

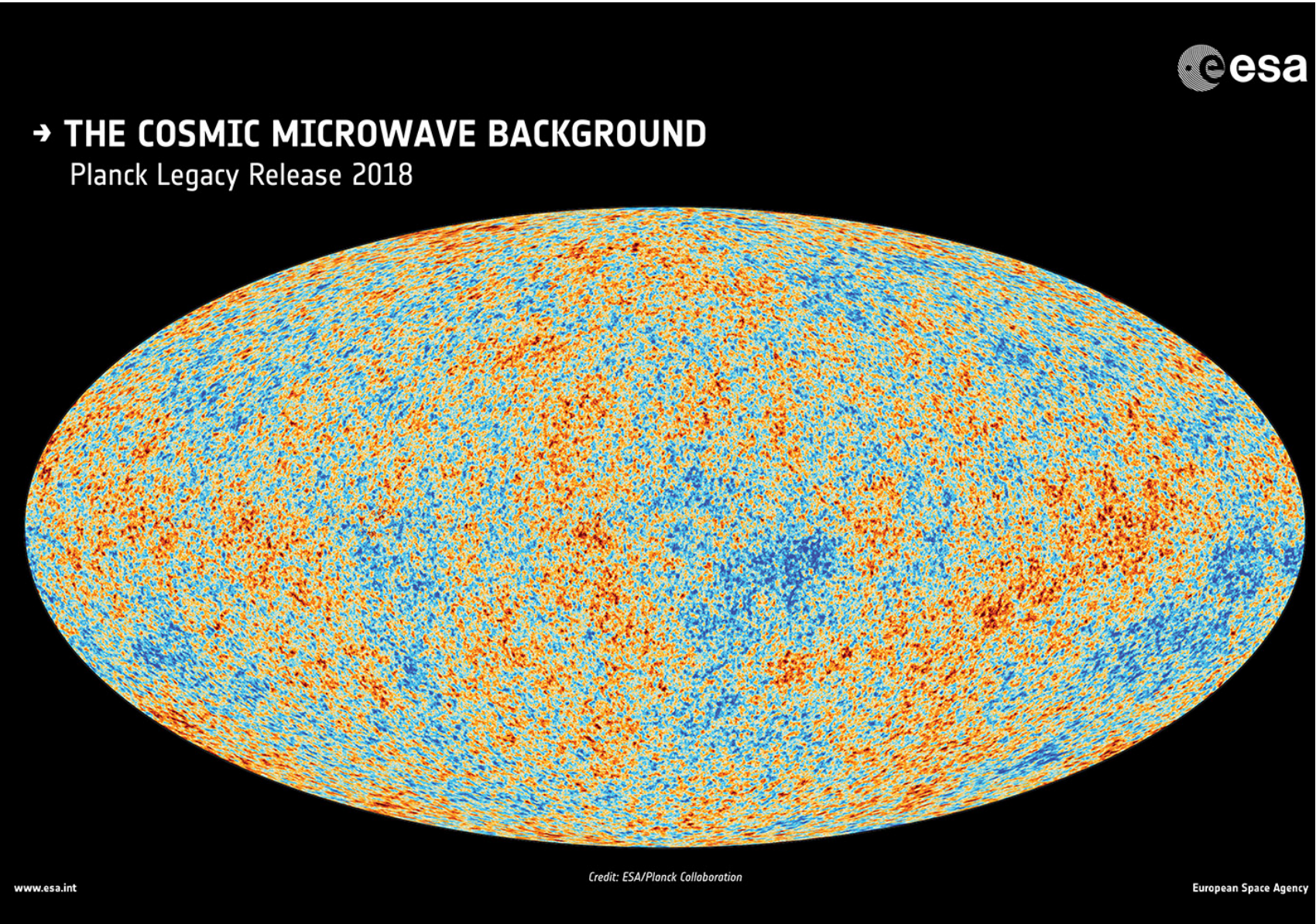
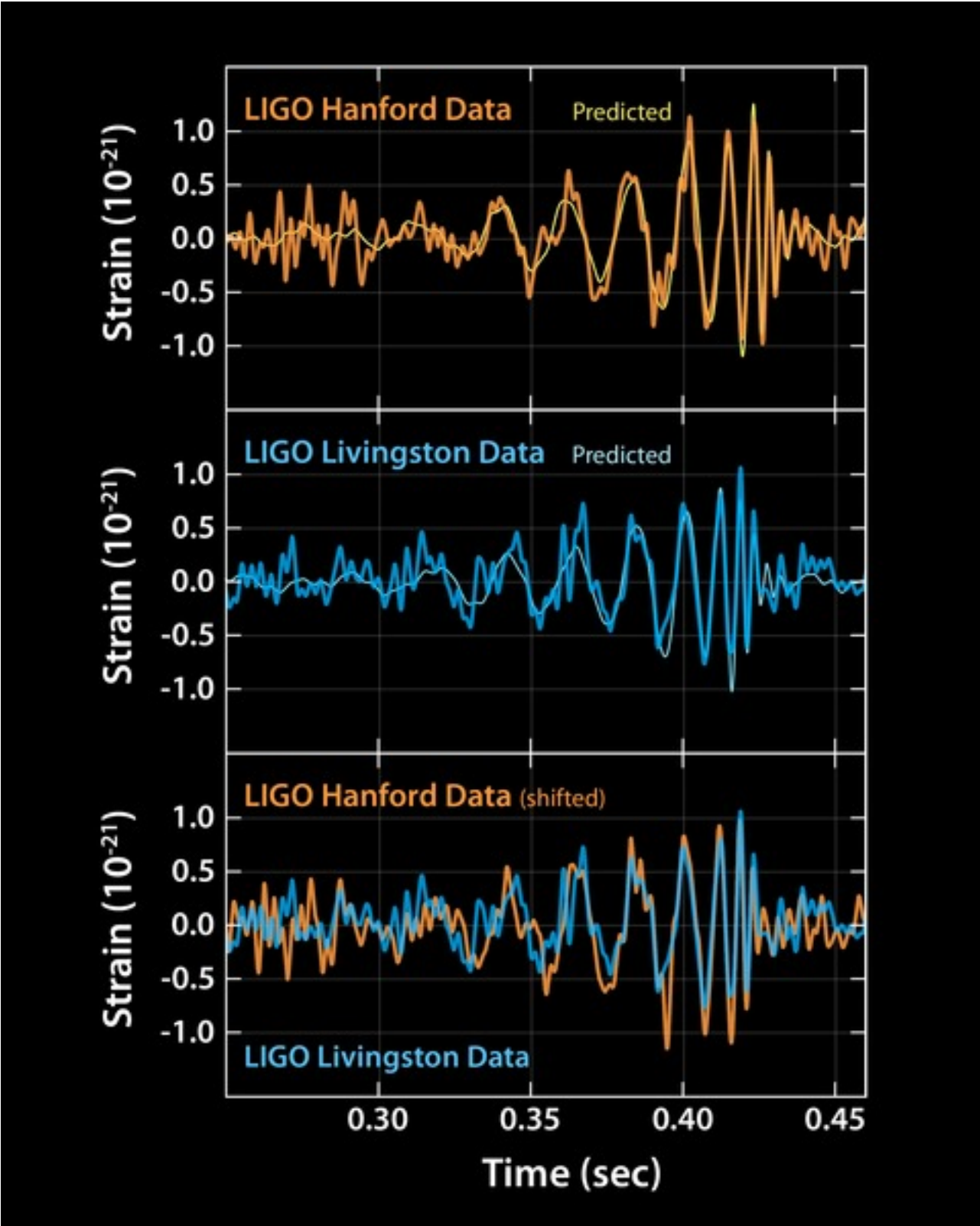
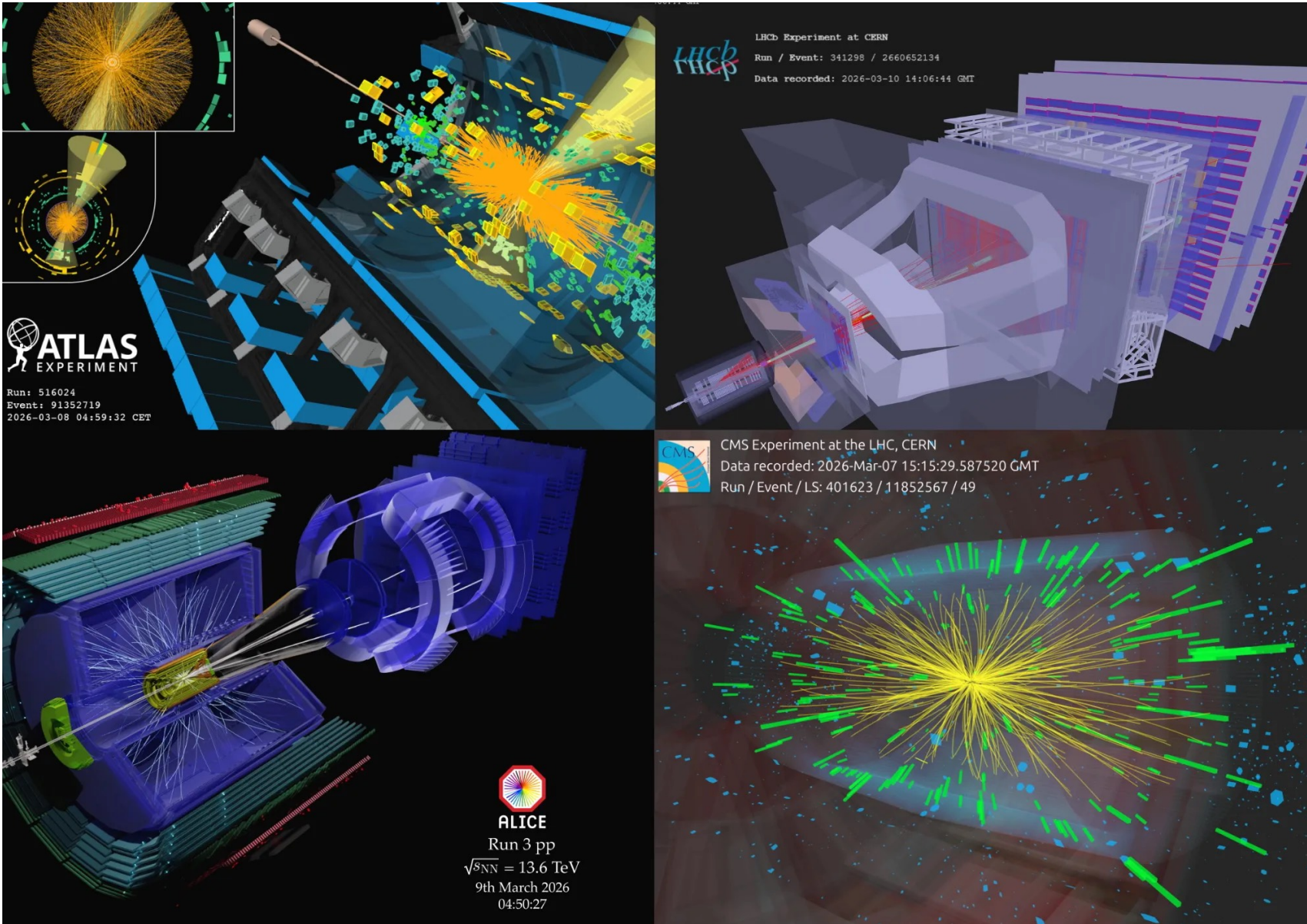
SPECIAL FUNCTIONS AND PHENOMENOLOGY

Elliptics and Beyond '26
Les Diablerets, September 7th 2026

Lorenzo Tancredi – Technical University Munich



PHENOMENOLOGY “TODAY” BEYOND COLLIDERS



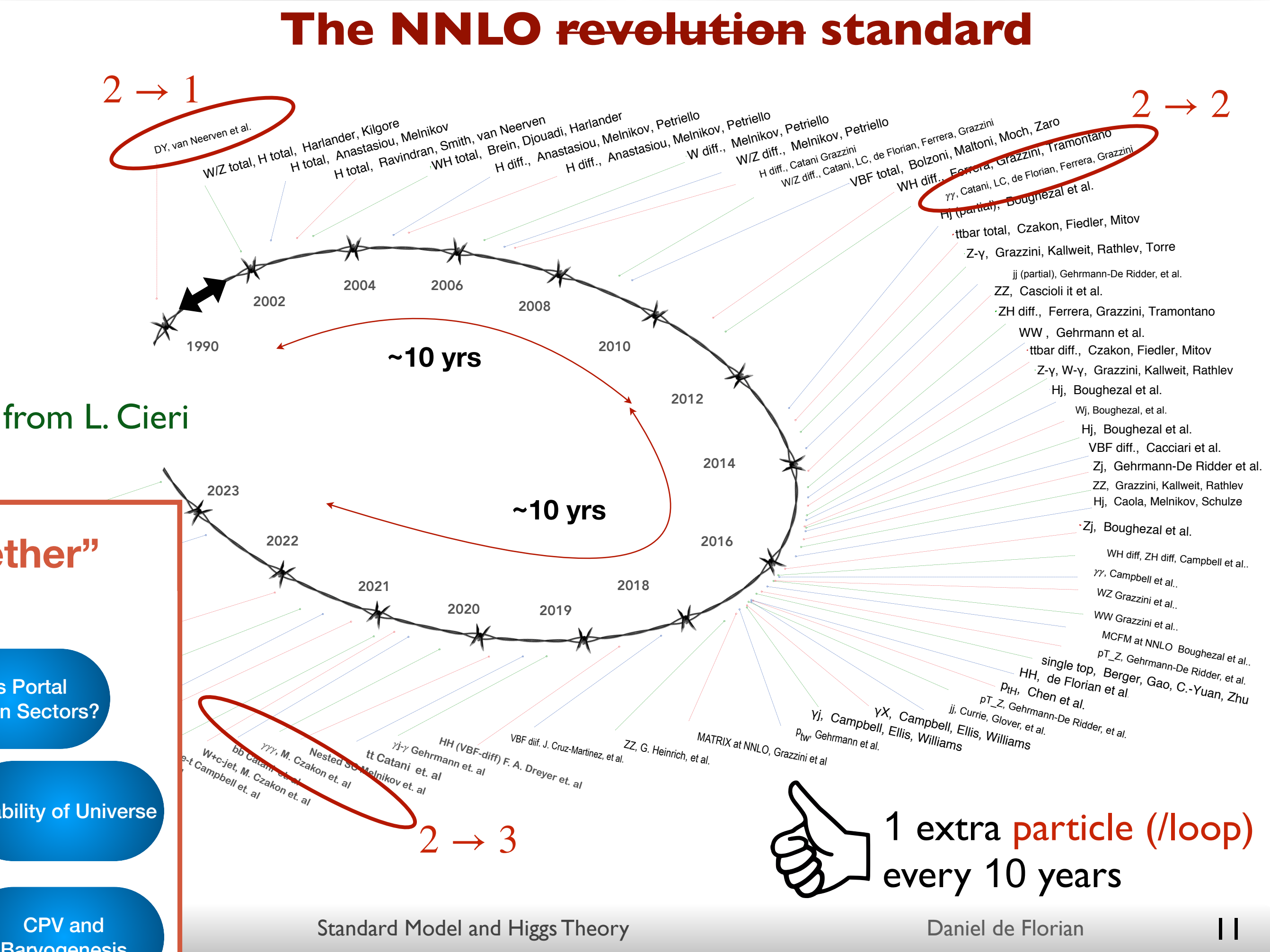
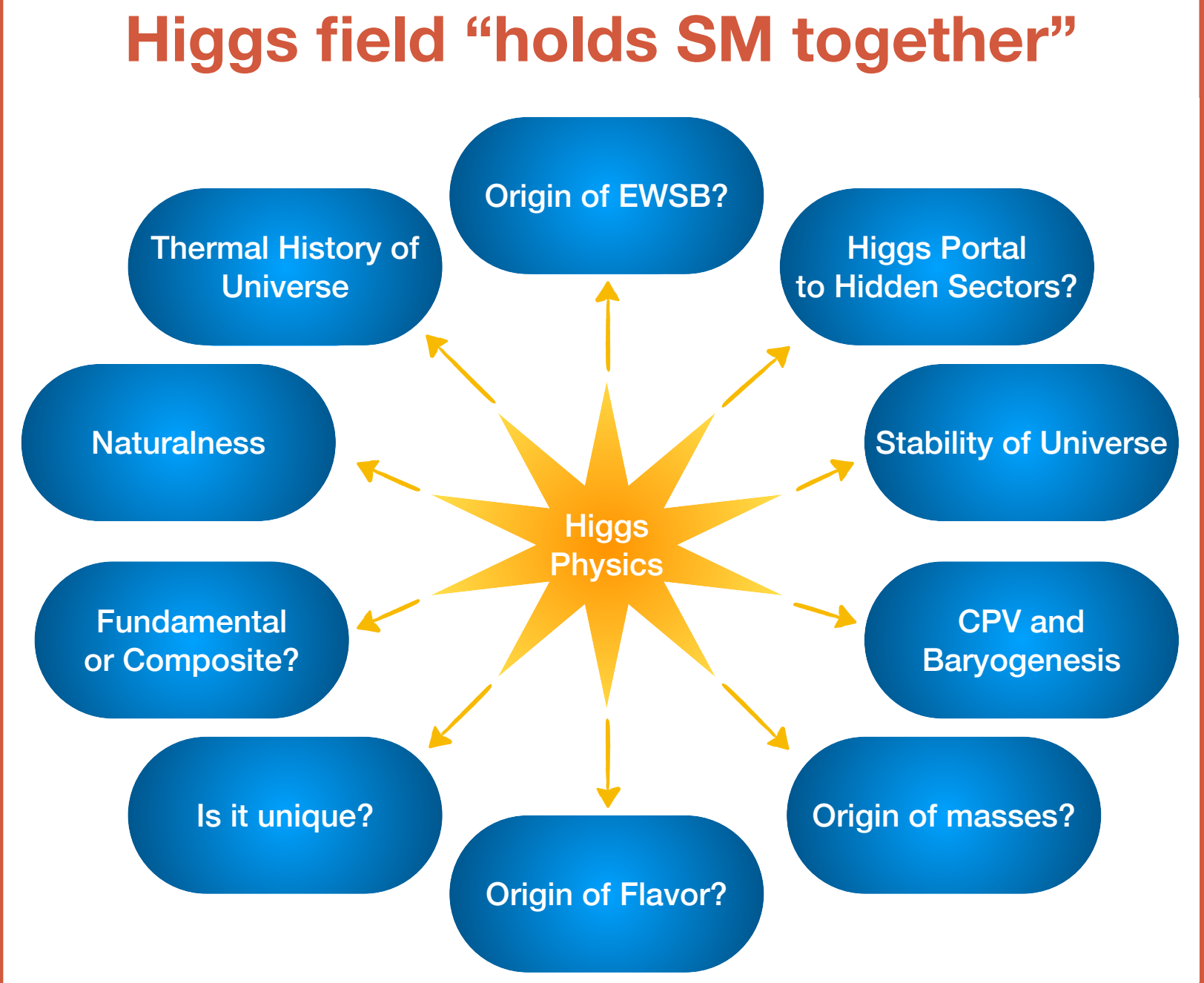
MULTISCALE AMPLITUDES

GRAVITATIONAL WAVE FORMS

COSMOLOGICAL CORRELATORS

PRECISION COLLIDER PARTICLE PHYSICS

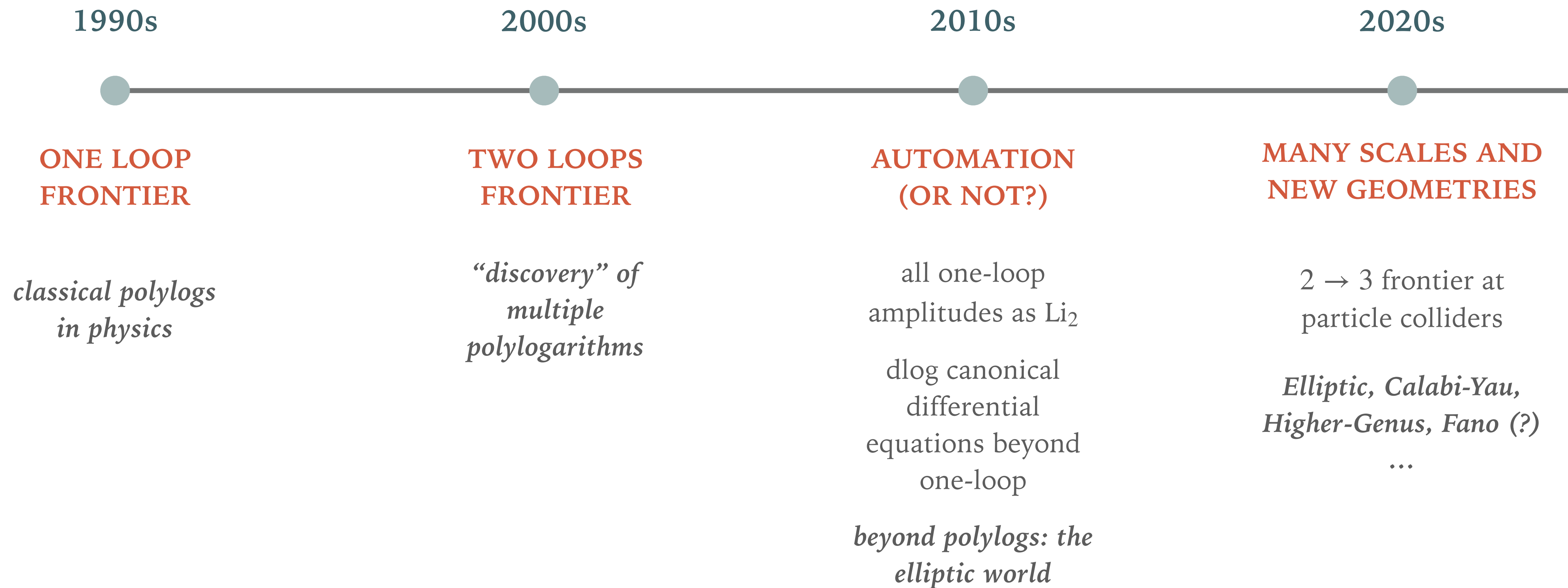
mass →	≈2.3 MeV/c ²	≈1.275 GeV/c ²	≈173.07 GeV/c ²	0	≈125 GeV/c ²
charge →	2/3	2/3	2/3	0	0
spin →	1/2	1/2	1/2	1	0
	u	c	t	g	H
	up	charm	top	gluon	Higgs boson
QUARKS	≈4.8 MeV/c ²	≈95 MeV/c ²	≈4.18 GeV/c ²	0	
	-1/3	-1/3	-1/3	0	
	1/2	1/2	1/2	1	
	d	s	b	γ	
	down	strange	bottom	photon	
LEPTONS	0.511 MeV/c ²	105.7 MeV/c ²	1.777 GeV/c ²	0	
	-1	-1	-1	0	
	1/2	1/2	1/2	1	
	e	μ	τ	Z	
	electron	muon	tau	Z boson	
LEPTONS	<2.2 eV/c ²	<0.17 MeV/c ²	<15.5 MeV/c ²		
	0	0	0		
	1/2	1/2	1/2		
	ν_e	ν_μ	ν_τ		
	electron neutrino	muon neutrino	tau neutrino		



Precision is essential for discoveries and exclusions ...

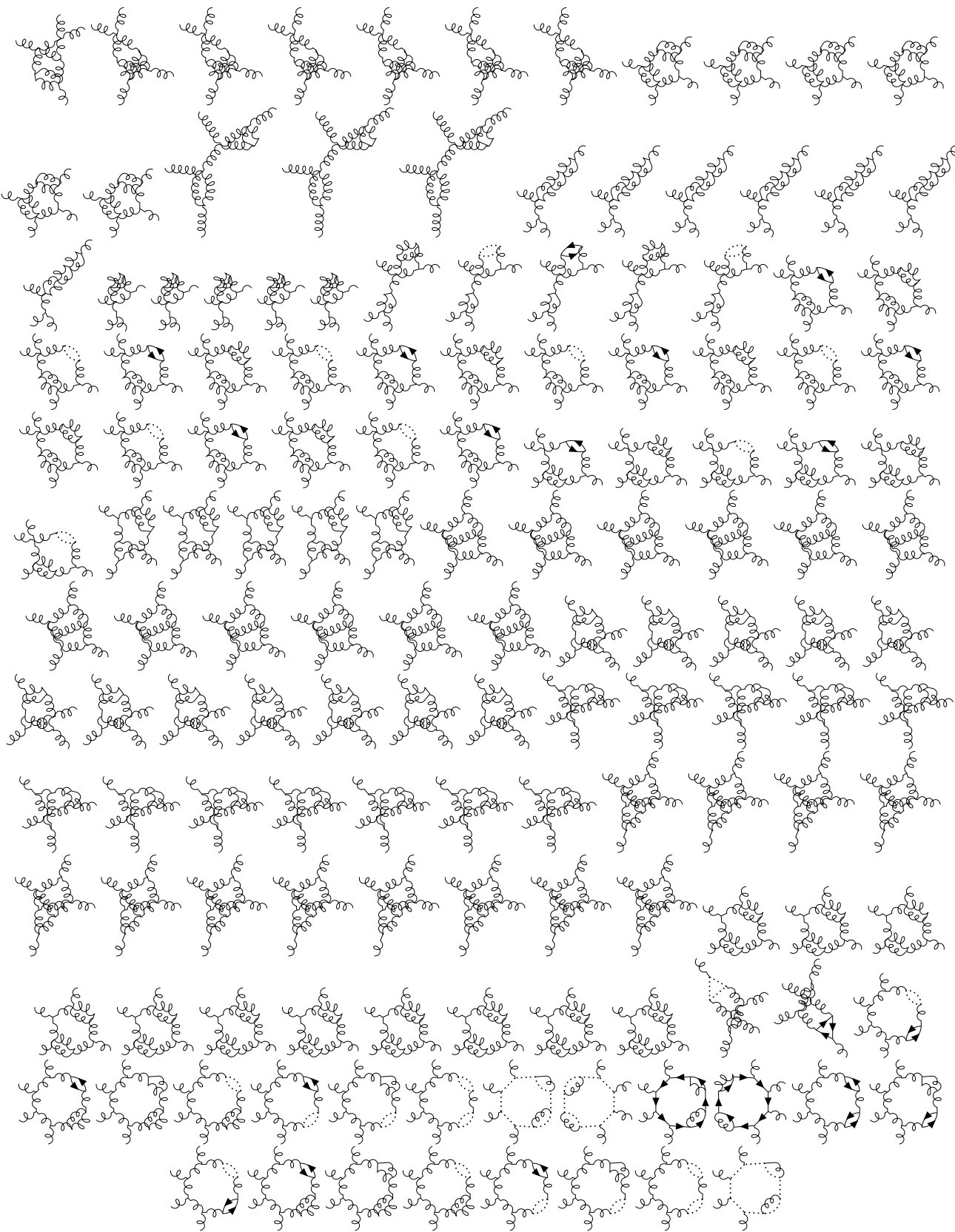
THE SPECIAL ROLE OF SPECIAL FUNCTIONS

Special function technology moved the perturbative frontier



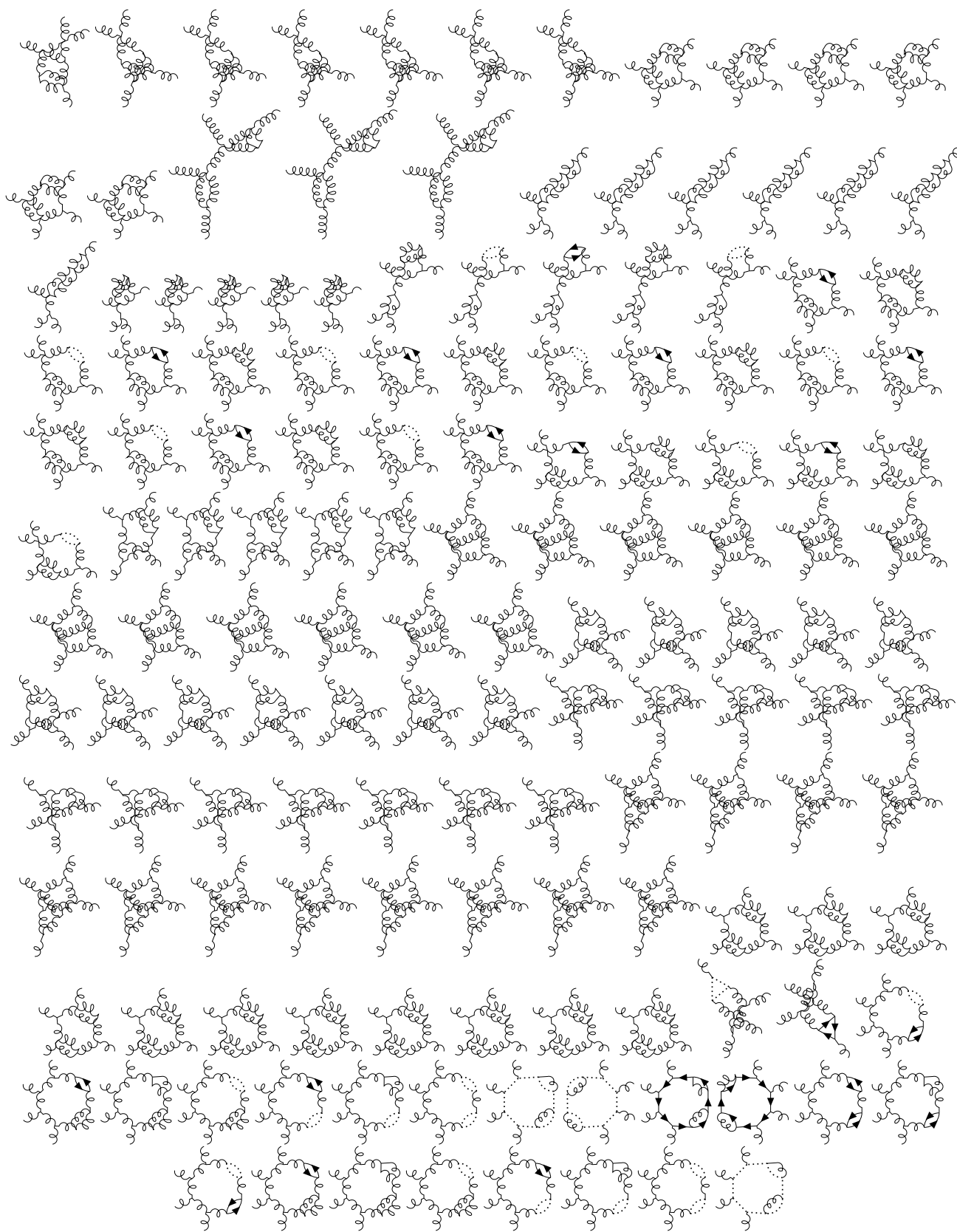
THE POLYLOG-PARADIGM: DIVIDE ET IMPERA

10^{4-7} integrals at 2-3 loops



THE POLYLOG-PARADIGM: DIVIDE ET IMPERA

10^{4-7} integrals at 2-3 loops $\longrightarrow$ Integration by Parts (IBPs)



$$\int \frac{\partial}{\partial k_i^\mu} \left(v^\mu \prod_{\ell=1}^L \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}} \right) = 0$$

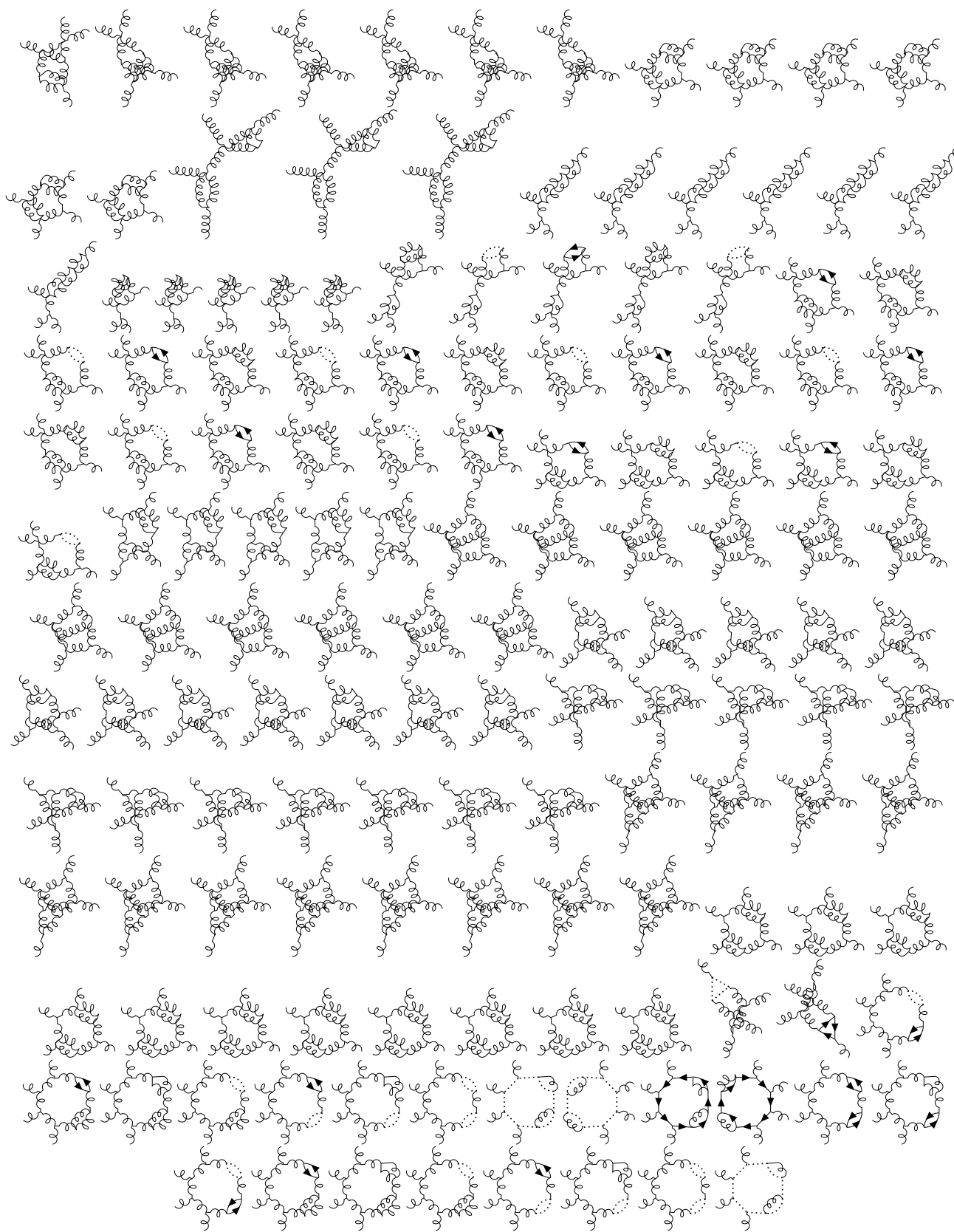
[Chetyrkin, Tkachov; Laporta; ...]

N master ints $\vec{I}$

$$I_j = \int \prod_{\ell=1}^L \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}}$$

THE POLYLOG-PARADIGM: DIVIDE ET IMPERA

10^{4-7} integrals at 2-3 loops $\longrightarrow$ Integration by Parts (IBPs) $\longrightarrow$ Special functions (polylogs)

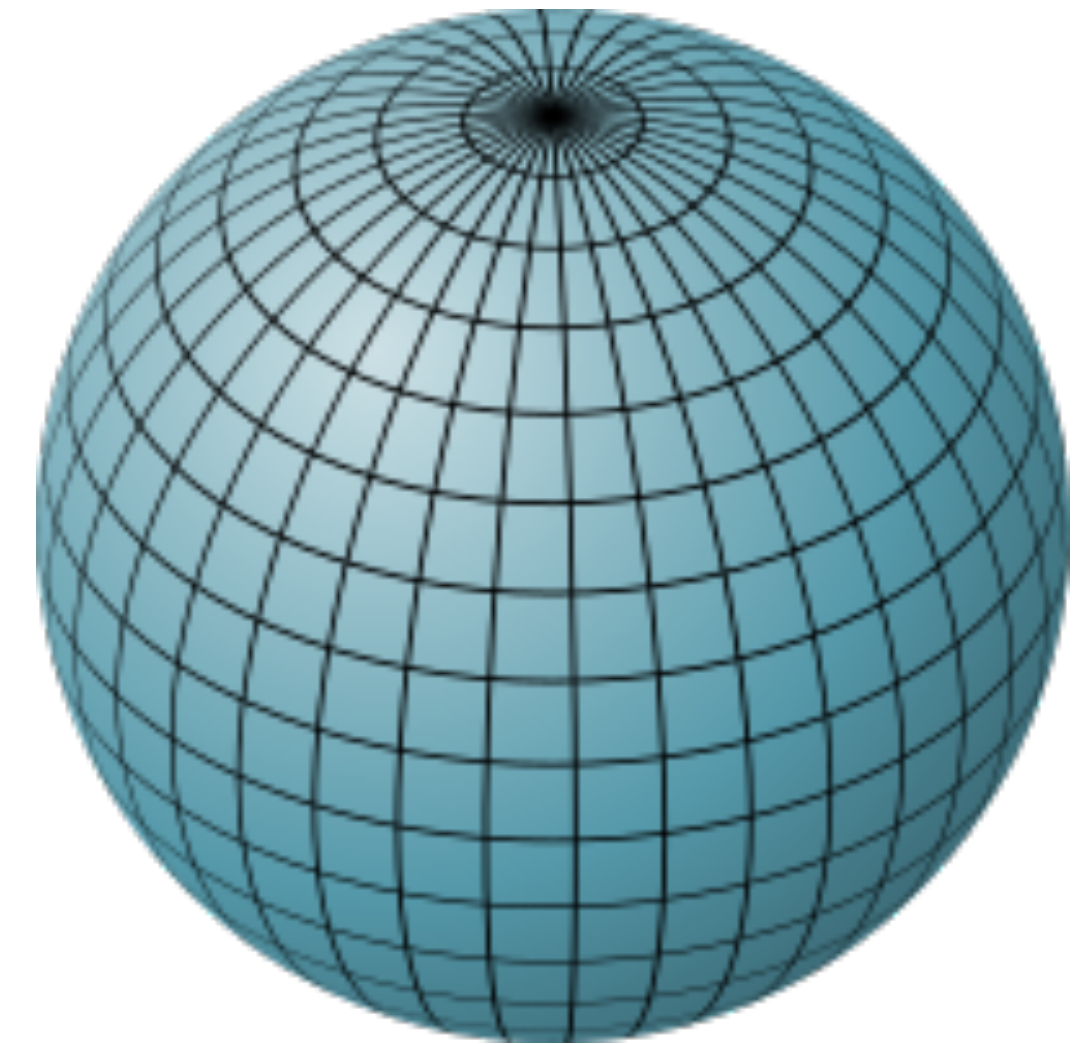


$$\int \frac{\partial}{\partial k_i^\mu} \left(v^\mu \prod_{\ell=1}^L \frac{d^D k_\ell}{(2\pi)^D} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_n^{b_n}} \right) = 0$$

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$$\begin{aligned} G(c_1, c_2, \dots, c_n, x) &= \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, \dots, c_n, t_1) \\ &= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \dots \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n} \end{aligned}$$

Decompose the problem into “*simple*” building blocks which naturally evaluate to “*good special functions*”

THE POLYLOG-PARADIGM: DIVIDE ET IMPERA

$$\mathcal{A} = \sum_{i=1}^N r_i f_i$$

r_i : **rational** (or algebraic)

poles from single particle
states going on-shell

f_i : **transcendental**

branch cuts from multiparticle
states going on-shell

good “special functions” should help manifest physical properties of observables

SPECIAL (POLYLOGARITHMIC) NUMBERS

This “idea”, first applied* in a systematic way in 1997 to reduce **3loop g-2 to 17 MIs**

$$a_e^{QED} = C_1 \left(\frac{\alpha}{\pi} \right) + C_2 \left(\frac{\alpha}{\pi} \right)^2 + C_3 \left(\frac{\alpha}{\pi} \right)^3 + C_4 \left(\frac{\alpha}{\pi} \right)^4 + C_5 \left(\frac{\alpha}{\pi} \right)^5 + \dots$$

$$C_3 = \begin{array}{c} \text{[Feynman diagrams: a bubble with two wavy lines, and two triangles with wavy lines]} \end{array} = \frac{83}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) + \frac{100}{3} \left[\left(\text{Li}_4 \left(\frac{1}{2} \right) + \frac{\ln^4 2}{24} \right) - \frac{\pi^2 \ln^2 2}{24} \right] \\ - \frac{239}{2160} \pi^4 + \frac{139}{18} \zeta(3) - \frac{298}{9} \pi^2 \ln 2 + \frac{17101}{810} \pi^2 + \frac{28259}{5184}$$

[Laporta, Remiddi '97]

* as far as I know...

POLYLOGARITHMIC AMPLITUDES: 3-JET PRODUCTION AT LEP

$$e^+(p_5) + e^-(p_6) \rightarrow \gamma^*(p_4) \longrightarrow q(p_1) + \bar{q}(p_2) + g(p_3)$$

[Gehrmann, Remiddi '99]
[Garland, Gehramnn, Glover, Koukoutakis, Remiddi '02]

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$$D_\alpha(y,z) =$$
$$\frac{1}{12y(1-z)}\mathrm{H}(0;z) - \frac{1}{12y(1-y-z)}\mathrm{H}(0;z) + \frac{z}{6(1-y-z)^2}\left[\frac{\pi^2}{6} + \mathrm{H}(0;z)\mathrm{G}(0;y) + \mathrm{H}(1,0;z) - \mathrm{G}(1,0;y)\right] + \frac{z}{4(1-y-z)}\left[-\frac{\pi^2}{3} + \mathrm{H}(0;z) - 2\mathrm{H}(0;z)\mathrm{G}(0;y) - 2\mathrm{H}(1,0;z) - \mathrm{G}(0;y) + 2\mathrm{G}(1,0;y)\right]$$
$$+ \frac{z^2}{4(1-y-z)^2}\left[-\frac{\pi^2}{6} - \mathrm{H}(0;z)\mathrm{G}(0;y) - \mathrm{H}(1,0;z) + \mathrm{G}(1,0;y)\right] - \frac{1}{12(1-y)}\mathrm{G}(0;y)$$
$$+ \frac{1}{72(1-z)^2}\left[\pi^2 + 25\mathrm{H}(0;z) - \frac{3}{2}\mathrm{H}(0;z)\mathrm{G}(0;y) - 9\mathrm{H}(0,0;z) + 6\mathrm{H}(1,0;z)\right]$$
$$+ \frac{1}{144(1-z)}\left[4\pi^2 + 38 + 37\mathrm{H}(0;z) - 6\mathrm{H}(0;z)\mathrm{G}(0;y) - 36\mathrm{H}(0,0;z) + 24\mathrm{H}(1,0;z) - 3\mathrm{G}(0;y)\right]$$
$$+ \frac{1}{12(1-y-z)}\left[\mathrm{H}(0;z) + 2\mathrm{G}(0;y)\right] + \frac{\pi^2}{72}\left[\frac{395}{12} - \mathrm{H}(0;z) + 2\mathrm{G}(1-z;y) - \mathrm{G}(0;y) - 2\mathrm{G}(1;y)\right]$$
$$- \frac{19}{36}\zeta_3 + \frac{1}{144}\left[\frac{3661}{18} - 25\mathrm{H}(0;z) + 24\mathrm{H}(0;z)\mathrm{G}(1-z,0;y) + 29\mathrm{H}(0;z)\mathrm{G}(0;y) - 36\mathrm{H}(0;z)\mathrm{G}(0,0;y) + 6\mathrm{H}(0;z)\mathrm{G}(1,0;y) - 28\mathrm{H}(0,0;z) - 36\mathrm{H}(0,0;z)\mathrm{G}(0;y) - 6\mathrm{H}(0,1,0;z) + 40\mathrm{H}(1,0;z) + 24\mathrm{H}(1,0;z)\mathrm{G}(1-z;y) - 30\mathrm{H}(1,0;z)\mathrm{G}(0;y) - 36\mathrm{H}(1,0,0;z) - 24\mathrm{G}(1-z,1,0;y) + 53\mathrm{G}(0;y) - 82\mathrm{G}(0,0;y) + 6\mathrm{G}(0,1,0;y) - 40\mathrm{G}(1,0;y) + 36\mathrm{G}(1,0,0;y) + 24\mathrm{G}(1,1,0;y)\right]$$
$$+ i\pi\left\{\frac{1}{8(1-z)^2}\mathrm{H}(0;z) + \frac{1}{8(1-z)}\left[1 + 2\mathrm{H}(0;z)\right] + \frac{1}{48}\left[\frac{8\pi^2}{3} - \frac{28}{3} + 13\mathrm{H}(0;z) + 12\mathrm{H}(0;z)\mathrm{G}(0;y) + 12\mathrm{H}(1,0;z) + 31\mathrm{G}(0;y) - 12\mathrm{G}(1,0;y)\right]\right\}$$

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f_i

r_i

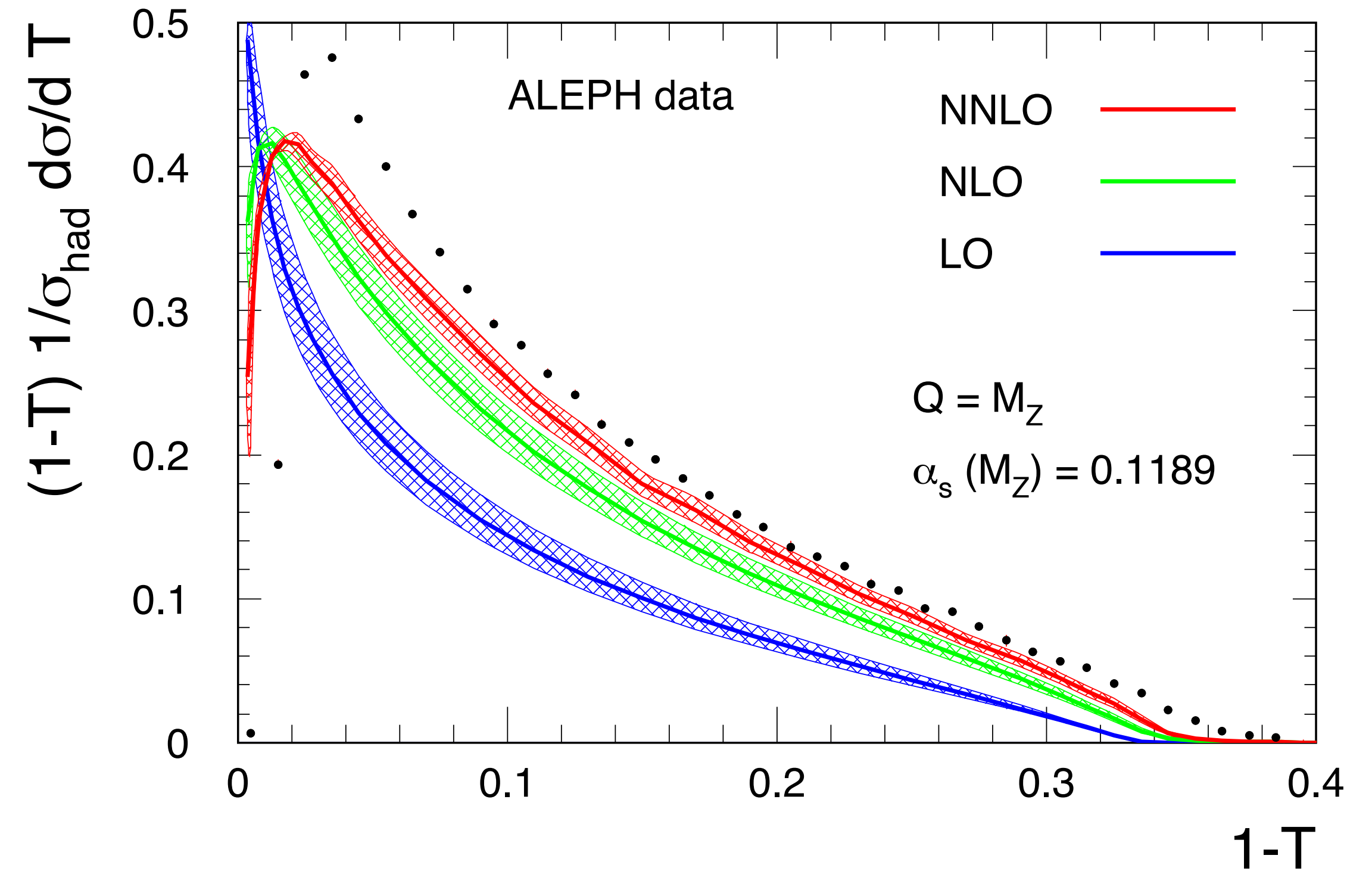
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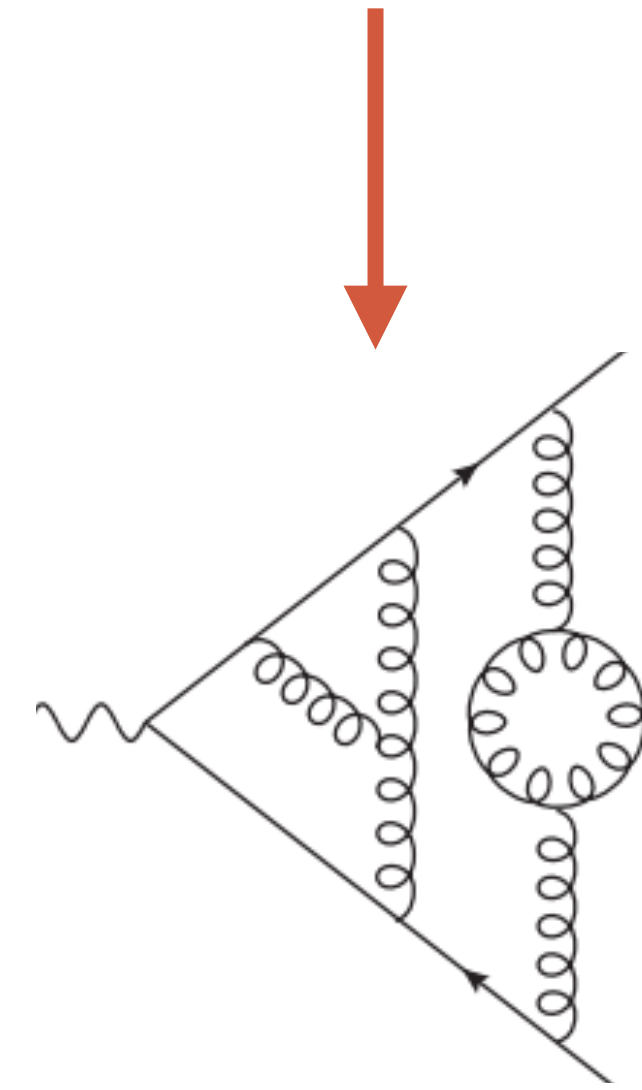
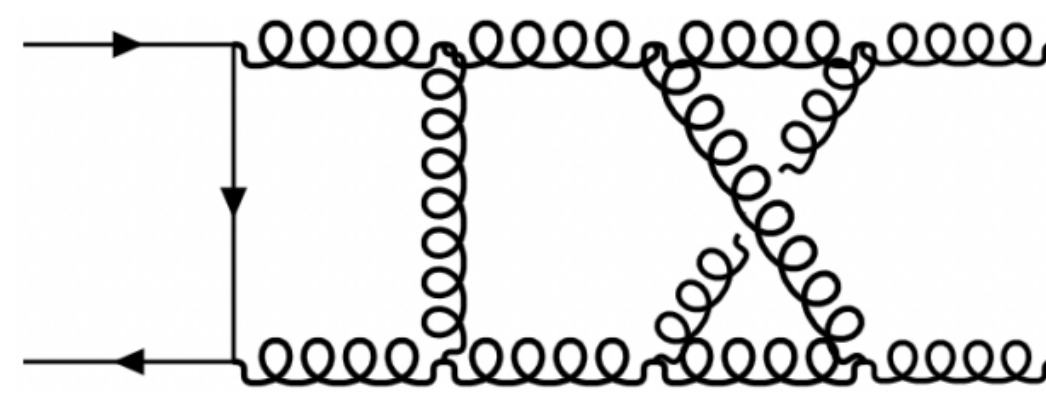
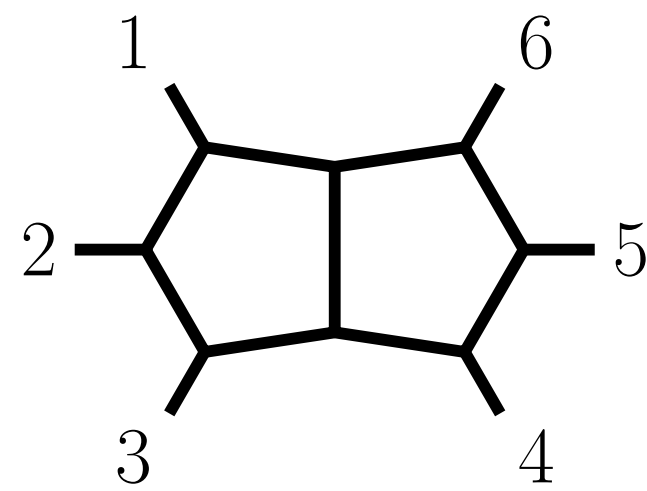
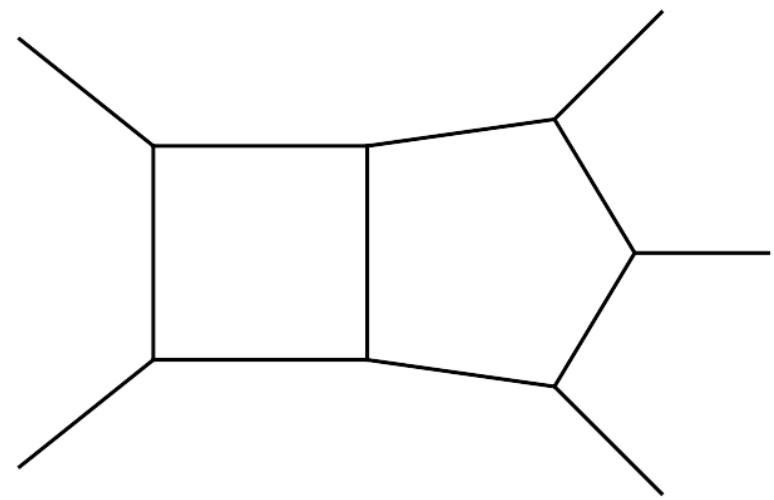
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 \end{aligned}$$



[Dissertori, Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, Stenzel '08]

SUCCESSSES OF POLYLOGS: STATE OF THE ART

Since then, things have changed a lot! State-of-the-art: **2 loop 5 & 6 point** - **3 loop 4 point** - **4 loop 3 point**



Abreu, Agarwal, Badger, Buccioni, Chawdhry, Chicherin, Czakon, de Laurentis, Febres-Cordero, Gambuti, Gehrmann, Henn, Ita, Lo Presti, Manteuffel, Ma, Mitov, Monni, Page, Peraro, Pochelet, Schabinger, Sotnikov, LT, Usovitch, Zhang, ...

Bargiela, Bobadilla, Canko, Caola, Jakubcik, Gambuti, Gehrmann, Guan, Ruf, Henn, Lim, Mella, Mistlberger, Wasser, Manteuffel, Syrrakos, Smirnov, LT ...

Henn, Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Stainhauser, Gehrmann, Sotnikov, Moch, Herzog, Yang...

Crucial: **massless** internal and **mostly massless** external legs

THE “DIFFERENTIAL” POINT OF VIEW

differential equations for Feynman integrals

POLYLOGS AND DIFFERENTIAL EQUATIONS

Due to integration by parts identities, master integrals satisfy linear systems of rational differential equations

[Kotikov '93] [Remiddi '97] [Gehrmann, Remiddi '99]

$$\frac{\partial}{\partial s_{ij}} \vec{I} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & 0 & \\ & & & & * \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & \ddots \\ & & & & & * \end{pmatrix} \vec{I}$$

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↓

rational functions of s_{ij}, m_i^2, D

lower triangular shape
coupled blocks in general non-trivial

essential (mysterious?) simplifications in the limit $D \rightarrow 4$

CANONICAL BASES IN THE POLYLOG CASE

Turn non-trivial “transport problem” into “trivial” iteration in the dimensional regulator $D = 4 - 2\epsilon$

[Kotikov '10] [Arkani-Hamed et al '10] [Henn '13]

$$d\vec{I} = \epsilon \left[\begin{array}{c} \epsilon\text{-indep} \end{array} \right] \vec{I} \quad \longrightarrow \quad \left[\begin{array}{c} \epsilon\text{-indep} \end{array} \right] = \sum_i B_i \omega_i$$

differential forms ω_i contain full information to all ϵ orders

Basis choice effectively reorganizes perturbation theory

The analytic problem becomes a controlled recursion in ϵ

PHYSICS

boundary data & connection matrix

note that $\epsilon = 0$ is physics!

CANONICAL BASES IN THE POLYLOG CASE

In the polylog case, differential forms are dlogs: pure single poles [Kotikov '10] [Arkani-Hamed et al '10] [Henn '13]

$$d\vec{I} = \epsilon \left[\epsilon\text{-indep} \right] \vec{I}$$

$$\left[\epsilon\text{-indep} \right] = \sum_i B_i d \log f_i$$

logarithmic singularities manifest in diff equations

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PURITY

result is unipotent: $\frac{d}{dx} \ln(x - a) = \frac{1}{x - a}$

TRANSCENDENTAL WEIGHT

uniform weight order by order in ϵ

PHYSICS

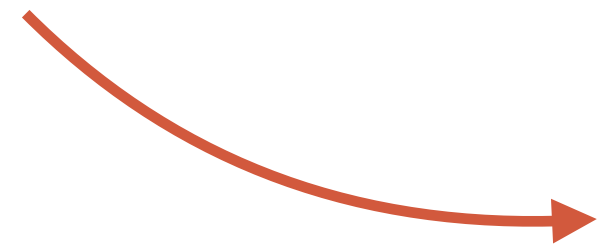
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PURITY: THE IMPORTANCE OF BEING UNIPOTENT

Solution in terms of iterated integrals over d-logs to all orders in ϵ are **unipotent**

$$I^{(n)}(f_1, \dots, f_n; x) = \int_{\gamma} d \log f_n \dots d \log f_1$$

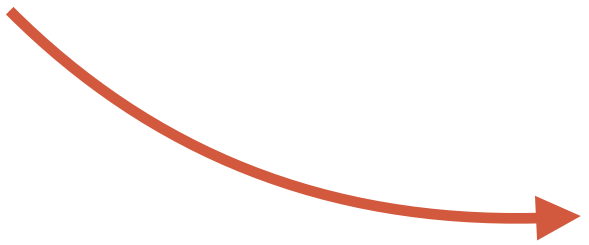


$$d I^{(n)}(f_1, \dots, f_n; x) \propto \sum_k d \log c_k \underbrace{I^{(n-1)}(f_{k_1}, \dots, f_{k_{n-1}}; x)}_{n-1}$$

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$$I^{(n)}(f_1, \dots, f_n; x) = \int_{\gamma} d \log f_n \dots d \log f_1$$


$$d I^{(n)}(f_1, \dots, f_n; x) \propto \sum_k d \log c_k \underbrace{I^{(n-1)}(f_{k_1}, \dots, f_{k_{n-1}}; x)}_{n-1}$$

Unipotency central property for definition of special functions: **multiple polylogarithms**

$$\begin{aligned} G(c_1, c_2, \dots, c_n, x) &= \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, \dots, c_n, t_1) \\ &= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} \dots \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n} \end{aligned}$$

Functions with **unipotent monodromy**
on the Riemann sphere

[E.E.Kummer 1840]

[Nielsen, Goncharov, Remiddi, Vermaseren] ...

Numerics: ... [Vollinga, Weinzierl 2004] ...

Result is trivially **MPLs** if f_i are all rational

HOW DO WE FIND THESE BASES?

integrand analysis, single poles and leading singularities

INTEGRAND ANALYSIS IN THE DLOG CASE

Starting from a **right basis at $\epsilon = 0$** (integrands in dlog form) $J_k = R_k \int_{\gamma} \mathrm{d} \log X_m \dots \mathrm{d} \log X_1$

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Diff equations become

$$d\vec{J} = (A + \epsilon B) \vec{J}$$

Solution matrix is **not unipotent** at $\epsilon = 0$

Its derivative contains an homogeneous term $A \neq 0$

Upon solving homogeneous equation at $\epsilon = 0$ $d\vec{J}_H = A\vec{J}_H$

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Equivalent to dividing out integrals by residues R_k : these residues are called **“leading singularities”**

[Cachazo '08] [Arkani-Hamed et al '10] [Henn '13] [Henn, Mistlberger, Smirnov, Wasser '20]

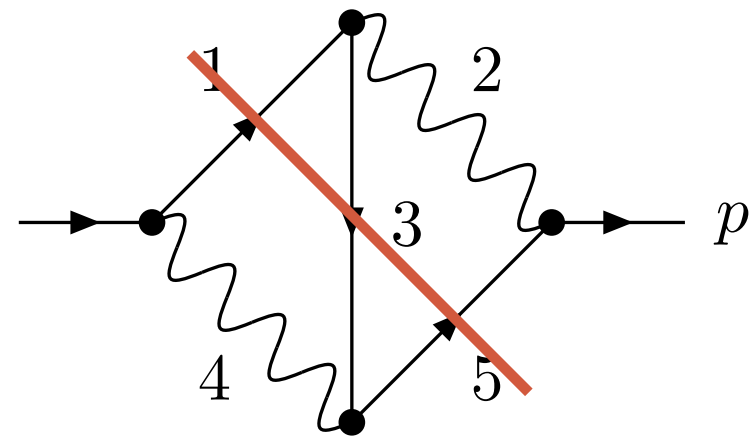
Integrals with *unit leading singularities* satisfy **unipotent differential equations** [Forner, Mella, Nega, Tancredi, Wagner '26]

FROM POLYLOGS TO ELLIPTICS (AND BEYOND)

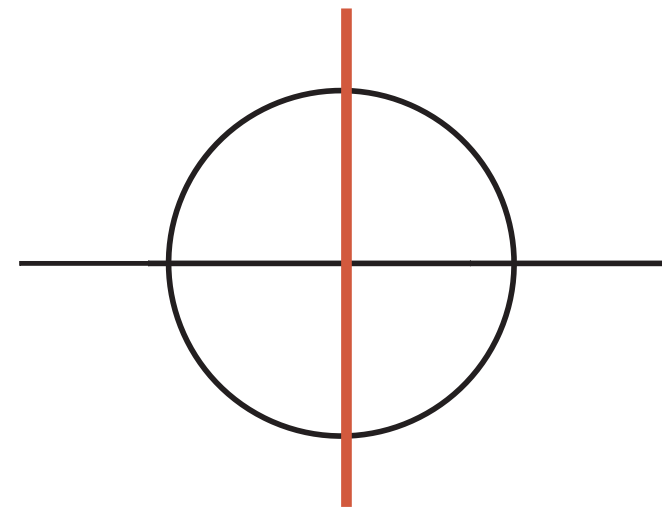
from unipotency and ϵ recursion to special functions?

ELLIPTIC FEYNMAN GRAPHS

Electron self-energy in QED @ 2 loops [first computation attempted in 1961 by A. Sabry]



The electron propagator in QED; A. Sabri 1962



$$= \frac{1}{\sqrt{(3m - \sqrt{s})(\sqrt{s} + m)^3}} K \left(\frac{16m^3 \sqrt{s}}{(3m - \sqrt{s})(\sqrt{s} + m)^3} \right)$$

The sunrise integral

It's imaginary part “contains” an elliptic integral

$$K(x) = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - x z^2)}}$$

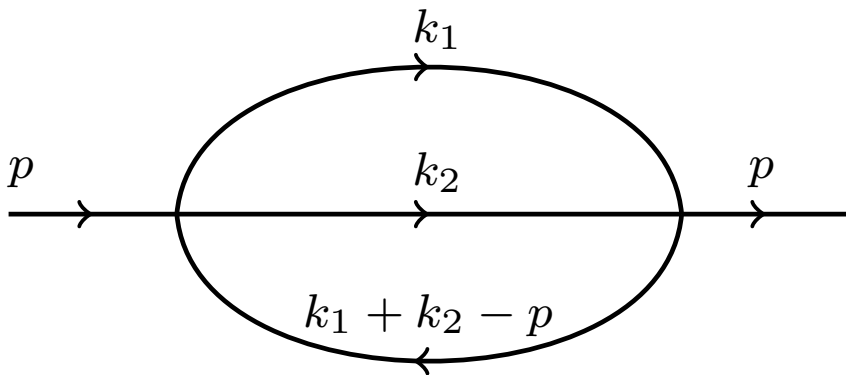
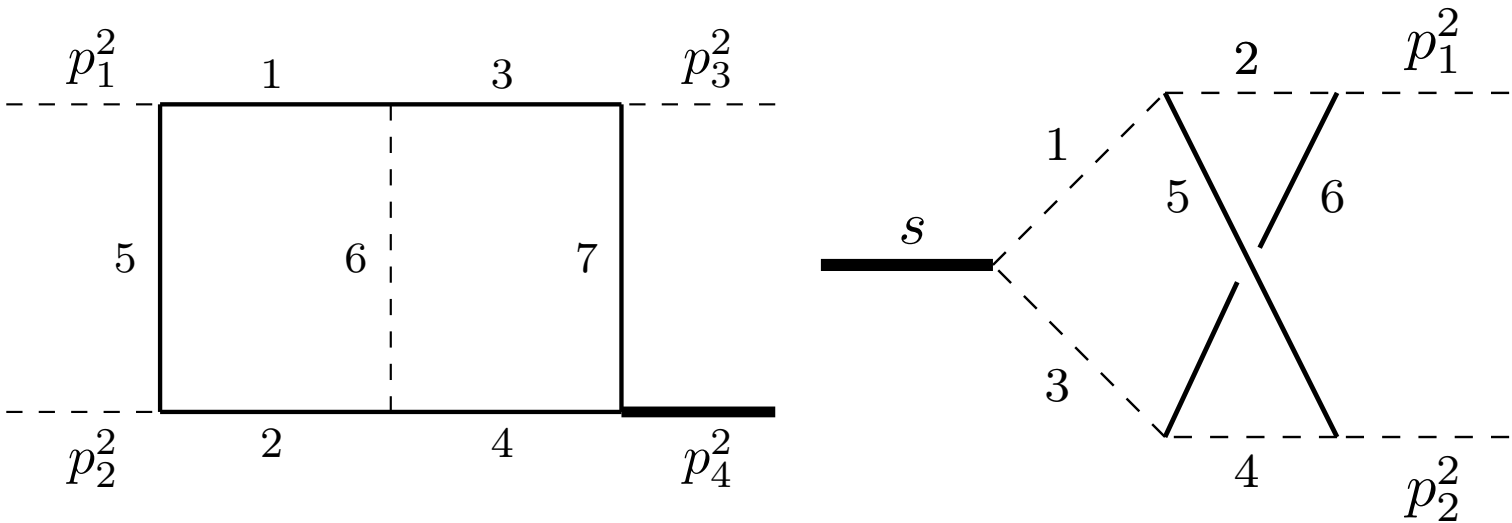
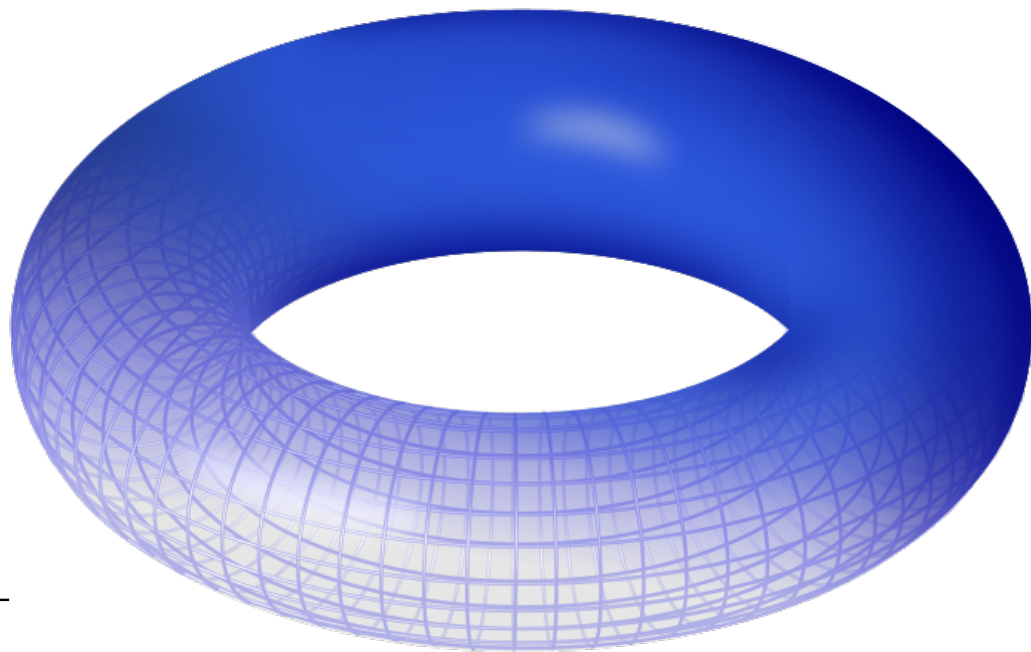
BEYOND ELLIPTICS IN GENUS AND DIMENSION

see [Bargiela, Frellesvig, Marzucca, Morales, Seefeld, Wilhelm, Yang '25] for full classification at 2 loops

Higgs physics

Top physics

EW physics



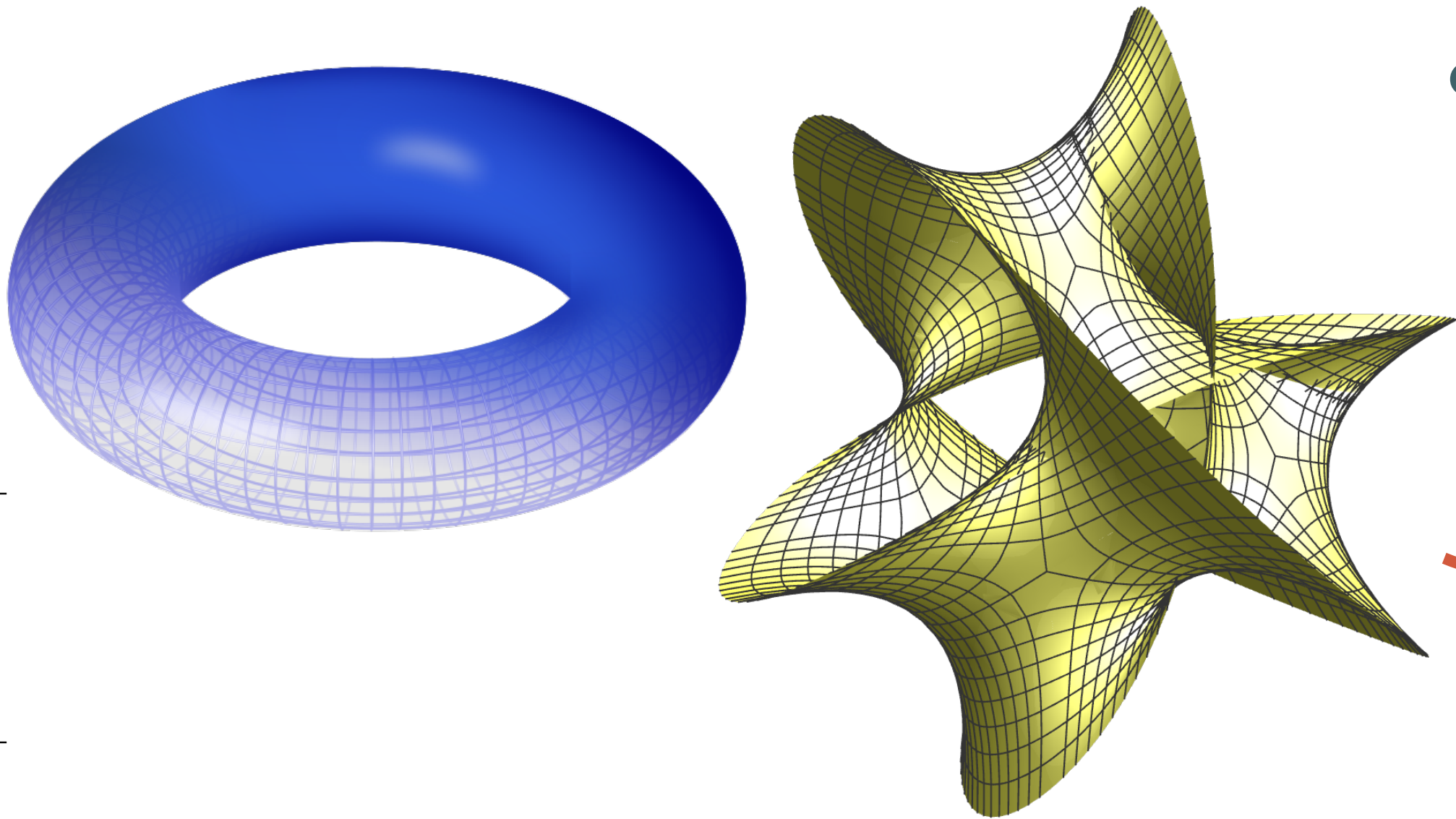
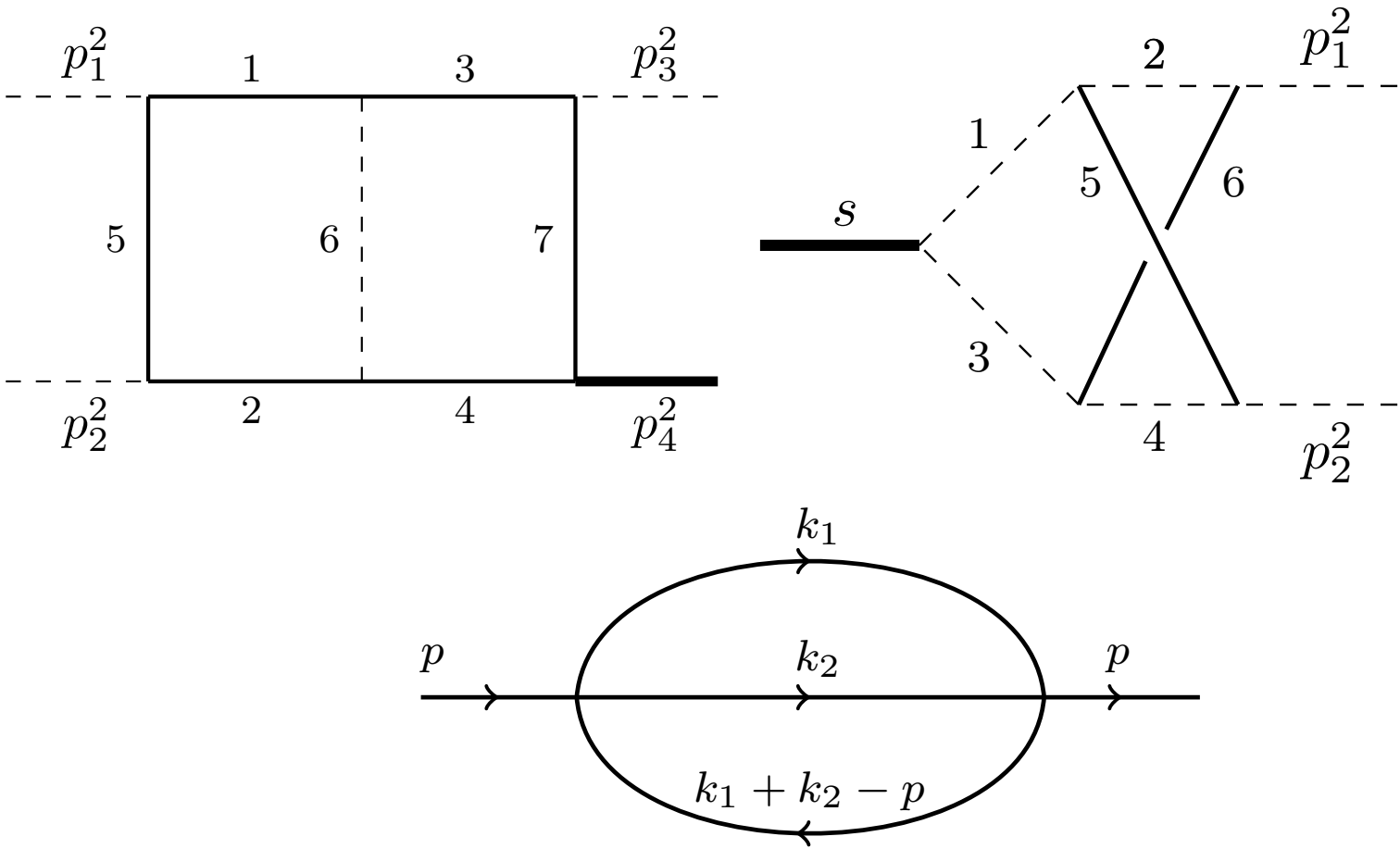
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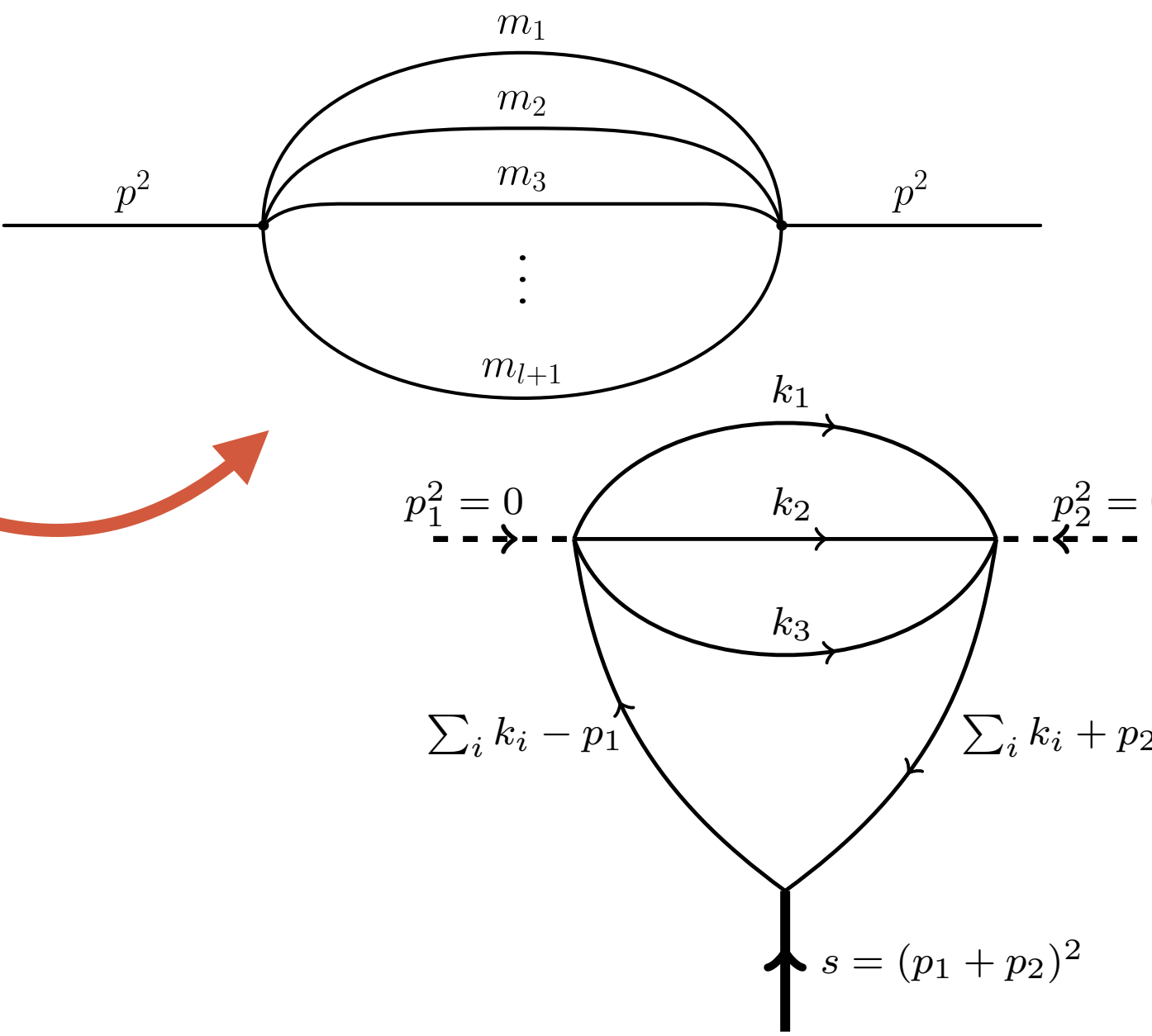
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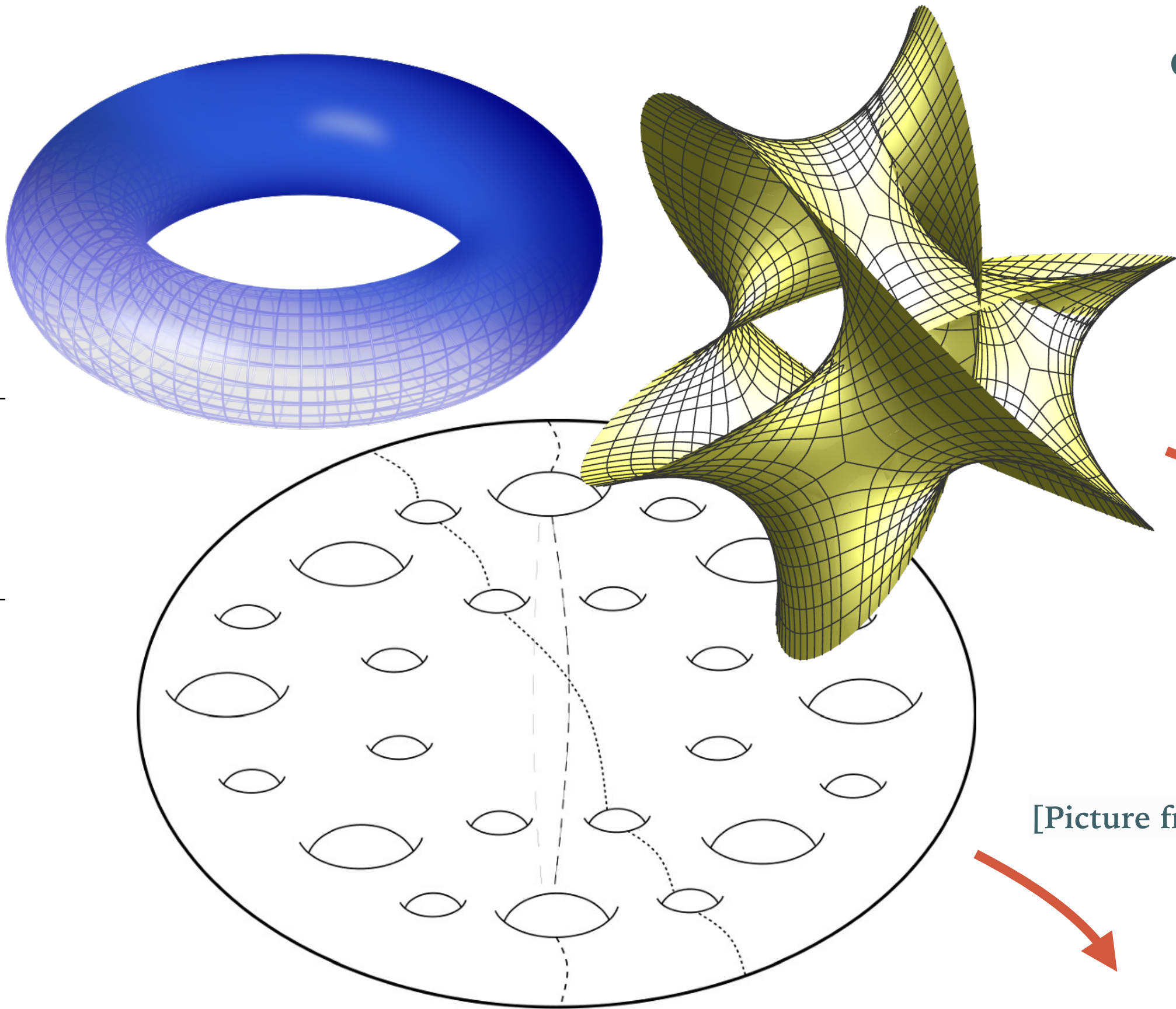
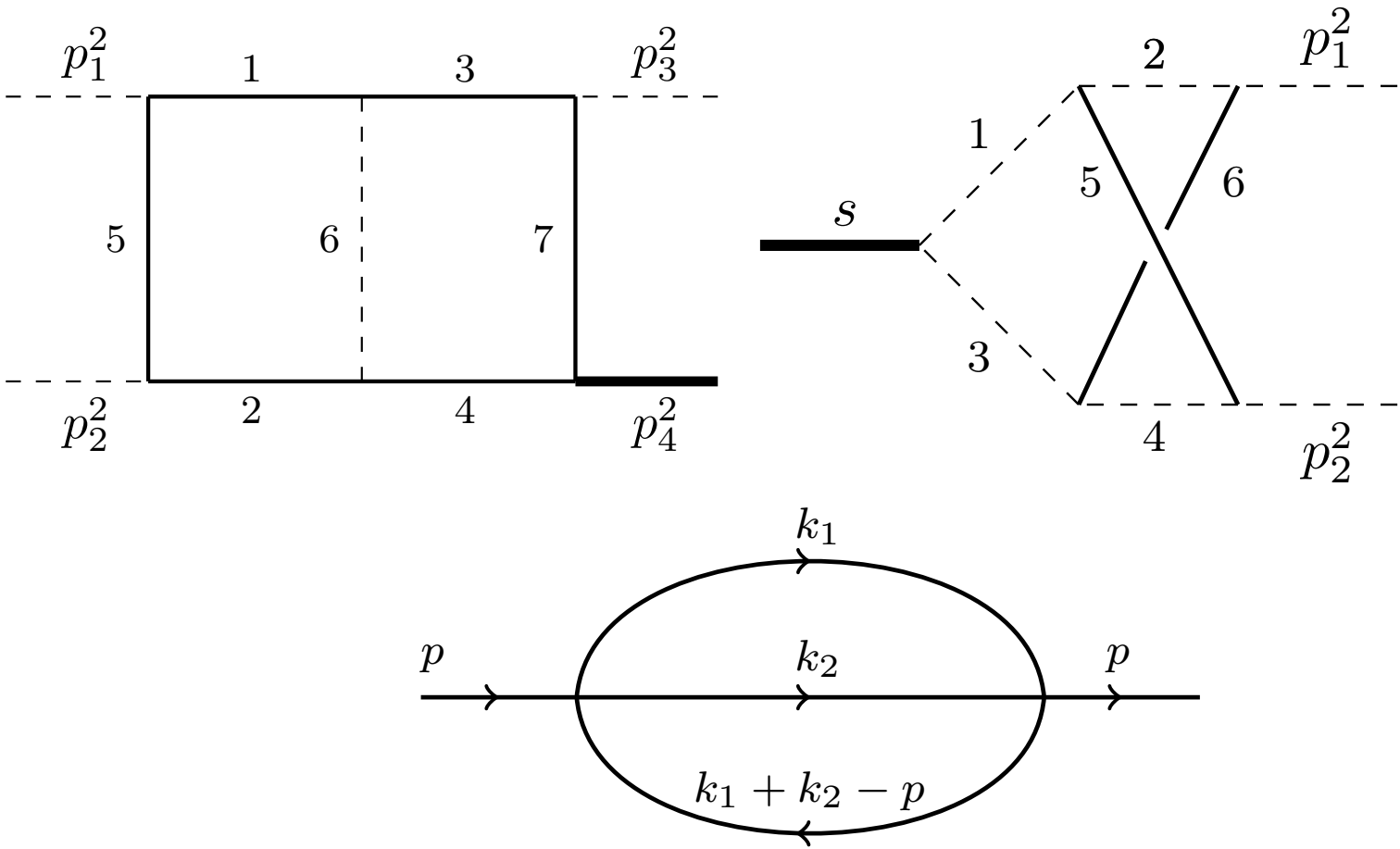
QED propagator, EW corrections



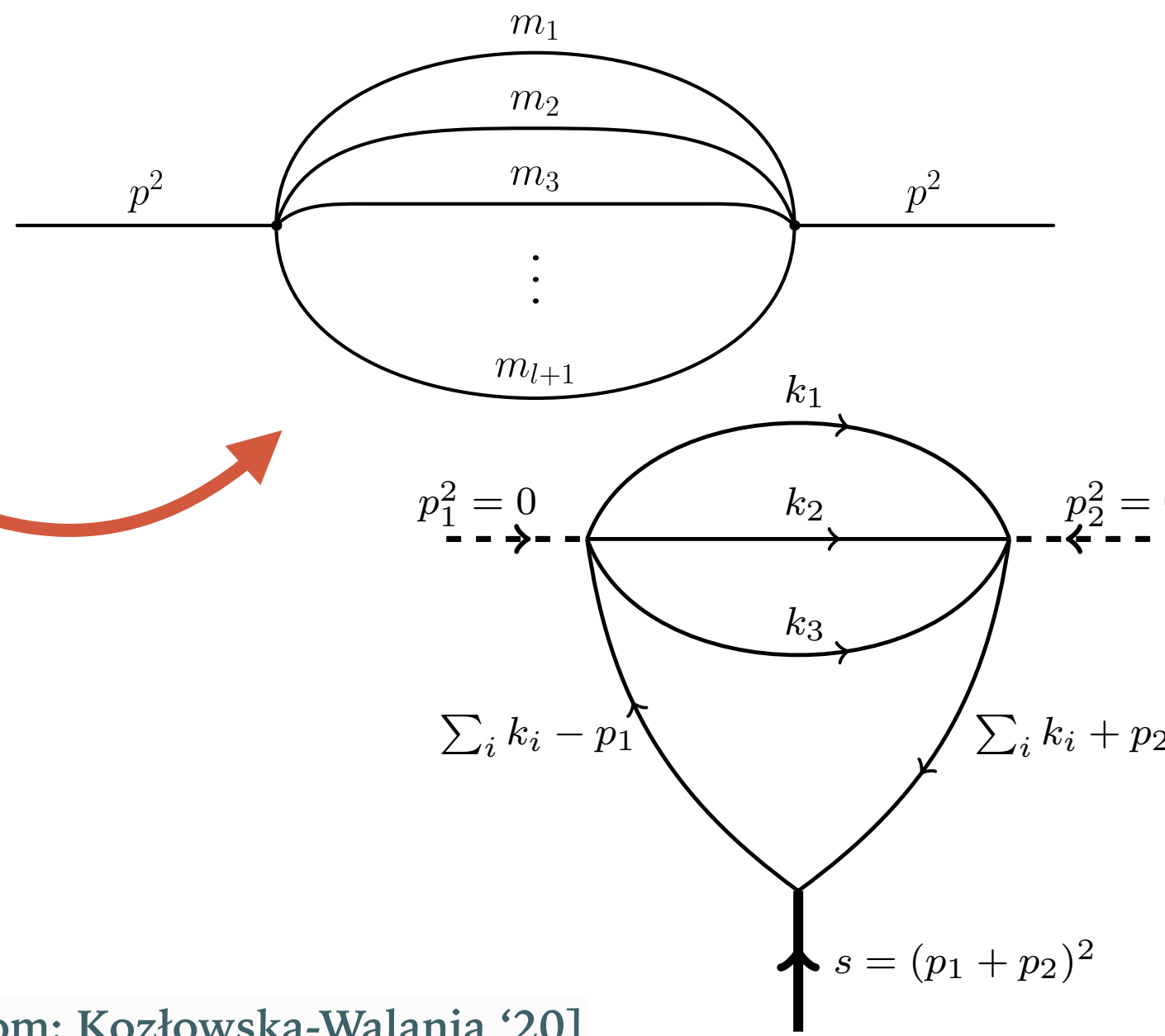
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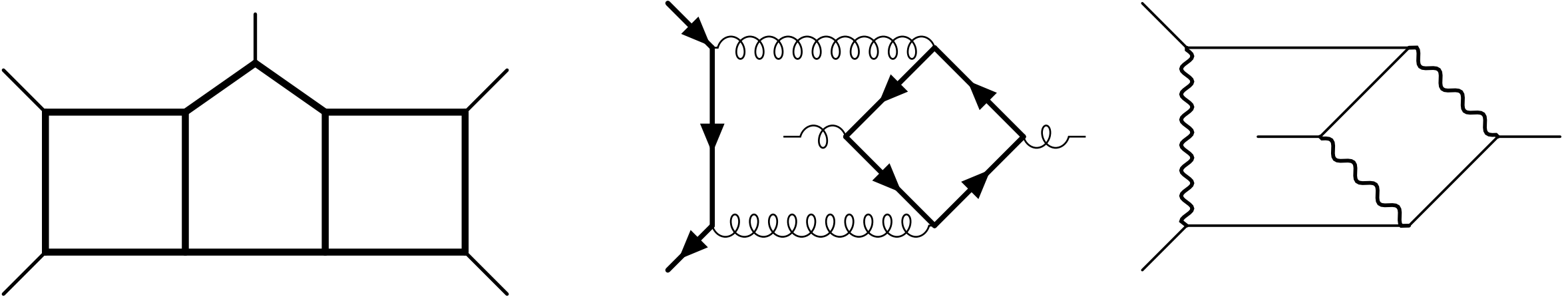


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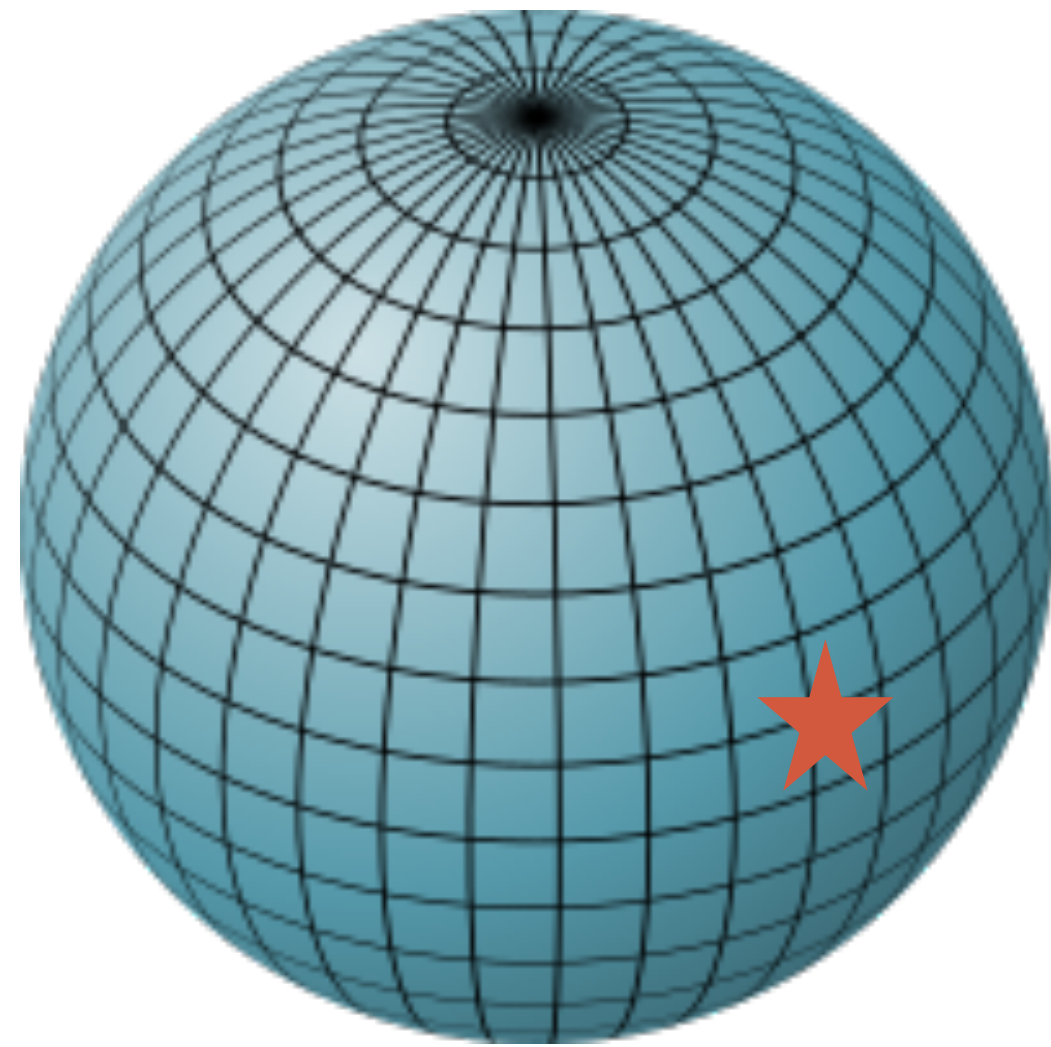


[Picture from: Kozłowska-Walania '20]

EW corrections
multi-particle scattering
 $t\bar{t} + X$ production



COHOMOLOGY BEYOND THE RIEMANN SPHERE

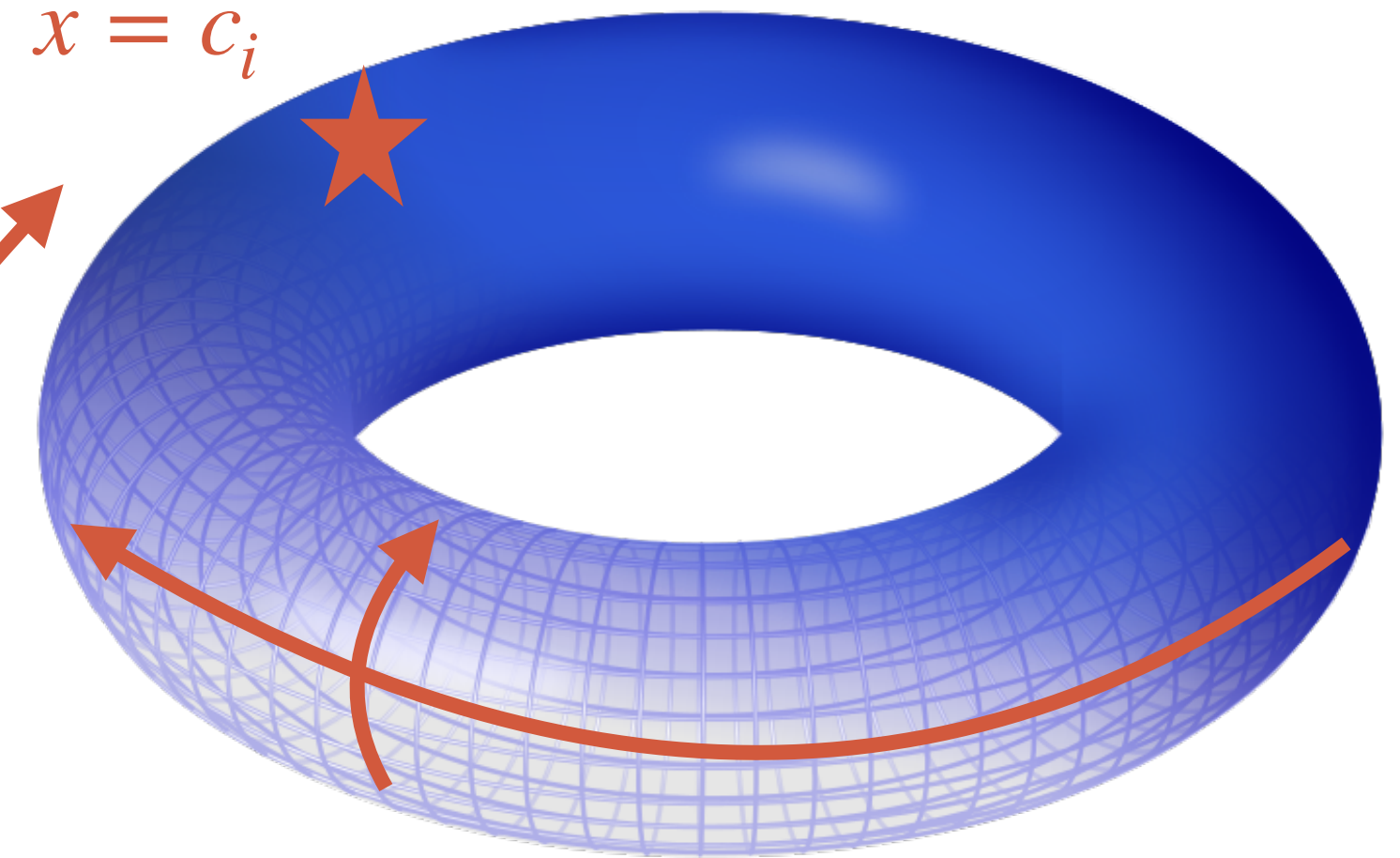


$x = a$

entire space of functions spanned by single poles

$$\log(1 - x/a) = \int_0^x \frac{dt}{t - a}$$

$x = c_i$



genus 1, elliptic curve; $y = \sqrt{P_3(x)}$

First kind

No poles $\omega \sim \int \frac{dx}{y}$

Second kind

double poles $\eta \sim \int \frac{dx x}{y}$

Third kind

single poles $g \sim \int \frac{dx}{(x - c_i)y}$

DIFFERENTIAL EQUATIONS AND GEOMETRY

$d\vec{J} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & 0 & \\ & & & & * \\ * & & & & \\ & & & & \ddots \\ & & & & & * \end{pmatrix} \vec{J}$

Solution of each hom “block” at $\epsilon = 0$ can be related to periods of some manifold

DIFFERENTIAL EQUATIONS AND GEOMETRY

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Solution of each hom “block” at $\epsilon = 0$ can be related to periods of some manifold

In a typical multiloop problem, different manifolds appear:

- dlogs** (possibly with roots)
- (multiple) **elliptic** curves
- different **CY** or **higher genus**
-

Geometry of the full problem is *(potentially)* much more complicated than individual blocks

Is it a problem? Differential equations nicely help “*separate*” blocks and reveal **unipotent part!**

ELLIPTIC CASE: A TOY EXAMPLE

$$I_n^C(a, \epsilon) = \frac{1}{\sqrt{\pi}} \int_C dx \frac{x^n}{\sqrt{x(1-x)(1-ax)}} (x(1-x)(1-ax))^\epsilon, \quad 0 < a < 1, \quad n \in \mathbb{Z}$$

Elliptic curve $y^2 = x(1-x)(1-ax)$

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Elliptic curve $y^2 = x(1-x)(1-ax)$ Two master integrals: $I_0(a, \epsilon), \quad I_1(a, \epsilon)$

$$\frac{\partial}{\partial a} \begin{pmatrix} I_0(a, \epsilon) \\ I_1(a, \epsilon) \end{pmatrix} = \begin{pmatrix} \frac{1}{2(1-a)} + \frac{\epsilon}{1-a} & -\frac{1}{2(1-a)} - \frac{3\epsilon}{1-a} \\ \frac{1}{2a(1-a)} + \frac{\epsilon}{a(1-a)} & -\frac{2-a}{2a(1-a)} - \frac{(2+a)\epsilon}{a(1-a)} \end{pmatrix} \begin{pmatrix} I_0(a, \epsilon) \\ I_1(a, \epsilon) \end{pmatrix}$$

Coupled system of differential equations as $D \rightarrow 4$, i.e. $\epsilon \rightarrow 0$

FROM LS TO SEMISIMPLE-UNIPOTENT SPLITTING

Period matrix: solution at $\epsilon = 0$ is not unipotent

$$W(a) = \begin{pmatrix} \varpi_0 & \varpi_1 \\ \varpi'_0 & \varpi'_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \varpi_0 & 0 \\ \varpi'_0 & \frac{\Delta(a)}{\varpi_0} \end{pmatrix}}_{W_{\text{ss}}} \underbrace{\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}}_{W_{\text{u}}},$$

$$W_{\text{ss}}^{-1} = \begin{pmatrix} \frac{1}{\varpi_0} & 1 \\ -\frac{\varpi'_0}{\Delta(a)} & \frac{\varpi_0}{\Delta(a)} \end{pmatrix}$$

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Rotating away *inverse semi-simple part of the period matrix*

[Broedel, Duhr, Dulat, Tancredi '18] [Görge, Nega, Tancredi, Wagner '23]

[Duhr, Maggio, Nega, Tancredi, Wagner '25]

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix} \begin{pmatrix} \frac{1}{\varpi_0} & 0 \\ -\frac{\varpi'_0}{\Delta(a)} & \frac{\varpi_0}{\Delta(a)} \end{pmatrix} \begin{pmatrix} I_1 \\ \partial_a I_1 \end{pmatrix}$$

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Rotation by the inverse semi-simple matrix *defines a basis with unit leading singularities* [Forner, Mella, Nega, Tancredi, Wagner '26]

See talks by Christoph and Felix

UNIPOTENT CANONICAL DIFFERENTIAL EQUATIONS

Choosing second integral with unit leading singularities to $\mathcal{O}(\epsilon^0)$, K_2 , diff equations become *ϵ -factorized*

$$\frac{\partial}{\partial a} \begin{pmatrix} J_1 \\ K_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{1}{a(1-a)} & -\frac{1}{a(1-a)\varpi_0^2} \\ -\frac{(1-a+a^2)\varpi_0^2}{a(1-a)} & -\frac{1}{a(1-a)} \end{pmatrix} \begin{pmatrix} J_1 \\ K_2 \end{pmatrix}$$

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In elliptic case, we can do it at the integrand level since the *generalization of “dlog”-forms are known* [L.L. Yang and Y. Zhang ’25]
[Forner, Mella, Nega, Tancredi, Wagner ’26]

→ *differential forms of pure eMPLs* [Brown, Levin ’11][Broedel, Matthes, Schlotterer ’14, ’15]
[Broedel, Duhr, Dulat, Penante, Tancredi ’17, ’18]

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Can be achieved even without knowing pure kernels both at integrand and diff eq level [Görge, Nega, Tancredi, Wagner ’23]
[Duhr, Maggio, Nega, Sauer, Tancredi, Wagner ’25]

Equivalent step-by-step approach by epsilon-collaboration [Pögel, Wang, Weinzierl ’22, ’23] [eps-collaboration ’25]

See talks by Pögel, Weinzierl, Melnichenko

UNIPOTENT CANONICAL DIFFERENTIAL EQUATIONS

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[Forner, Mella, Nega, Tancredi, Wagner '26]

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Observation: *differential forms proportional to ϖ_0^2 consistently drop from physical results in $D = 4$*

[Duhr, Gasparotto, Nega, Tancredi, Weinzierl '24]

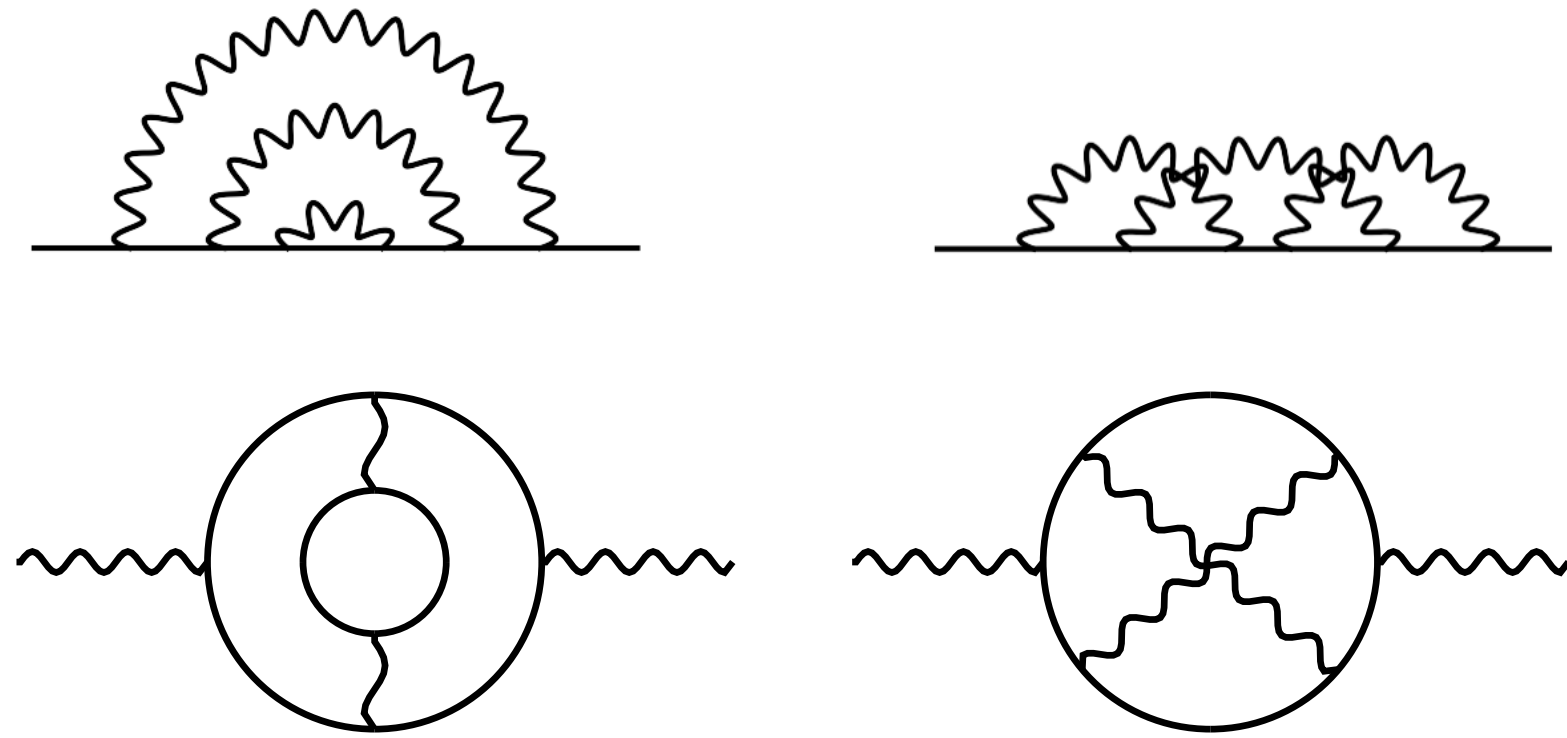
[Forner, Nega, Tancredi, '24]

[Becchetti, Coro, Nega, Tancredi, Wagner '25]

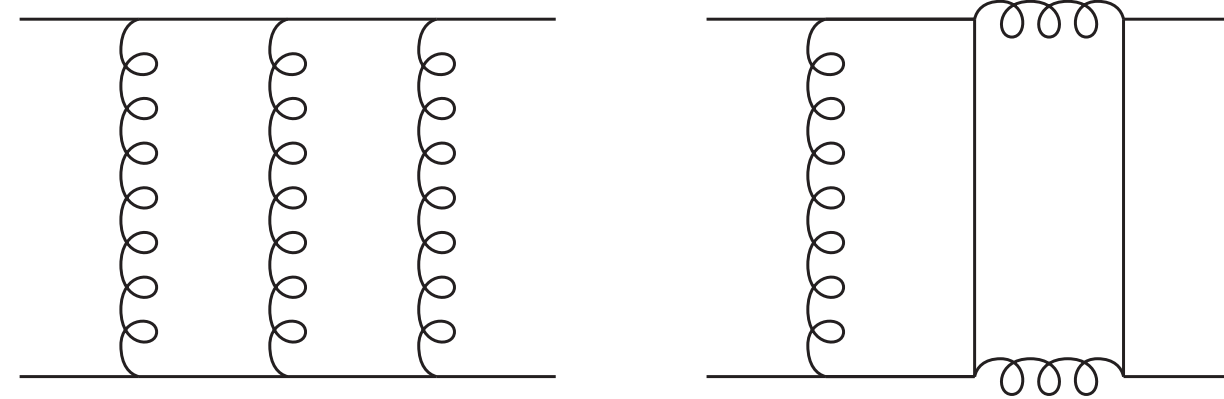
Same pattern observed in GW! Mathematical or physical mechanism behind these cancellations?

NUMERICAL RESULTS FOR ELLIPTIC AND K3 “AMPLITUDES”

Three-loop QED correlators



2 loop QED Bhabha scattering



$pp \rightarrow \gamma\gamma$ and $pp \rightarrow jj, \gamma j$

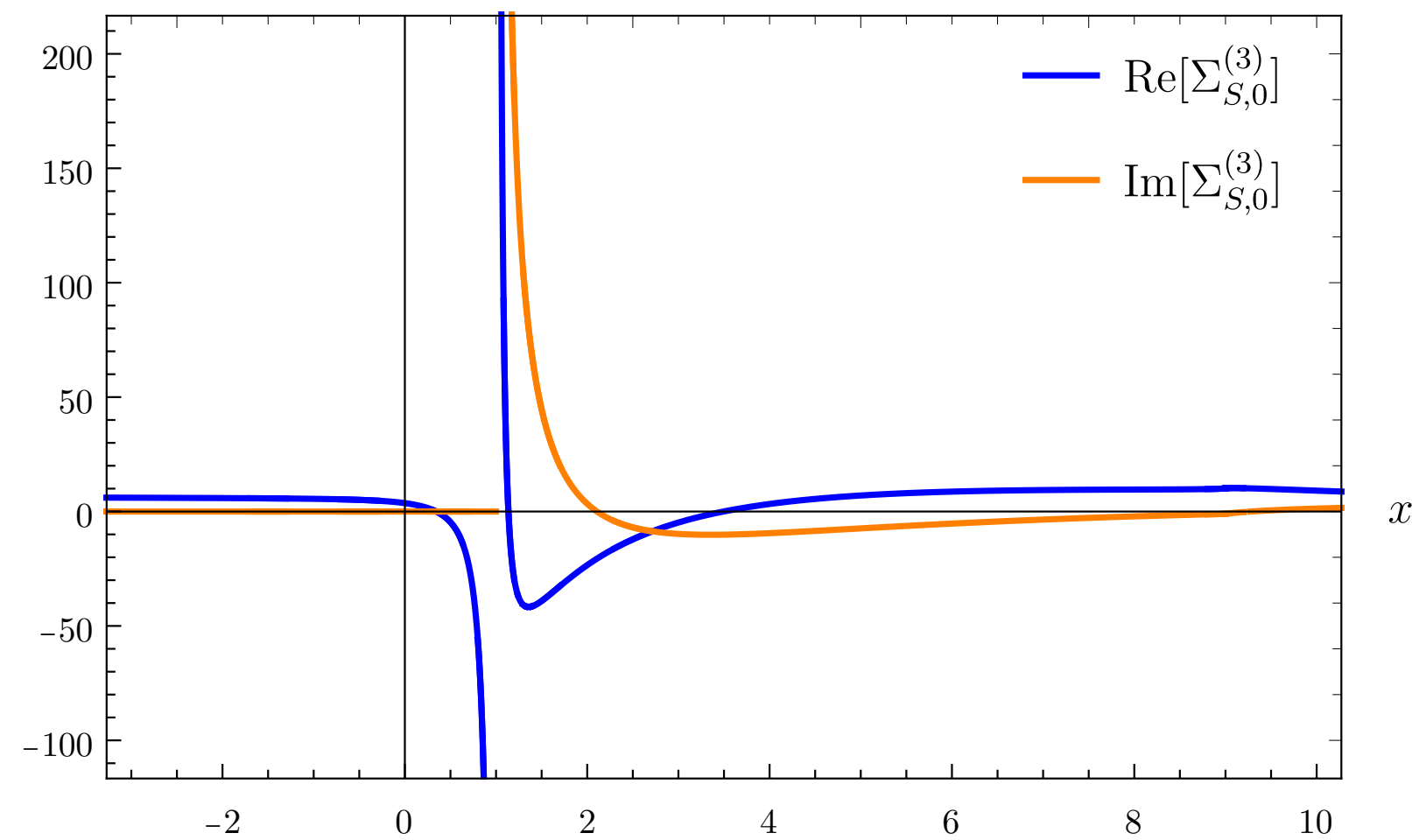
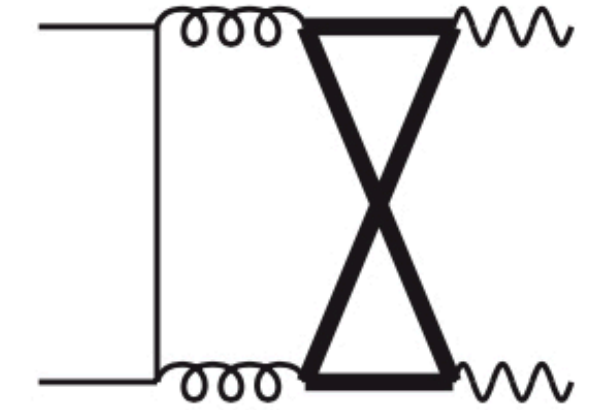
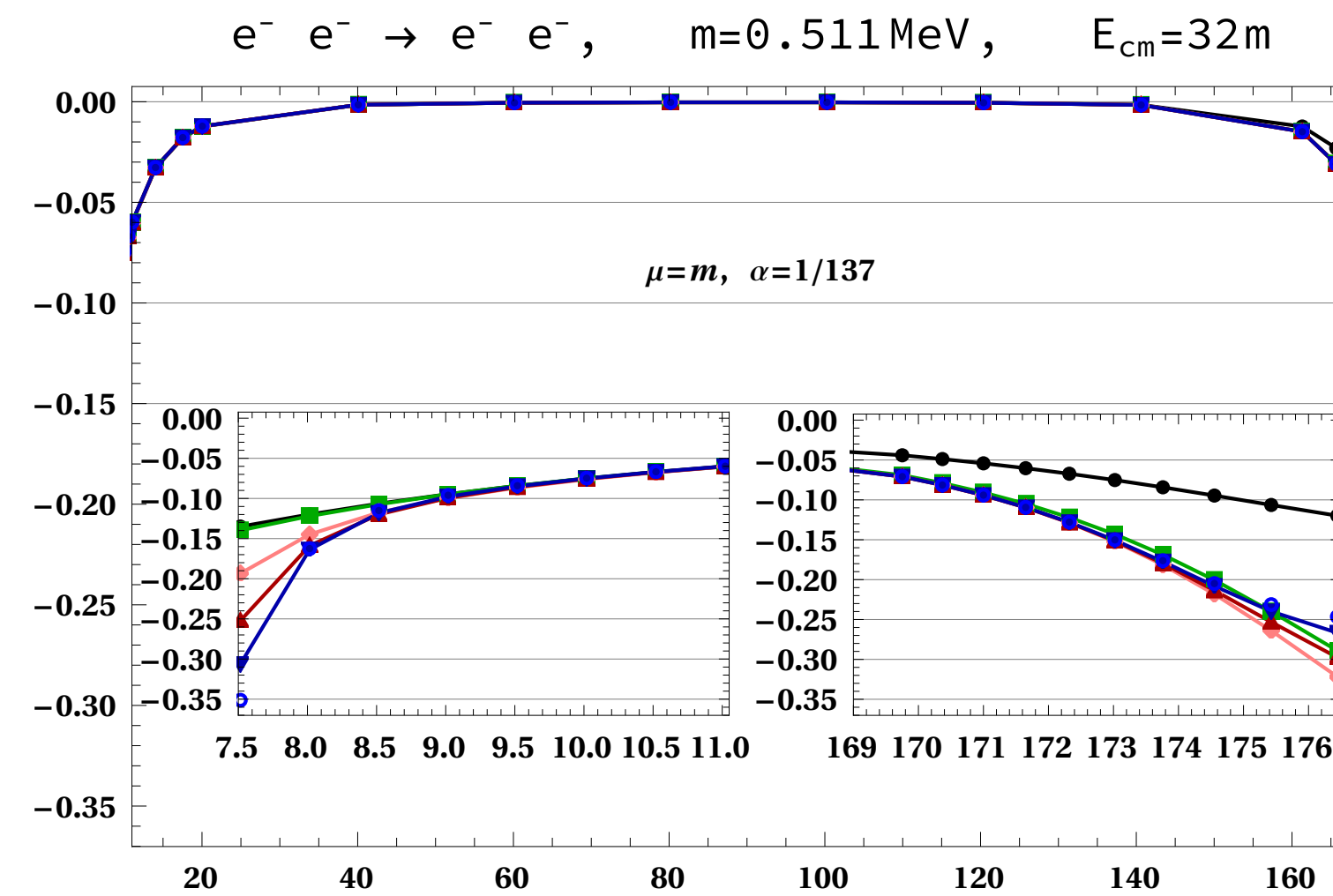
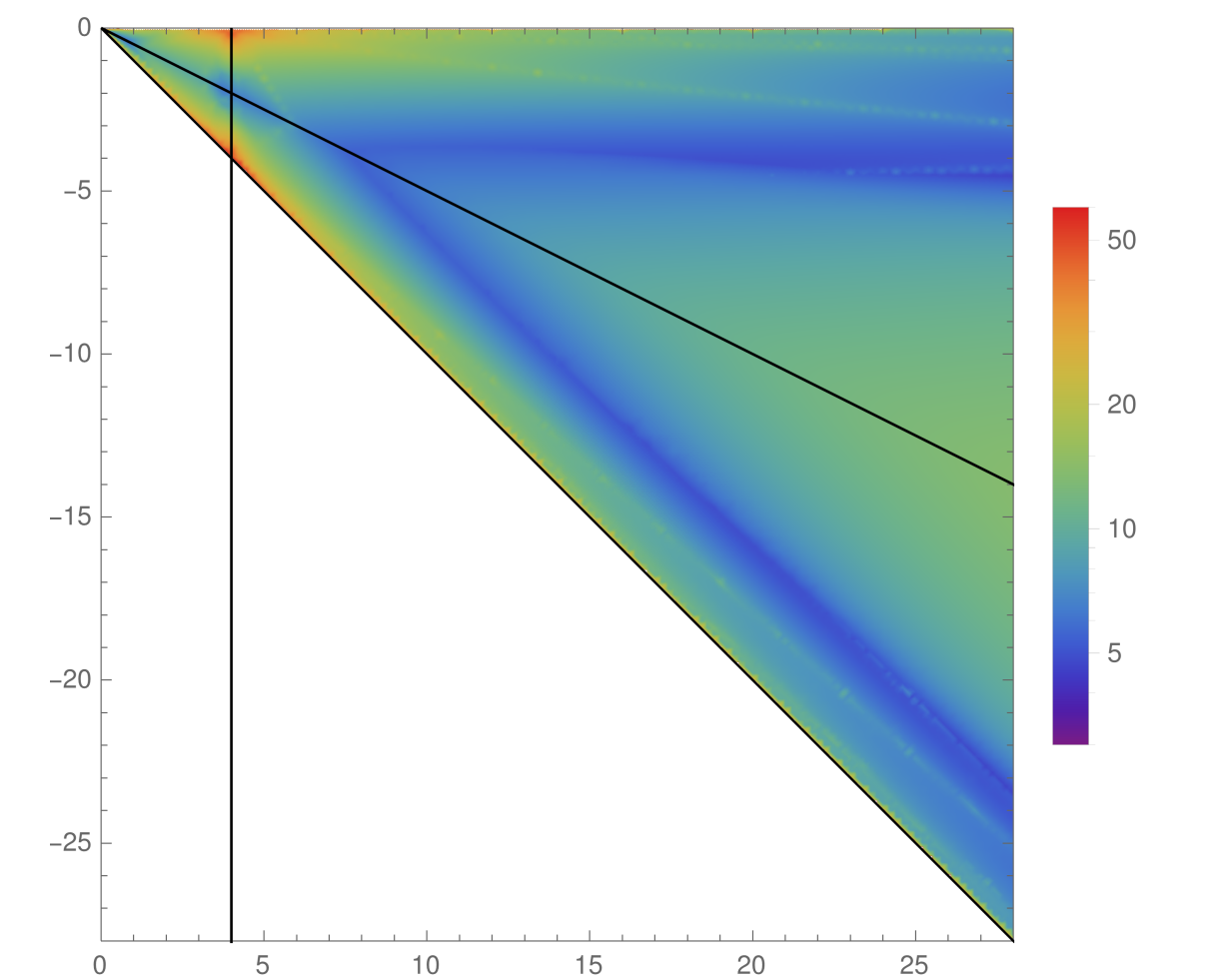


Figure 4: Real and imaginary part of $\Sigma_{S,0}^{(3)}$ (for $\xi = 0$).



- x^0
- x^4
- x^8
- x^{12}
- x^{16}
- x^{20}



(a) Real part of the coefficient of $\mathcal{C}_3$ of the $\mathcal{O}(\epsilon^0)$ part of $\mathbf{f}_{gg}^{--++}(s, t, m^2)$

NUMERICAL EVALUATION

Canonical differential equations can be solved numerically using *algorithms for rational solutions of diff equations*

[Czakov, Tancredi '26]

Simplest example: **square root** $r = \sqrt{f(x)}$, $f(x) \in \mathbb{Q}(x)$ $\rightarrow \frac{d}{dx}r = \frac{1}{2} \frac{1}{f(x)} r$

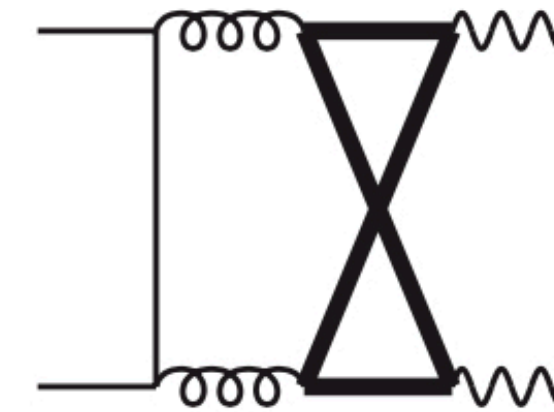
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Applied to **non trivial case** $q\bar{q} \rightarrow jj$ @ 2 loops: 16 square roots, 1 elliptic curve



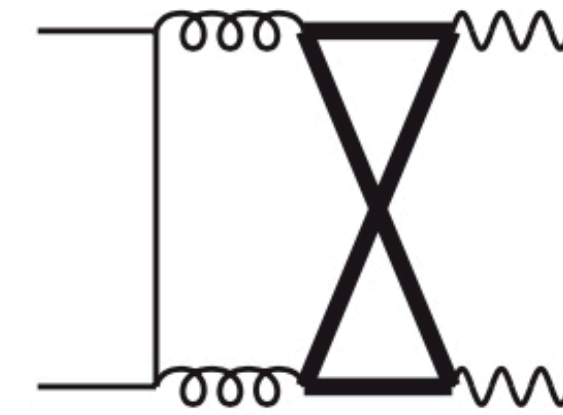
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$$\begin{aligned}
 r_1 &= \sqrt{s(s-4m^2)}, & r_2 &= \sqrt{t(t-4m^2)}, \\
 r_3 &= \sqrt{(s+t)(4m^2+s+t)}, & r_4 &= \sqrt{s(s+4m^2)}, \\
 r_5 &= \sqrt{t(t+4m^2)}, & r_6 &= \sqrt{(s+t)(s+t-4m^2)}, \\
 r_7 &= \sqrt{st(st-4m^2(s+t))}, & r_8 &= \sqrt{s(s+t)(s(s+t)-4m^2t)}, \\
 r_9 &= \sqrt{t(s+t)(t(s+t)-4m^2s)}, & r_{10} &= \sqrt{s(-4m^2(s+t)^2+s(m^2+s+t)^2)}, \\
 r_{11} &= \sqrt{s(-4m^2t^2+s(-m^2+t)^2)}, & r_{12} &= \sqrt{(s+t)(4m^2t^2+(-m^2+t)^2(s+t))}, \\
 r_{13} &= \sqrt{t(-4m^2s^2+(-m^2+s)^2t)}, & r_{14} &= \sqrt{(s+t)(4m^2s^2+(-m^2+s)^2(s+t))}, \\
 r_{15} &= \sqrt{t(-4m^2(s+t)^2+t(m^2+s+t)^2)}, & r_{16} &= \sqrt{-(m^2st(s+t))},
 \end{aligned}$$

$$\frac{\partial^2 \varpi_0(s, m^2)}{\partial (m^2)^2} = -\frac{4 \varpi_0(s, m^2)}{m^2(16m^2 + s)} - \frac{32m^2 + s}{m^2(16m^2 + s)} \frac{\partial \varpi_0(s, m^2)}{\partial m^2},$$

$$\frac{\partial \varpi_0(s, m^2)}{\partial s} = -\frac{\varpi_0(s, m^2)}{s} - \frac{m^2}{s} \frac{\partial \varpi_0(s, m^2)}{\partial m^2},$$

$$\frac{\partial^2 \varpi_0(s, m^2)}{\partial s \partial m^2} = \frac{4 \varpi_0(s, m^2)}{s(16m^2 + s)} - \frac{1}{16m^2 + s} \frac{\partial \varpi_0(s, m^2)}{\partial m^2},$$

$$\frac{\partial G(s, t, m^2)}{\partial s} = \frac{m^2(16m^2s - t(3s + 2t)) \varpi_0(s, m^2) r_9(s, t, m^2)}{(s + t) \Delta(s, t, m^2)^2}$$

$$- \frac{m^2(16m^2 + s) r_9(s, t, m^2)}{2(s + t) \Delta(s, t, m^2)} \frac{\partial \varpi_0(s, m^2)}{\partial m^2},$$

$$\begin{aligned}
 \frac{\partial G(s, t, m^2)}{\partial t} &= -\frac{m^2s^2(16m^2 + s) \varpi_0(s, m^2) r_9(s, t, m^2)}{t(s + t) \Delta(s, t, m^2)^2} \\
 &+ \frac{m^2s(16m^2 + s) r_9(s, t, m^2)}{2t(s + t) \Delta(s, t, m^2)} \frac{\partial \varpi_0(s, m^2)}{\partial m^2},
 \end{aligned}$$

$$\frac{\partial G(s, t, m^2)}{\partial m^2} = \frac{s(s + 2t) \varpi_0(s, m^2) r_9(s, t, m^2)}{\Delta(s, t, m^2)^2},$$

ITERATED INTEGRALS WITH UNIPOTENT MONODROMY

Result in terms of iterated integrals with unipotent monodromy, which can be evaluated numerically:
see also [Baur, Duhr '26] and **Gideon's talk on Thursday**

ITERATED INTEGRALS WITH UNIPOTENT MONODROMY

Result in terms of iterated integrals with unipotent monodromy, which can be evaluated numerically:
see also [Baur, Duhr '26] and **Gideon's talk on Thursday**

For specific cases, one can prove one obtains:

dlogs of rational functions: **Multiple polylogarithms**

$$G(c_1, \dots, c_k; x) = \int_0^x dt \, r(c_1, t) G(c_2, \dots, c_k; t), \quad r(c, t) = \frac{1}{t - c} \quad c \in \mathbb{C}$$

one single elliptic curve: **Elliptic multiple polylogarithms**

$$\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x, \vec{a}\right) = \int_0^x dt \, \Psi_{n_1}(c_1, t, \vec{a}) \, \mathcal{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; t, \vec{a}\right)$$

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z_{k+1}, \tau\right) = \int_0^{z_{k+1}} dw \, g^{(n_1)}(w - z_1; \tau) \, \tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; w, \tau\right)$$

$\Gamma(N)$ modular functions: **Iterated integrals of Eisenstein series**

$$\begin{aligned} I\left(\begin{smallmatrix} n_1 & N_1 \\ r_1 & s_1 \end{smallmatrix} \mid \dots \mid \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau\right) &\equiv I(h_{N_1, r_1, s_1}^{(n_1)}, \dots, h_{N_k, r_k, s_k}^{(n_k)}; \tau) \\ &= \int_{i\infty}^{\tau} d\tau' \, h_{N_1, r_1, s_1}^{(n_1)}(\tau') \, I\left(\begin{smallmatrix} n_2 & N_2 \\ r_2 & s_2 \end{smallmatrix} \mid \dots \mid \begin{smallmatrix} n_k & N_k \\ r_k & s_k \end{smallmatrix}; \tau'\right) \end{aligned}$$

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Already in the pure dlog case, but with algebraic kernels,
it's not clear what the special functions are

$$\int_{\gamma} \omega_1 \dots \omega_n, \quad \omega_i = d \log f_i \quad f_i \notin \mathbb{Q}(x)$$

MPLs with algebraic kernels?
eMPLs if roots generate elliptic curve?

$\Gamma(N)$ modular functions: **Iterated integrals
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TWO POINTS OF VIEW

GEOMETRY $\rightarrow$ FUNCTIONS

From geometry to “polylogs”:

- Iterated integrals
- Pure kernels with single poles
- Functions with unipotent monodromy

Results obtained for

- Multiple polylogarithms
- Elliptic multiple polylogarithms
- Dual picture as integrals on elliptic curve
- Recent generalization to higher genus

See Oliver’s lecture on Wednesday!

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INTEGRALS $\rightarrow$ FUNCTIONS

From Feynman integrals to special functions:

- Differential equations
- Semisimple-unipotent splitting
- ϵ -factorized forms for iteration

Results obtained for

- Iterated integrals over dlogs (beyond MPLs)
- Elliptic kernels
- Mix of elliptic and algebraic dlogs
- CY, genus 2, elliptic, algebraic dlogs ...

See Oliver's lecture on Wednesday!

A CONNECTION THROUGH CANONICAL BASES

GEOMETRY $\rightarrow$ FUNCTIONS

multiple polylogs on arbitrary geometries

INTEGRALS $\rightarrow$ FUNCTIONS

system of canonical differential equations



Canonical differential equations produce by construction functions which are **unipotent** in sense above

Is this enough to guarantee that results will be “polylogarithms” on whatever geometry we might encounter?

MY MAIN OPEN QUESTIONS FOR ELLIPTICS AND BEYOND '26

WHAT IS AN INTRINSIC MATHEMATICAL CHARACTERIZATION OF CANONICAL BASES?

Unipotent monodromy? Logarithmic singularities? ...

WHICH SPECIAL FUNCTIONS?

Can a “correct” canonical basis guarantee that results “land” on the correct special functions?

CAN PHYSICS CONSTRAIN THE GEOMETRIES WE NEED?

Do geometries always have to admit logarithmic degenerations? Is there always a MUM point?

Are some differential forms “unphysical”?

How to embed simplifications in $D = 4$ in the space of special functions we choose?

THANK YOU!