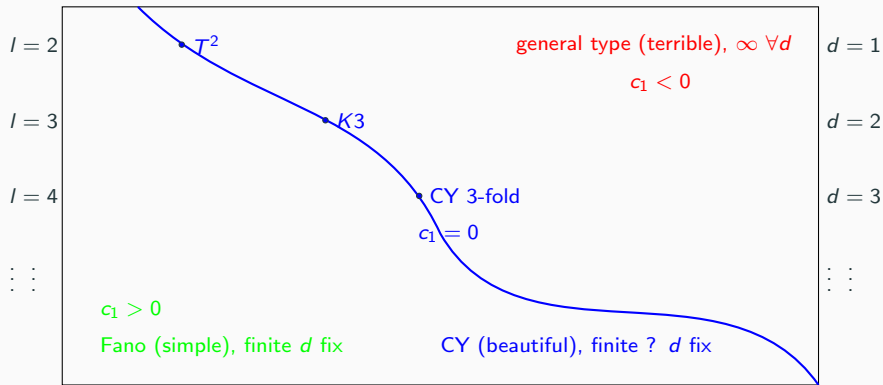


Additional Philosophical Remarks: Kodaira map of algebraic varieties

$$\begin{array}{ccccccc} \ell = 0 & \ell = 1 & \ell = 2 & \ell = 3 & \dots \\ g = 0 & g = 1 & g = 2 & g = 3 & \dots \end{array}$$



Additional Philosophical Remarks: Kodaira map of algebraic varieties



Monodromy and iteration over dimensions

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One-parameter K3 operators $\mathcal{L}^3(K3)$ of type $(1, 1, 1)$ are locally symmetric squares of operators associated with elliptic curves E :

$$\mathcal{L}^3(K3) = \mathrm{Sym}^2(\mathcal{L}^2(E))$$

Monodromy and iteration over dimensions

One-parameter CY fourfold operators $\mathcal{L}^5(M_4)$ of type $(1, 1, 1, 1, 1)$ are, up to a $\mathbb{Z}_2$ choice related to a lift to the spin representation, given by the nontrivial rank-five summand of

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In many slightly more complicated examples one has

$$\mathcal{L}^4(M_3) = \text{Hadamard}^{(2)} \mathcal{L}^2(E) .$$

For example, in the 5PM 1SF calculation all CY operators are obtainable from $\mathcal{L}^2(y^2 = x(x-1)(x-z))$ by the above constructions. In which sense do we need higher-dimensional CY period geometries?

A hint from Topological string theory

CY threefolds are special for topological field theory, since $F_g \neq 0$ for all g , due to $\dim_{vir} \overline{M}_g(hol. maps, M_3) = 0$ for all g .

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$$\begin{aligned} S^{zz} &= -\frac{1}{C} \left(\frac{w_{0,2}}{w_{0,1}} - q_{zz}^z \right), & \tilde{S}^z &= -\frac{1}{C} \left(\frac{w_{1,2}}{w_{0,1}} - q_{zz} \right), \\ \tilde{S} &= -\frac{1}{2C} \left(\frac{w_{1,3}}{w_{0,1}} - \partial_z q_{zz} - q_{zz} q_{zz}^z \right) + \frac{\partial_z C}{2C^2} \left(\frac{w_{1,2}}{w_{0,1}} - q_{zz} \right) - \frac{q_z^z}{2}. \end{aligned} \tag{8}$$

with $w_{i,j}$ minors of the Wronskian and $C, q_* \in \mathbb{Q}(\underline{z})$ [Gu, Kashani-Poor,](#)

[Marino, AK, e-Print: 2305.19916.](#)

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Maybe the topological string provides a toy explanation for the conundrum: why do higher-dimensional CY n -folds appear instead of higher-genus Riemann surfaces?