Effective Theories and Topological Strings

Maximilian Schwick

New directions in the large charge expansion,

June 23rd, 2026

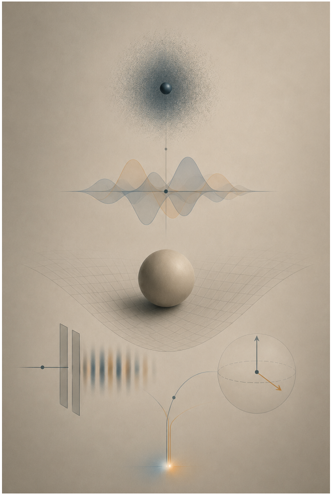
UNIVERSITÄT BERN

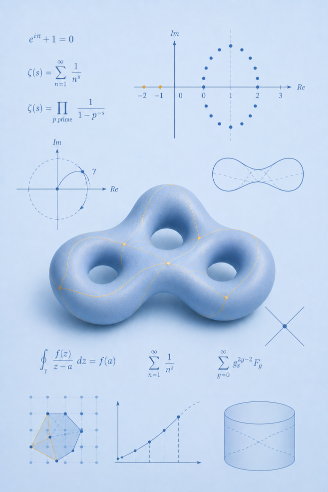
Bibliography

D. Orlando, S. Reffert, MS, to appear

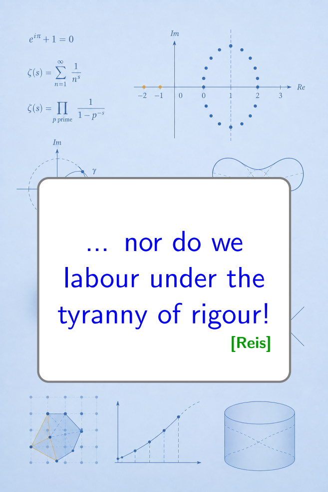
Overview

Mathematical Physics





Mathematical Physics



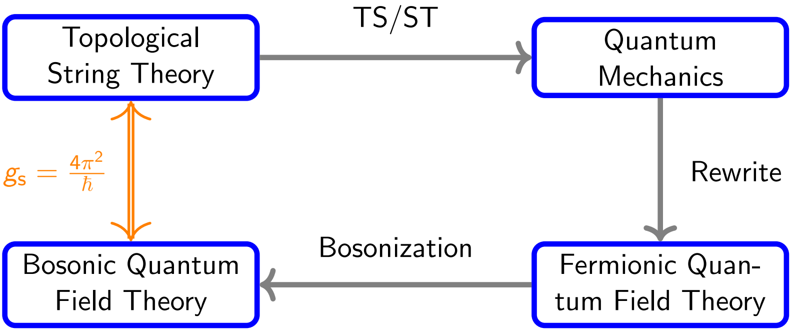
Quo Vadis?

- Physical Setting: 1D QM Fermi Gas but for topological strings!
- Bosonization in 1D $\Rightarrow$ Coadjoint orbit path integral!
- Construct a $1+1$D bosonic QFT for local topological strings.
- We need a large $\hbar$ expansion to match with topological strings!

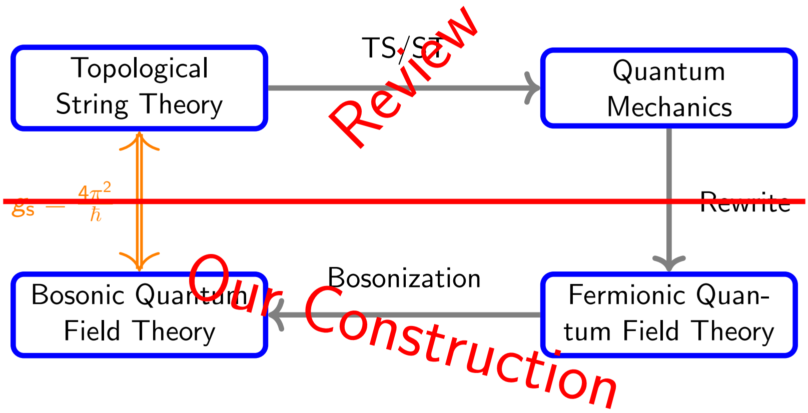
$\Rightarrow$ Very familiar formalism in a very hard example!

$\Rightarrow$ Can we learn something new along the way?

Plan: Let's Take a Trip Through Physics

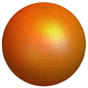
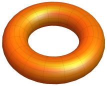
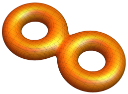


Plan: Let's Take a Trip Through Physics



The TS/ST correspondence

- Topological string free energy (specific non-perturbative completion)

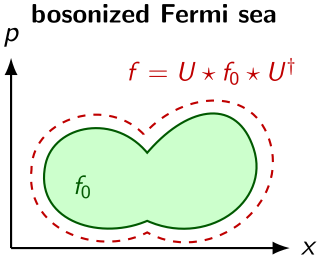
$F \simeq \frac{1}{g_s^2}$  $+$  $+\ g_s^2$  $+ \cdots.$

- Quantum mechanical determinant of a one particle Hamiltonian $H : \mathcal{H} \to \mathcal{H}$

$$\det_{\mathcal{H}} \left(1 + e^{-(H-\mu)}\right), \qquad g_s = \frac{4\pi^2}{\hbar}$$

Mathy version: There is a precise correspondence between enumerative geometry on toric CY manifolds and spectral theory of trace class operators. [Grassi-Hatsuda-Marino]

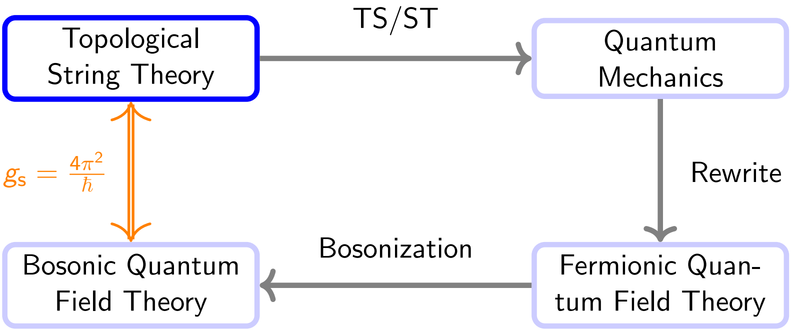
1+1D Bosonic Quantum Field Theory



$$S_{\rm orb} = \int d\tau \; \mathrm{Tr}\left[f_0 \star U^\dagger \star (\partial_\tau + H) \star U\right]$$

- Here $\hbar$ dependence is exact in $\star$.
- Extend TS/ST to a correspondence between (toric) TS and bosonic quantum field theory.

Overview


TS/ST
Topological String Theory
Quantum Mechanics
Rewrite
$g_s = \frac{4\pi^2}{\hbar}$
Bosonization
Bosonic Quantum Field Theory
Fermionic Quantum Field Theory

Topological String A-Model

Associate to a Calabi-Yau (CY) threefold M the Free Energy series

$$F(t) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}$$

- g_s string coupling
- t large radius coordinate
- Factorially growing genus expansion
- Generating function of Gromow-Witten invariants that "count" holomorphic curves of degree d and genus g

$$F_g(t) = \sum_{d=0}^{\infty} N_{g,d} e^{-d \cdot t}$$

Topological String B-Model

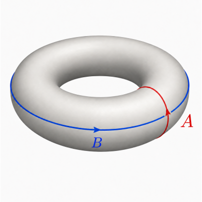
- Simplest (yet non-trivial) CY threefolds are toric CYs. Their mirrors (the B-model) reduce to algebraic curves

$$W\left(e^x, e^y\right) = 0$$

- Compute the prepotential from periods of the curve

$$t = \oint_A y(x)\mathrm{d}x$$

$$\frac{\partial F_0}{\partial t} = \oint_B y(x)\mathrm{d}x$$



$\Rightarrow$ choice of cycles is not unique! Various coordinate frames:

- large radius frame t
- conifold frame t_c
- orbifold frame t_o

- higher g_s order can be computed via topological recursion

[Bouchard-Klemm-Mariño-Pasquetti]

Topological String on Local $\mathbb{P}^2$

- Total space of the canonical bundle over $\mathbb{P}^2$

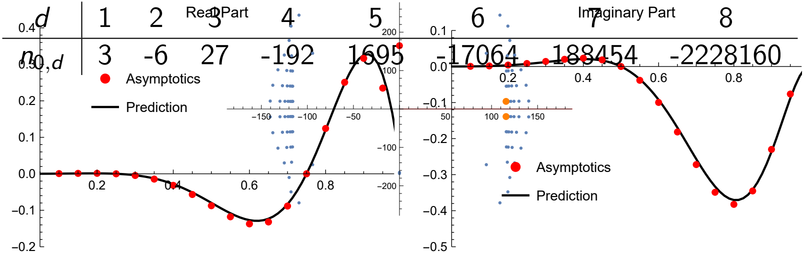
$$X = \mathcal{O}_{\mathbb{P}^2}(-3) \quad \longrightarrow \quad \mathbb{P}^2$$

- The mirror B-model is encoded by the mirror curve

$$W_{\mathbb{P}^2}(x,p) = e^x + e^p + e^{-x-p} + \kappa \qquad \longleftarrow \quad \text{like } t$$

- Canonical Example: Lots has been computed!!

d	1	2	3	4	5	6	7	8
$n_{0,d}$	3	-6	27	-192	1695	-17064	188454	-2228160



A Comment on the Conifold Frame

- We will focus on the conifold frame. Here

$$F_g(t_c) = F_g^{\text{con}}(t_c) + \widehat{F}_g(t_c)$$

where

$$F_0^{\text{con}}(t_c) = \frac{t_c^2}{2}\left(\log(t_c) - \frac{3}{2}\right), \quad F_1^{\text{con}}(t_c) = -\frac{1}{12}\log(t_c),$$

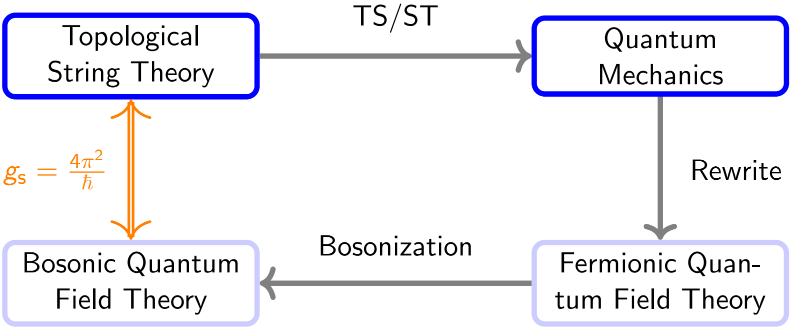
$$F_g^{\text{con}}(t_c) = \frac{B_{2g}}{2g(2g-2)} t_c^{2-2g}, \quad g \geq 2$$

- Convenient 'finite-N' reorganization (think of N particles)

$$t_c = g_s N, \qquad N \text{ fixed}$$ [Mariño, MS]

- 'Finite-N' local $\mathbb{P}^2$ Topological String

$$Z_N^{\mathbb{P}^2} = \exp\left[F^{\mathbb{P}^2}(g_s N)\right] \simeq \underbrace{\exp\left[F^{\text{con}}(g_s N)\right]}_{Z_N^{\text{con}} = \frac{g_s^{N^2/2}}{(2\pi)^{N/2}} G_2(1+N)} \left(1 + g_s \frac{N(5-2N^2)}{72\sqrt{3}} + \mathcal{O}(g_s^2)\right)$$


Topological String Theory
TS/ST
Quantum Mechanics
Rewrite
Fermionic Quantum Field Theory
Bosonization
Bosonic Quantum Field Theory
$g_s = \frac{4\pi^2}{\hbar}$

TS/ST: Quantizing the Mirror Curve

For a toric CY X, write the mirror curve as

$$W_X(\mathrm{e}^x, \mathrm{e}^p) = \mathrm{O}_X(x, p) + \kappa = 0.$$

Quantize the curve (Weyl ordering prescription)[Grassi-Hatsuda-Marino]

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar, \qquad \mathrm{e}^{rx+sp} \longrightarrow \mathrm{e}^{r\hat{x}+s\hat{p}}.$$

QM Spectral Problem:

$$\widehat{\mathrm{O}}_X|\psi_n\rangle = \mathrm{e}^{E_n}|\psi_n\rangle, \qquad \kappa = -\mathrm{e}^{E}.$$

For local $\mathbb{P}^2$:

$$\widehat{\mathrm{O}}_{\mathbb{P}^2} = \mathrm{e}^{\hat{x}} + \mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{x}-\hat{p}}.$$

Density Operator

Quantum mechanical Hamiltonian:

$$\widehat{\mathcal{O}}_X = \mathrm{e}^{\widehat{H}_X}, \qquad \widehat{\rho}_X = \widehat{\mathcal{O}}_X^{-1} = \mathrm{e}^{-\widehat{H}_X}.$$

- $\widehat{\rho}_X$ is expected to be positive and trace class [Grassi-Hatsuda-Marino]
- its eigenvalues are e^{-E_n}

$\Rightarrow$ We can compute the spectral determinant

$$\Xi_X(\mu, \hbar) = \det(1 + \kappa \widehat{\rho}_X) = \prod_{n=0}^{\infty} \left(1 + \kappa \mathrm{e}^{-E_n}\right), \qquad \kappa = \mathrm{e}^{\mu}.$$

- This is ordinary quantum mechanics with a complicated Hamiltonian.

TS/ST Correspondence

topological string non-perturbative completion $\longleftrightarrow$ spectral determinant $\Xi(\mu, \hbar)$

[Grassi-Hatsuda-Marino]

- Identity between two functions
- We just need the asymptotic version

$$\exp\left[\sum_{g=0}^{\infty} F_g(t) g_s^{2g-2} + (\text{non-perturbative...})\right] \simeq \underbrace{\det\left(1 + e^{\mu}\widehat{\rho}_X\right)}_{=\,\Xi_X(\mu,\hbar)}$$

- Parameters: $t \leftrightarrow \mu$ (quantum mirror map) and $g_s = \frac{4\pi^2}{\hbar}$.

Local $\mathbb{P}^2$: Factorized Density Operator

For local $\mathbb{P}^2$, [Kashaev-Mariño]

$$\boxed{\widehat{\rho}_{\mathbb{P}^2} = \hat{\mathrm{O}}^{-1} = \Psi^*(\widehat{\mathrm{p}})\, G(\widehat{\mathrm{q}})\, \Psi(\widehat{\mathrm{p}})}$$

with the Faddeev quantum dilog appearing

$$\Psi(p) = \frac{e^{\frac{\pi b}{3} p}}{\Phi_b\left(p - \frac{\mathrm{i} b}{3}\right)},$$

and

$$G(x) = \frac{e^{\frac{\pi b}{3} x}}{2\cosh(\pi b x)}.$$

Comment: To write this we used normalized operators

$$\widehat{x} = \frac{2\pi b}{3}\left(2\widehat{\mathrm{p}} + \widehat{\mathrm{q}}\right), \qquad \widehat{p} = -\frac{2\pi b}{3}\left(\widehat{\mathrm{p}} + 2\widehat{\mathrm{q}}\right), \qquad \hbar = \frac{2\pi b^2}{3}.$$

Sidequest: Faddeev's Quantum Dilogarithm

For $b > 0$, Faddeev's quantum dilogarithm is the meromorphic function [Faddeev-Kashaev]

$$\Phi_b(z) = \exp\left[\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{-2\mathrm{i}zw}}{\sinh(bw)\sinh(w/b)}\frac{\mathrm{d}w}{w}\right].$$

Basic properties:

- Classical limit:

$$\Phi_b\left(\frac{x}{2\pi b}\right) \sim \exp\left[\frac{1}{2\pi\mathrm{i}b^2}\,\mathrm{Li}_2(-\mathrm{e}^x)\right], \qquad b \to 0.$$

- Pentagon identity: if

$$[\hat{p}, \hat{q}] = \frac{1}{2\pi\mathrm{i}},$$

  then

$$\Phi_b(\hat{p})\Phi_b(\hat{q}) = \Phi_b(\hat{q})\Phi_b(\hat{p}+\hat{q})\Phi_b(\hat{p}).$$

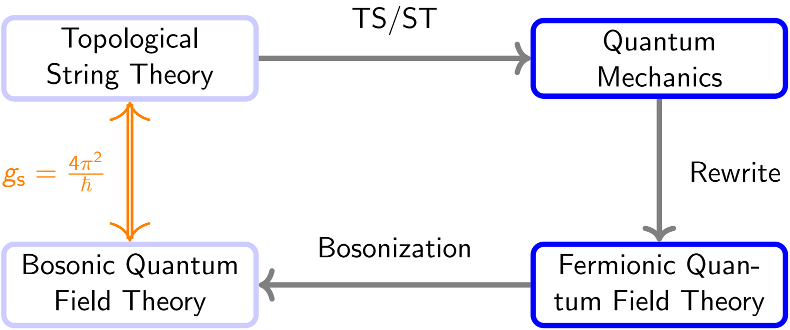
A Subtle Problem in Quantum Mechanics

- Evaluating the grand-canonical determinant at finite μ is hard!
- Compute fermionic spectral traces

$$\det\left(1+\mathrm{e}^{\mu}\widehat{\rho}_{\mathbb{P}^2}\right)=\sum_{N=0}^{\infty}\mathrm{e}^{N\mu}Z_N$$

- At finite N the spectral traces can be computed from the Mariño Zakani matrix models [Kashaev-Mariño, Mariño-Zakani, Mariño-Zakani]
  $\Rightarrow$ Non-perturbative computations can still be hard for generic N.
- Here we want to explore another way of computing these traces!

Overview


Topological String Theory
TS/ST
Quantum Mechanics
Rewrite
Fermionic Quantum Field Theory
Bosonization
Bosonic Quantum Field Theory
$g_s = \frac{4\pi^2}{\hbar}$

Determinant as a $1+1$D Fermion Theory

Let $\widehat{H}$ act on $\mathcal{H} = L^2(\mathbb{R}_x)$. Then the grand-canonical partition function of free fermions reads

$$\det_{\mathcal{H}}\left(1 + \mathrm{e}^{-\beta(\widehat{H}-\mu)}\right) \propto \int_{\mathrm{AP}} \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \mathrm{e}^{-S_E[\bar{\psi},\psi]},$$

with

$$S_E = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x\, \bar{\psi}(\tau, x)\left[\partial_\tau + \widehat{H}\left(x, p = -\mathrm{i}\hbar\partial_x\right) - \mu\right]\psi(\tau, x).$$

and boundary conditions

$$\psi(\tau + \beta, x) = -\psi(\tau, x), \qquad \bar{\psi}(\tau + \beta, x) = -\bar{\psi}(\tau, x).$$

Mechanism: Summing over Matsubara Modes

$$\int_{\mathrm{AP}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\left[-\int_0^\beta \mathrm{d}\tau\, \bar{\psi}\left(\partial_\tau + \widehat{H} - \mu\right)\psi\right]$$

$$= \mathrm{Det}_{\mathrm{AP}}\left(\partial_\tau + \widehat{H} - \mu\right) \quad = \quad \prod_a \prod_{n\in\mathbb{Z}} \left(-\mathrm{i}\omega_n + E_a - \mu\right)$$

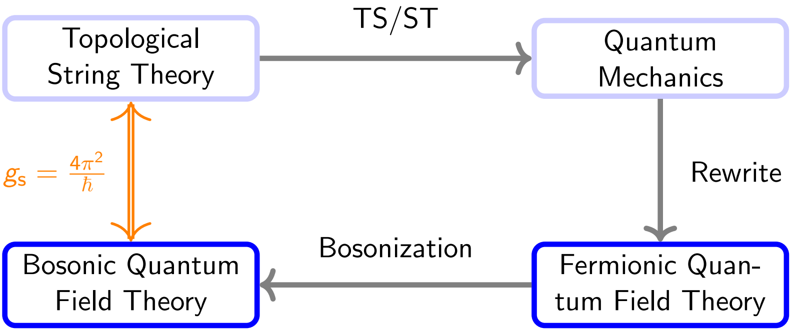
Matsubara $\omega_n = \frac{(2n+1)\pi}{\beta}$

Energy modes

$$\propto \prod_a \left(1 + \mathrm{e}^{-\beta(E_a - \mu)}\right) \quad = \quad \det_{\mathcal{H}}\left(1 + \mathrm{e}^{-\beta(\widehat{H}-\mu)}\right)$$

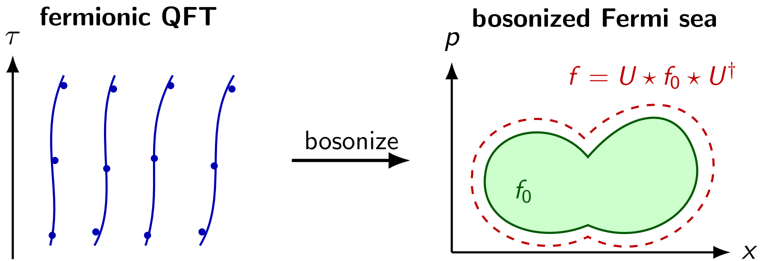
Contact with the topological string:

$$\hat{H} = -\log\left(\hat{\rho}\right) \qquad \text{(operator identity)}$$



Topological String Theory
TS/ST
Quantum Mechanics
$g_s = \frac{4\pi^2}{\hbar}$
Rewrite
Bosonization
Bosonic Quantum Field Theory
Fermionic Quantum Field Theory

Idea: Fermions vs. Bosonized Fermi Sea



$$S_{\mathrm{f}} = \int \mathrm{d}\tau \, \mathrm{d}x \, \bar{\psi} \left(\partial_\tau + \widehat{H} \right) \psi \qquad S_{\mathrm{orb}} = \int \mathrm{d}\tau \ \mathrm{Tr} \left[f_0 \star U^\dagger \star (\partial_\tau + H) \star U \right]$$

Finite-N Coadjoint Orbit

- finite one-particle Hilbert space

$$\mathcal{H}_1 = \text{span}\left\{|0\rangle, \ldots, |K-1\rangle\right\} \simeq \mathbb{C}^K, \qquad \mathcal{U}(K) \text{ symmetry.}$$

- with Fock space

$$\mathcal{H}_\text{F} = \bigoplus_{N=0}^{K} \wedge^N \mathcal{H}_1$$

- N-fermion fermi surface fills an N-dimensional subspace:

$$f_0 = \text{diag}(\underbrace{1, \ldots, 1}_{N}, 0, \ldots, 0).$$

- All equivalent fillings are generated by

$$f = U f_0 U^\dagger, \qquad U \in \mathcal{U}(K), \qquad f^2 = f, \qquad \text{Tr}\, f = N.$$

Finite-N Coadjoint Orbit Action

$$S_{\text{orb}}[U] = \int \mathrm{d}\tau \; \mathrm{Tr}\left[f_0 U^\dagger \left(\partial_\tau + \widehat{H}\right) U\right]$$

Dynamical Term $\qquad\qquad$ $\mathrm{Tr}\left[f\,\widehat{H}\right]$

- Sum over occupied energy states $\mathrm{Tr}\left[f\,\widehat{H}\right]$
- Single time derivative: Like QM [Kirillov]
- Path integral over the orbit parameterized by U [Szabo, Dhar-Mandal-Wadia]

$$\int \mathcal{D}U \begin{pmatrix}\text{measure}\\ \text{factors}\end{pmatrix} e^{-S_{\text{orb}}[U]}$$

Coadjoint Orbit

- Recover infinite Hilbert space $K \to \infty$

$$u(K) \xrightarrow{K\to\infty} \text{Moyal algebra}$$

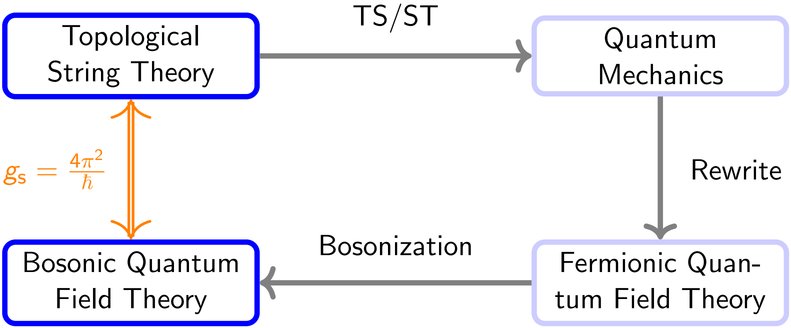
- Moyal algebra on phase space

$$A(x,p)\star B(x,p) = A(x,p)\exp\left[\frac{i\hbar}{2}\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)\right]B(x,p)$$

- Coadjoint orbit action [Delacrétaz-Du-Mehta-Son, Chen-Delacrétaz]

$$S_{\text{orb}} = \int d\tau \ \mathrm{Tr}\left[f_0 \star U^\dagger \star (\partial_\tau + H) \star U\right].$$

- Remark: We could have arrived here without heuristics by writing a coherent state path integral for the fermionic spectral traces. This will precisely produce the orbit and it is how we can fix normalizations.[Szabo]


Topological String Theory
TS/ST
Quantum Mechanics
Rewrite
Fermionic Quantum Field Theory
Bosonization
Bosonic Quantum Field Theory
$g_s = \frac{4\pi^2}{\hbar}$

Topological String as Bosonic Quantum Field Theory

$$\int \mathcal{D}U \left(\begin{smallmatrix}\text{measure}\\ \text{factors}\end{smallmatrix}\right) \exp\left\{-\int \mathrm{d}\tau \ \mathrm{Tr}\left[f_0 \star U^\dagger \star \left(\partial_\tau + H(x,p)\right) \star U\right]\right\}$$

$$N = \mathrm{Tr}\left[f_0\right] = \int \frac{\mathrm{d}x\mathrm{d}p}{2\pi\hbar} f_0(x,p) \qquad -\log_\star\left(\rho(x,p)\right)$$

Finite N: $x \sim \sqrt{\hbar}, p \sim \sqrt{\hbar}$

Plan: Match against the 'finite-N' reorganization of the topological string with $g_s = \frac{4\pi^2}{\hbar}$!

Deep Quantum Limit

Large $\hbar$ limit!

Local $\mathbb{P}^2$ at large $\hbar$

- Compute the Weyl symbol for $\widehat{\rho}_{\mathbb{P}^2}$ via Moyal products

$$\rho_{\mathbb{P}^2} = \Psi^*(P) \star G(X) \star \Psi(P)$$

$$\mathrm{q} = \sqrt{\hbar} X, \;\; \mathrm{p} = \sqrt{\hbar} P$$
$$[X, P] = \frac{\mathrm{i}}{\hbar}\alpha$$
$$\Psi(P) = \mathrm{e}^{\hbar\phi_0(P) + \frac{1}{\hbar}\phi_2(P) + \dots}$$

- Moyal shifts are easy under Fourier transform

$$\rho_{\mathbb{P}^2} = \int \frac{\mathrm{d}k}{2\pi} \hat{G}(k) \mathrm{e}^{\hbar\left(\phi_0^\star\left(P + \frac{\alpha k}{2}\right) + \phi_0\left(P - \frac{\alpha k}{2}\right) + \mathrm{i}kX\right) + \mathcal{O}\left(\hbar^{-1}\right)}$$

where

$$G(X) = \int \frac{\mathrm{d}k}{2\pi} \hat{G}(k) \mathrm{e}^{\mathrm{i}kX}$$

- Integral has large $\hbar$ saddles $\Rightarrow$ Harmonic oscillator + corrections

$$\boxed{\rho_{\mathbb{P}^2} = \rho_{\mathrm{h.o.}} + \rho_{\mathrm{int}}, \qquad \rho_{\mathrm{int}} = \mathcal{O}\left(\frac{1}{\hbar}\right)}$$

- We know how to deal with the harmonic oscillator!
- Hamiltonian can be computed as a large $\hbar$ expansion around the harmonic oscillator
- Compute partition function around the harmonic oscillator saddle ($\beta = 1$)

$$Z_N = \mathrm{Tr}_{\wedge^N \mathcal{H}} \left[\mathrm{e}^{-\beta \widehat{H}} \right] = \mathrm{Tr}_{\wedge^N \mathcal{H}} \left[\mathrm{e}^{-\beta \left(\widehat{H}_{\text{h.o.}} + \widehat{H}_{\text{int}} \right)} \right] =$$

$$= Z_N^{\text{gaussian}} \left\langle \text{interaction terms} \right\rangle_N^{\text{gaussian}}$$

- compute exactly for local $\mathbb{P}^2$:

$$Z_N = Z_N^{\text{gaussian}} \left(1 + g_{\text{s}} \frac{N(5 - 2N^2)}{72\sqrt{3}} + \dots \right),$$

$$Z_N^{\text{gaussian}} = \frac{g_{\text{s}}^{N^2/2}}{(2\pi)^{N/2}} G_2\left(1 + N\right)$$

This matches exactly!

Overview

Conclusion

[Wikipedia]

Constructed a bosonic 1+1D quantum field theory that is dual to a topological string theory

Open Questions

- Can we compute non-perturbative sectors of the topological string? $\Rightarrow$ Other saddles of Fourier transform?
- Can we compute at large N?
- Can we do a grandcanonical computation with μ?

Thank You!