On Instantons @ LARGE Charge

Raffaele Savelli
University of Rome "Tor Vergata"

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w/ A. CIPRIANI

$\mathcal{N}=2$ 4d SCFTs

- Amazing context to explore the wonders of LARGE charge

- Interesting (sometimes intrinsic) strong coupling dynamics, yet tractable due to symmetries, dualities, recursion relations...

- Focus is on 2-point functions of chiral primaries

$\Rightarrow$ Ring generated by Coulomb-branch operators

- Restrict to rank 1 $\Rightarrow$ 1d CB

hard!

$$\langle \mathcal{O}^n(x)\, \bar{\mathcal{O}}^n(0) \rangle = \frac{G_{2n}}{|x|^{2dn}}$$

$U(1)_R$ charge

conformal dim.

Large n

Work of Hellerman, Maeda, Orlando, Reffert, Watanabe ('17 – '21) showed

$$\log G_{2n} \underset{n\to\infty}{\cong} A\,n + \tilde{B} + \Gamma(dn + \alpha - 1)$$

"non-universal"

$2(a^{CFT} - a^{EFT})$

$A \longrightarrow$ normalization of $\mathcal{O}$

$\tilde{B} \longrightarrow$ normalization of $\mathbb{1}$

$$\tilde{B} = B - \log Z$$

"scheme"-dependent

- Prediction of EFT on the C.B.
- insensitive to $\mathrm{Exp}(-n)$ terms
- NO Reference to any other parameter except n (e.g. gauge coupling)

SU(2) SQCD

- The only Lagrangian case @ rank 1
- Great playground for a microscopic test of HMORW
- Possibly teaches lessons about isolated models
- Key technical tool : SUSY LOCALIZATION

  Expresses 2-pt funct. of 1/2 BPS operators

  as $G_{2n}(\tau, \bar{\tau})$ $\qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$

  expanded for $g_{YM} \to 0$

  ¿ $n \to \infty$ vs $g_{YM} \to 0$?

- Numerical tests already made by HMORW
- Analytic tests so far only by

  Grassi, Komargodski, Tizzano '19

  limited to one-loop

- We include instantons ($q \equiv e^{2\pi i \tau}$ - expansion)

  NOT suppressed for $n \to \infty$

Large vevs

- Large n naturally involves another expansion

$$\langle \Phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

large n kicks $a \to \infty$

Comes from a saddle of the free energy

$$\log Z \equiv F \approx -2ca^2 + 4n \log a$$

SW prepotential

$c \underset{g_{YM} \to 0}{\sim} \frac{1}{g_{YM}^2}$

from $2n$ insertions of $\mathcal{O} \sim \mathrm{Tr}\,\Phi^2$

$\Rightarrow$ $a \sim \sqrt{n/c}$ $\underset{n \to \infty}{\rightsquigarrow}$ heavy electric and magnetic charges

g_{YM} fixed!

- Are small vevs irrelevant @ large n ??

NO!

@ 1 loop: $\Gamma(n+1)\Gamma(n+2) \rightsquigarrow \Gamma\left(2n + \frac{5}{2}\right)$

(GKT '19)

inclusion of small a

The ε - expansion

@ instantons : small a enter $\log(Z\, G_{2n})$
only as $\mathrm{Exp}(-n)$

But what's large vs small a in practice?

Nekrasov Partition Function : ('02)

$$Z = Z_{tree}\; Z_{1L}\; Z_{inst.}$$

$\mathrm{Exp}(-4\pi\, \mathrm{Im}\,\tau\; a^2)$ Barnes G, τ - indep $(1 + \mathcal{O}(q))$

Localization implemented using Ω - background

Here : $\varepsilon_1 = \varepsilon_2 = \frac{1}{R}$

Reorganize weak coupling expansion of Z
in terms of expansion for $aR \to \infty$

$$\log Z \equiv F = \sum_{g=0}^{\infty} F_g(\tau,\bar\tau)\,(aR)^{2-2g}$$

$F_0 \to$ SW prep. ; F_1 contains logarithms !

$F_0, F_1 \Rightarrow$ "large vevs" ; $F_{g\geq 2} \Rightarrow$ "small vevs"

Microscopic computation

- Recipe by Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu '16
- Based on Pestun prescription for Z_{S^4} ('07)

$$Z_{S^4} = \int_{\mathbb{R}} da\, (2a)^2 |Z|^2$$

Define the matrices: $\dfrac{M_{ij}^{(n)}(\tau,\bar\tau)}{Z_{S^4}} = \dfrac{\partial_\tau^i \, \partial_{\bar\tau}^j \, Z_{S^4}}{Z_{S^4}}$

$\cancel{U(1)}_R$ on S^4 $\Longrightarrow$ mixing of operators w/ $\neq R$!

$$\Longrightarrow \quad G_{2n}(\tau,\bar\tau) = \frac{16^n}{Z_{S^4}(\tau,\bar\tau)} \frac{\det M^{(n)}}{\det M^{(n-1)}}(\tau,\bar\tau)$$

Gram - Schmidt orthogonalization

Ratio of det's difficult to handle for $n \to \infty$

help from matrix - model techniques

(even when not exactly solvable)

Universality @ large n

Recall :

$$\log Z_{S^4} G_{2n} \underset{n\to\infty}{=} A(\tau,\bar\tau)\, n + B(\tau,\bar\tau) + \Gamma\left(2n+\tfrac{5}{2}\right)$$

$$+ \mathcal{O}\left(\mathrm{Exp}(-\sqrt{n})\right)(\tau,\bar\tau)$$

No higher-derivative F-terms $\Rightarrow$ recursion relations $\rightsquigarrow$ set of algebraic constraints

$\Rightarrow$ Entire $\Gamma\left(2n+\frac{5}{2}\right)$ deduced from 3 coeff.

$$n\log n \;,\quad \log n \;,\quad \frac{1}{n}$$

τ-independence surprising from microscopic p.o.v.

Strategy :

1) Work order by order in q to see patterns.

2) Generalize to closed-form argument

Instantons @ large n (I)

$$Z_{\text{inst.}} = 1 + \frac{q}{2}(a^2-3) + \frac{q^2}{4}\frac{8a^8 + a^6 - 91a^4 - 60a^2 + 132}{(4a^2+9)^2} + O(q^3)$$

$$= 1 + \frac{q}{2}(a^2-3) + \frac{q^2}{4}\boxed{\left[\underbrace{\left(\frac{a^4}{2} - \frac{35}{16}a^2 + \frac{13}{8}\right)}_{F_0,\ F_1} + \underbrace{\frac{3(a^2+2)}{16(4a^2+9)^2}}_{F_{g\geq 2}}\right]}_{*} + O(q^3)$$

Taking derivatives inside the integral :

$$M_{ij}^{(n)}(\tau,\bar\tau) = \int_{\mathbb{R}} da\ 4a^2 |Z_{1L}|^2 (-1)^j (2\pi i)^{i+j}\ e^{-4\pi \operatorname{Im}\tau\, a^2}$$

$$\cdot\left[a^{2i} + \frac{q}{2}(a^2+1)^i(a^2-3) + \frac{q^2}{4}(a^2+2)^i\,[*] + O(q^3)\right]\cdot\left[\begin{matrix} i\to j \\ q\to \bar q\end{matrix}\right]$$

Key technical device : Andreief identity

det of matrix of integrals $\rightsquigarrow$ multivariate integral of det's

$$\det_{ab}\int d\mu(y)\, f_a(y)\, f_b(y) = \frac{1}{(n+1)!}\int \prod_{j=0}^{n} d\mu(y_j)\, \det_{ab}\big(f_a(y_b)\big)\det_{cd}\big(f_c(y_d)\big)$$

$$\Rightarrow \det M_{ij}^{(n)}(\tau,\bar\tau) = \frac{(2\pi)^{n(n+1)}}{(n+1)!}\int_{\mathbb{R}^{n+1}} \prod_{k=0}^{\infty} d\mu(a_k)\prod_{i<j}(a_i^2-a_j^2)^2\left|\det M(\tau)\right|^2$$

Instantons @ large n (II)

with $\quad d\mu(a) = da \, 4a^2 |Z_{1L}|^2 e^{-4\pi \operatorname{Im}\tau \, a^2}$

and $\quad \mathcal{M}(\tau) = \sum_{I=0}^{\infty} K^{(I)}(a) \left(\frac{q}{2}\right)^I$

Expanding in q : $\det \mathcal{M}(\tau) = 1 + \frac{q}{2} \operatorname{Tr} K^{(1)}(a)$

$$+ \frac{q^2}{4} \left[\operatorname{Tr}\left(K^{(2)}_{g=0,1}(a) + K^{(2)}_{g\geq 2}(a) - \frac{K^{(1)}(a)^2}{2} \right) + \frac{\left(\operatorname{Tr} K^{(1)}(a)\right)^2}{2} \right.$$

$$+ \mathcal{O}(q^3)$$

$$\operatorname{Tr} K^{(1)}(a) = \frac{1}{2}(n+1)(n+2) + \sum_{\alpha=0}^{n} a_\alpha^2$$

$$\operatorname{Tr}\left(K^{(2)}_{g=0,1}(a) - \frac{K^{(1)}(a)^2}{2} \right) + \frac{\left(\operatorname{Tr} K^{(1)}(a)\right)^2}{2} = P_4(n) + P_2(n) \sum_{\alpha=0}^{n} a_\alpha^2 + P_0(n) \left(\sum_{\alpha=0}^{n} a_\alpha^2 \right)^2$$

We are led to evaluate moments in a statistical ensemble

Instantons @ large n (III)

Ensemble defined by :

$$X_n(t) \equiv \frac{1}{(n+1)!} \int \prod_{k=0}^{n} d\mu(a_k) \prod_{i<j} \left(a_i^2 - a_j^2\right)^2$$

$$t \equiv 4\pi \operatorname{Im}\tau + 8\log 2$$

$$\log\det M_{ij}^{(n)}(\tau,\bar\tau) = \overbrace{\log\left(4^{n+1}(2\pi)^{n(n+1)}\right) + \log X_n(t)}^{\text{GKT '19}}$$

$$+ \frac{q+\bar q}{2}\left(\frac{(n+1)(n-6)}{2} + \langle \operatorname{Tr}\mathcal{X}\rangle_c\right)$$

$$+ \frac{q^2+\bar q^2}{64}\left((9n-46)(n+1) + 13\langle \operatorname{Tr}\mathcal{X}\rangle_c\right) + \frac{(q+\bar q)^2}{8}\langle(\operatorname{Tr}\mathcal{X})^2\rangle_c$$

$$+ \mathcal{O}(q^3)$$

$\mathcal{X} \longrightarrow$ matrix w/ eigenvalues $\{a_\alpha^2\}_{\alpha=0,\dots,n}$

$\langle(\operatorname{Tr}\mathcal{X})^p\rangle \longrightarrow$ expectation value in $X_n(t)$

$$\hookrightarrow = (-1)^p \frac{\partial_t^p X_n(t)}{X_n(t)} \underset{n\to\infty}{\simeq} \frac{\left[(n+1)(n+2)\right]_p}{t^p} \quad \text{Pochhammer}$$

$\langle(\operatorname{Tr}\mathcal{X})^p\rangle_c \longrightarrow$ cumulants

Instantons @ large n (IV)

E.g. $\langle \mathrm{Tr}\, X \rangle_c = \langle \mathrm{Tr}\, X \rangle$

$$\langle (\mathrm{Tr}\, X)^2 \rangle_c = \langle (\mathrm{Tr}\, X)^2 \rangle - (\langle \mathrm{Tr}\, X \rangle)^2$$

and so on ......

Striking fact : $\langle (\mathrm{Tr}\, X)^p \rangle_c$ quadratic in n $\forall p$

Final result : $\log(Z_{S^4}\, G_{2n}) \underset{n\to\infty}{\cong} \log(Z_{S^4}\, G_{2n})\Big|_0$

Contributions only to A, B

$$+ \frac{q+\bar{q}}{2}\left(n - 3 + \frac{2(n+1)}{t}\right)$$

$$+ \frac{q^2+\bar{q}^2}{32}\left(9n - 23 + \frac{13(n+1)}{t}\right) + \frac{(q+\bar{q})^2}{4}\,\frac{n+1}{t^2}$$

$$+ O(q^3)$$

In fact, comparing w/ exact expressions of $A(\tau,\bar{\tau})$, $B(\tau,\bar{\tau})$ by HMORW ('20, '21)

A, B <u>do not</u> receive contributions from $F_{g\geq 2}$

Small vevs ?

Long (and intricate) story short....

$$\mathrm{Tr}\, K^{(2)}_{g\geq 2}(a) \propto \sum_{p=0}^{\infty} f^{(p)}(n)\, P_p(a_\alpha^2)$$

degree-p symmetric polyn.
(S_{n+1} - invariant)

$$f^{(0)}(n) = \sum_{r=0}^{n} c_r\, 2^{r+1} \binom{n+2}{r+2}$$

$$f^{(1)}(n) = \sum_{r=0}^{n+1} c_r\, 2^{r} \binom{n+1}{r}$$

⋮

similar structures

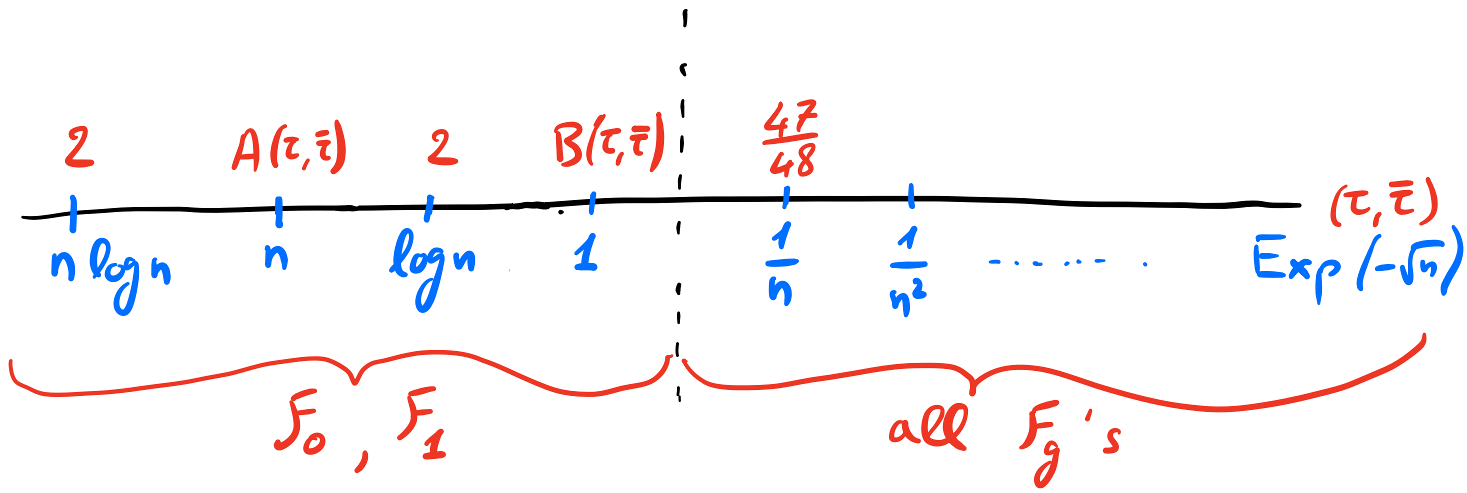
Computing moments in $X_n(t)$:

$$\log\left(Z_{S^4}\, G_{2n}\right)\Big|_{g\geq 2} \underset{n\to\infty}{\simeq} (q^2+\bar{q}^2)\sum_{p=0}^{\infty} \frac{F^{(p)}(n)}{t^p} + O(q^3)$$

with $F^{(p)}(n) \sim e^{-2n\log 3} \quad \forall p$

$\Rightarrow$ τ dependence of $F_{g\geq 2}$ is $\exp(-n)$ suppressed!

Large vevs, exactly (I)



Truncate Z_{S^4} at $\mathcal{F}_1$ and see .....

$$Z_{S^4}\Big|_{g=0,1} = \int_{\mathbb{R}} da \; e^{2\pi i \mathcal{F}_0 - 2\pi i \bar{\mathcal{F}}_0} \; e^{\mathcal{F}_1 + \bar{\mathcal{F}}_1}$$

where
$$\mathcal{F}_0 = \frac{1}{2} \overbrace{F(\tau)}^{\tau_{IR}} a^2$$

$$\mathcal{F}_1 = \frac{1}{12} \log \Delta^{SW}(u) - \frac{1}{2} \log \frac{\partial a}{\partial u}(u)$$

Manschot Moore Zhang '19

Matone : $u = \frac{\partial}{\partial \tau} \mathcal{F}_0 = \frac{F'(\tau)}{2} a^2$

$F \equiv \lambda^{-1}$ modular lambda

$$q = \lambda(\tau_{IR})$$

$$\lambda = \frac{\vartheta_2^4}{\vartheta_3^4}$$

Large vevs, exactly (II)

- Now one can compute a-integral!

$$\boxed{Z_{S^4} \propto \frac{|F'(\tau)|^2}{(F(\tau)-\bar{F}(\bar\tau))^2}} \boxed{= \partial\bar\partial \log\left(F(\tau)-\bar{F}(\bar\tau)\right)}$$

- Remarkably: $\mathbb{H}_{UV} \xrightarrow{F} \mathbb{H}_{IR}$

$$\det M^{(n)}_{ij}(\tau,\bar\tau) = \det \partial^i\bar\partial^j Z_{S^4} = a_{n+1} Z_{S^4}^{\frac{(n+1)(n+2)}{2}}$$

with $a_{n+1} = (-1)^n\, n!\,(n+1)!\, a_n$, $a_1 \equiv 1$

$$\Rightarrow \quad \log G_{2n}(\tau,\bar\tau) = \log\left(n!(n+1)!\right) \supset \begin{cases} 2n\log n \\ 2\log n \end{cases}$$

$$+\, n\left[\log|F'(\tau)|^2 - \log\left(-4\left(F(\tau)-\bar{F}(\bar\tau)\right)^2\right)\right]$$

$A(\tau,\bar\tau)$ of HMORW ('20)

$$+\ \frac{1}{2}\log\frac{\pi}{8} \qquad \tilde{B} = B(\tau,\bar\tau) - \log Z_{S^4}(\tau,\bar\tau)$$

@ large vevs $\tilde{B}$ is independent of τ!

Outlook

Given the irrelevance of instantons to fix the universal large-charge behavior

$$\Gamma\left(2n+\frac{5}{2}\right) \subset G_{2n} \quad \text{for } n\to\infty$$

and given that the 3 Argyres-Douglas theories of rank 1 arise as strong-coupling points of SU(2) SQCD w/ $1 \le N_f \le 3$

Speculate

that we can derive the EFT-predicted

$$\Gamma\left(dn+\frac{3d-1}{2}\right) \qquad d=\frac{6}{5},\frac{4}{3},\frac{3}{2}$$

from asymptotically-free SU(2) SQCD.

@ Large vevs : $\mathcal{F}_0 \propto a^2 = u^{\frac{2}{d}}$; $\mathcal{F}_1 \sim \frac{3}{2}(d-1)\log a$

@ small vevs : $Z_{\text{inst}} \sim 1+\mathcal{O}(\Lambda R)$

[Grassi, Fucito, Morales, RS '23] $Z_{1L} \longrightarrow \text{const.}$

Interpolating : $|Z|^2\Big|_{g=0,1,\infty} = e^{(\mathcal{F}_0+\bar{\mathcal{F}}_0)R^2}\left(a^2+\text{const}\right)^{\frac{3}{2}(d-1)}$

$$M^{(n)}_{ij}\Big|_{g=0,1,\infty} = \int_{\mathbb{R}} da \ (a^2)^{\frac{d}{2}(i+j)} \ e^{-ca^2} \left(a^2 + \text{const}\right)^{\frac{3}{2}(d-1)} \sim$$

$$\sim \int_{\mathbb{R}_+} dx \ x^{\frac{1}{2}((i+j)d-1)} \ e^{-x} \left(x + \text{const}\right)^{\frac{3}{2}(d-1)}$$

the universal behavior should not depend on this constant !!

Note, as an aside, that :

$$M^{(0)} = \sqrt{\pi} \ \text{HypergeometricU}\left[-\frac{3}{2}(d-1), -\frac{3}{2}d+2, \text{const}\right]$$

Challenge :

Verify, even only numerically, that :

$$\frac{\det M^{(n)}_{ij}}{\det M^{(n-1)}_{ij}} \underset{n\to\infty}{\supset} \Gamma\left(dn + \frac{3d-1}{2}\right)$$