Spinning the Large-Charge Bootstrap

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New directions in the large-charge expansion
Les Diablerets, Switzerland
June 23, 2026

Motivation

Large charge success story

- Two heavy external operators

$$\Phi_Q, \qquad \Phi_{-Q}, \qquad Q \gg 1$$

- Plain vanilla setup: a superfluid EFT on the cylinder

$$\mathbb{R} \times S^2$$

- The lowest operator dimension

$$\Delta_Q = \alpha |Q|^{3/2} + \beta |Q|^{1/2} + \gamma + \cdots .$$

EFT = systematic way for computations in $1/|Q|$

$$\langle Q | \, \mathcal{O}_{\text{light}} \ldots \mathcal{O}_{\text{light}} | Q \rangle$$

Motivation

Large charge is not always the same state of matter

- CFT can realize large charge through qualitatively different states

superfluid,   Fermi liquid on S^2,   conformal solids

- In large-N holographic CFTs, the large-charge state can also be dual to a bulk object such as

AdS boson star (CFT superfluid)   or   charged black hole

Questions

- How generic is the superfluid EFT description?
- What assumptions about the CFT lead to the EFT description?
- Crossing is the most generic constraint in CFT
- All CFTs in question have

$$\phi_q, \quad J^a, \quad T^{ab}$$

Large charge bootstrap

- The scalar-scalar sector was done in JHEP 05 (2018) 043 by Daniel Jafferis, Baur Mukhametzhanov, Alexander Zhiboedov

Assumptions

1. The ground state Φ_Q is unique
2. The large charge-Q spectrum is parity even and

$$\Delta_\ell \neq \Delta_Q + \ell, \qquad \Delta_\ell \neq \Delta_Q + \ell - 2, \qquad \ell \geq 2$$

3. The macroscopic limit exists

$$Q \to \infty, \quad R \to \infty$$

$$\varepsilon = \Delta_Q R^{-3} = \text{fixed}, \quad \rho = Q R^{-2} = \text{fixed}$$

4. The same order in the s-channel expansion:

$$\text{first descendant of } \Phi_Q \qquad + \qquad \text{new primary states}$$

5. Only finitely many Regge trajectories contribute

There is at least one Goldstone Regge trajectory

Assumptions

Assuming also

5. The large charge-Q spectrum is non-degenerate

Only the Goldstone trajectory contributes to

$$\langle \Phi_{-Q} \mathcal{O}\, T^{ab} \Phi_Q \rangle$$

6. Only one Regge trajectory reaches zero at $\ell = 0$:

Only the Goldstone trajectory contributes to

$$\langle \Phi_{-Q} \mathcal{O}\, J^{a} \Phi_Q \rangle$$

Macroscopic limit (EFT anzatz)

The limit

$$Q \to \infty, \quad R \to \infty$$

$$\varepsilon = \frac{\Delta_Q}{R^3} = \text{fixed}, \quad \rho = \frac{Q}{R^2} = \text{fixed}$$

implies that

$$R = \sqrt{\frac{Q}{\rho}}, \qquad \Delta_Q = \alpha |Q|^{3/2} + \dots$$

The cylinder coordinates

$$ds^2 = R^2(d\tau^2 + d\theta^2) + \dots = \text{fixed}$$

Coordinates scale as

$$\tau,\ \theta \sim \frac{1}{\sqrt{|Q|}}$$

Scalar-scalar channel

Consider for simplicity just the scalar-scalar sector

$$\langle \Phi_{-Q}(x_4)\,\phi_{-q}(x_3)\,\phi_q(x_2)\,\Phi_Q(x_1)\rangle$$

$$\Downarrow$$

$$|z|^{\Delta_Q}\,G^{(Q)}_{SS}(z,\bar z) = \langle Q|\phi_{-q}(x_3)\,\phi_q(x_2)|Q\rangle$$

The s-channel expansion for $z = e^{\tau}\cos\theta$

$$\begin{aligned}|z|^{\Delta_Q}\,G^{(Q)}_{SS}(z,\bar z) &= \sum_{\mathcal{O}} |\lambda_{\mathcal{O}}|^2 g_{\mathcal{O}}(z,\bar z)\\ &= \sum_{\mathcal{O}} |\lambda_{\mathcal{O}}|^2 |z|^{\Delta_{\mathcal{O}}}\left(1 + A_{\mathcal{O}}|z|\cos\theta + \ldots\right)\end{aligned}$$

with

$$A_{\mathcal{O}} = \frac{\Big[\Delta_Q - \Delta_{\mathcal{O}} + O(1)\Big]\Big[\Delta_Q - \Delta_{\mathcal{O}} + O(1)\Big]}{2\Delta_{\mathcal{O}}}$$

Conformal block expansion

The leading contribution comes from the ground state

$$\Phi_{Q+q}$$

Corrections are suppressed by the relative factor $|z|^{\Delta_{\mathcal{O}}-\Delta_{Q+q}}$

$$|\lambda_{Q+q}|^2 |z|^{\Delta_{Q+q}} \left(1 + A_{Q+q}|z|\cos\theta + \ldots\right)$$

$$+\, |\lambda_{\mathcal{O}}|^2 |z|^{\Delta_{\mathcal{O}}} \underbrace{C_\ell^{(1/2)}(\cos\theta)}_{\text{spin } \ell \text{ primary}} \left(1 + A_{\mathcal{O}}|z|\cos\theta + \ldots\right) + \ldots$$

Operators with

$$\Delta_{\mathcal{O}} \gg \Delta_{Q+q}$$

are only relevant for

$$\tau < \frac{1}{\Delta_{\mathcal{O}}} \ll \frac{1}{\sqrt{|Q|}}$$

Large charge anzatz

For other operators the descendant contribution is suppressed

$$A_{\mathcal{O}} \approx \frac{\left[\Delta_{Q+q} - \Delta_{\mathcal{O}}\right]\left[\Delta_{Q+q} - \Delta_{\mathcal{O}}\right]}{2\Delta_{\mathcal{O}}} \approx \frac{1}{\sqrt{|Q|}}$$

It is an assumption that

$$\frac{\lambda_{\mathcal{O}}}{\lambda_{Q+q}} \sim \frac{1}{\sqrt{|Q|}}$$

Therefore the large charge expansion gives

$$G_{SS}^{(Q)}(z,\bar{z}) = |\lambda_{Q+q}|^2 |z|^{\Delta_{Q+q}-\Delta_Q} \left(1 + A_{Q+q}|z|\cos\theta\right) + \sum_{\mathcal{O}} |\lambda_{\mathcal{O}}|^2 |z|^{\Delta_{\mathcal{O}}-\Delta_Q} C_\ell^{(1/2)}(\cos\theta)$$

Crossing LO

The crossing equations can be obtained by equating the s- u-channels

$$\langle \Phi_Q(x_1)\,\phi_{-q}(x_3)\,\phi_q(x_2)\,\Phi_{-Q}(x_4)\rangle = \langle \Phi_{-Q}(x_4)\,\phi_{-q}(x_3)\,\phi_q(x_2)\,\Phi_Q(x_1)\rangle$$

$$\Downarrow$$

$$G_{SS}^{(-Q)}\left(\frac{1}{z},\frac{1}{\bar{z}}\right) = G_{SS}^{(Q)}\left(z,\bar{z}\right)$$

At LO it implies that

$$|\lambda_{Q+q}|^2 = |\lambda_{-Q+q}|^2$$

and

$$\Delta_{Q+q} - \Delta_Q = \Delta_Q - \Delta_{Q-q}$$

Crossing NLO

At NLO, we get

$$e^{(\Delta_{Q+q}-\Delta_Q)\tau}\left(1+\frac{9\alpha q^2}{8\sqrt{Q}}\,e^{\tau}\cos\theta\right)+\sum_{\mathcal{O}}|\lambda_{\mathcal{O}}|^2 e^{(\Delta_{\mathcal{O}}-\Delta_Q)\tau}C_\ell^{(1/2)}(\cos\theta)$$

$$=e^{(\Delta_Q-\Delta_{Q-q})\tau}\left(1+\frac{9\alpha q^2}{8\sqrt{Q}}\,e^{\tau}\cos\theta\right)+\sum_{\mathcal{O}}|\lambda_{\mathcal{O}}|^2 e^{-(\Delta_{\mathcal{O}}-\Delta_Q)\tau}C_\ell^{(1/2)}(\cos\theta)$$

Defining

$$\omega_\ell=\Delta_{\mathcal{O}}-\Delta_Q,\qquad |\lambda_{\mathcal{O}}|^2=(2\ell+1)\frac{3\alpha q^2}{8\sqrt{Q}}\frac{|\lambda_\ell|^2}{\omega_\ell}$$

$$h(\tau,\cos\theta)=\tau+3\,e^{\tau}\cos\theta+\sum_{\ell}\frac{2\ell+1}{\omega_\ell}|\lambda_\ell|^2\,e^{\omega_\ell\tau}\,C_\ell^{(1/2)}(\cos\theta)$$

which satisfies

$$h(-\tau,\cos\theta)=h(\tau,\cos\theta)$$

EFT

In EFT, for $\tau < 0$, defining

$$J_\ell^2 = \frac{\ell(\ell+1)}{2}$$

we obtain

$$h_{EFT}(\tau, \cos\theta) = \tau + 3\, e^{\tau} \cos\theta + \sum_{\ell=2}^{\infty} \frac{2\ell+1}{J_\ell}\, e^{J_\ell \tau}\, C_\ell^{(1/2)}(\cos\theta)$$

Formally the derivative

$$\partial_\tau h_{EFT}(0^-, \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, C_\ell^{(1/2)}(\cos\theta) = 4\pi \delta_{S^2}(\vec{n})$$

Higher derivatives

$$\partial_\tau^{2n+1} h_{EFT}(0^-, \cos\theta) = 4\pi \sum_{k=0}^{n} c_k^{EFT} (\vec{\nabla}^2)^k \delta_{S^2}(\vec{n})$$

This suggests a way to solve the crossing equation

Bootstrap equations

The t-channel singularity is the only singularity of

$$h(\tau, \cos\theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{\omega_\ell} A_\ell \, e^{\omega_\ell \tau} \, C_\ell^{(1/2)}(\cos\theta)$$

We define the integral I_n

$$I_n = \int_0^\pi \left[\partial_\tau^{2n+1}\Big|_{\tau=-\varepsilon} h(\tau,\theta) - \partial_\tau^{2n+1}\Big|_{\tau=\varepsilon} h(\tau,\theta) \right] C_\ell^{(1/2)}(\cos\theta) \sin\theta d\theta$$

From the spectral sum we get

$$I_n = 2A_\ell \, \omega_\ell^{2n}$$

Bootstrap equations

On the other hand, the existence of the macroscopic limit implies

$$h(\tau, \cos\theta) \underset{\tau,\theta\to 0}{=} \frac{a_0\left(\frac{\tau}{\theta}\right)}{\left(\tau^2+\theta^2\right)^{\frac{1}{2}}} + \text{less singular terms,}$$

Therefore, every odd derivative is a distribution

$$\partial_\tau^{2n+1} h(0^\mp, \cos\theta) = \pm \sum_{k=0}^{n} c_k (\vec{\nabla}^2)^k \delta_{S^2}(\vec{n})$$

As a result,

$$I_n = 2\int_0^\pi d\theta\, C_\ell^{(1/2)}(\cos\theta)\sin\theta \sum_{k=0}^{n} c_k (\vec{\nabla}^2)^k \delta_{S^2}(\vec{n}) = 2P_n(J_\ell^2)$$

Therefore

$$A_\ell \omega_\ell^{2n} = P_n(J_\ell^2)$$

Regge trajectories

For $\ell \geq 2$

$$\sum_{i=1}^{N} |\lambda_{\ell,i}|^2 \, \omega_{\ell,i}^{2n} = P_n(J_\ell^2)$$

For $\ell = 1$

$$\underbrace{1}_{\text{descendant}} + \sum_{i=1}^{N} |\lambda_{1,i}|^2 \, \omega_{1,i}^{2n} = P_n(1)$$

For $\ell = 0$

$$\underbrace{\delta_{n,0}}_{\text{dimension mismatch}} + \sum_{i=1}^{N} |\lambda_{0,i}|^2 \, \omega_{0,i}^{2n} = P_n(0)$$

Unknowns: fusion coefficients $|\lambda_\ell|^2$, energies ω_ℓ, and polynomials $P_n(J_\ell^2)$

Regge trajectories

The spectrum is analytic for $\ell \geq 2$

We introduce the variables

$$w = J_\ell^2, \quad x_i(w) = \omega_{\ell,i}^2, \quad S_i(w) = |\lambda_{\ell,i}|^2$$

As a result, for $w \geq 3$

$$\sum_{i=1}^{N} S_i(w)\, x_i^n(w) = P_n(J_\ell^2)$$

For $w = 1$

$$1 + \sum_{i=1}^{N} S_{1,i}\, x_{1,i}^n = P_n(1)$$

For $w = 0$

$$\delta_{n,0} + \sum_{i=1}^{N} S_{0,i}\, x_{0,i}^n = P_n(0)$$

Regge trajectories

The partition function

$$W(t,w) = \sum_{n=0}^{\infty} t^n P_n(w)$$

For $w \geq 3$

$$W(t,w) = \sum_{i=1}^{N} \frac{S_i(w)}{1 - t\, x_i(w)}$$

For $w = 1$

$$W(t,1) = \frac{1}{1-t} + \sum_{i=1}^{N} \frac{S_{1,i}}{1 - t\, x_{1,i}}$$

For $w = 0$

$$W(t,1) = 1 + \sum_{i=1}^{N} \frac{S_{0,i}}{1 - t\, x_{0,i}}$$

Consistency

$$x_{0,i} \in \{x_1(0), \ldots x_N(0)\}, \quad x_{1,i} \in \{x_1(1), \ldots x_N(1)\}$$

Solutions

The strategy

- Solve for $w \geq 3$
- Find $P_n(w)$
- Analytically continue the rhs $P_n(w)$ to $w = 0, 1$

Example: $N = 1$

For $w \geq 3$

$$S(w)x^n(w) = P_n(w)$$

The solution

$$S(w) = P_0, \qquad x(w) = \frac{P_1(w)}{P_0} = c^2 w + m^2, \qquad P_n(w) = P_0\left(c^2 w + m^2\right)^n$$

For $w = 0$

$$\left.\begin{array}{ll} n = 0: & 1 + S_0 = P_0 \\ n \geq 1: & S_0 x_0^n = P_0\, m^{2n} \end{array}\right\} \Rightarrow m^2 = x_0 = S_0 = 0, \quad P_0 = 1$$

For $w = 1$

$$1 + S_1 x_1^n = c^{2n} \quad \Rightarrow \quad c^2 = 1, \quad S_1 = 0$$

The only solution is the Goldstone

$$P_n(w) = w^n, \qquad n \geq 0$$

$N = 2$

For $N = 2$

$$\begin{aligned} P_{n+2} &= \left(S_1 x_1^{n+1} + S_2 x_2^{n+1}\right)(x_1 + x_2) - \left(S_1 x_1^n + S_2 x_2^n\right) x_1 x_2 \\ &= P_{n+1}(x_1 + x_2) - P_n x_1 x_2 \end{aligned}$$

Equivalently,

$$P_{n+2}(w) = P_{n+1}(w) Q_1(w) - P_n(w) Q_2(w),$$

with $x_{1,2}(w)$ roots of

$$x^2 - Q_1(w)x + Q_2(w) = 0$$

In particular for $Q_1(w) = w$ and $Q_2(w) = (1 - a^2)w^2 + a^2 w$

$$x_{1,2} = \frac{w \pm a\sqrt{w(w-1)}}{2}$$

Arbitrary N

In general, we can show that polynomials satisfy order N recursion

$$P_{n+N}(w) = Q_1(w)P_{n+N-1}(w) + \cdots + (-1)^{n+1}Q_N(w)P_n(w)$$

with Q_n symmetric polynomials of x_i. Equivalently,

$$\prod_{n=1}^{N}(x - x_i) = x^N - Q_1 x^{n-1} + \cdots + (-1)^N Q_N$$

At the same time, Q_n is an order n polynomial of w

Any solution is specified by: $P_{n-1}(w)$ and $Q_n(w)$, with $n = 1, \ldots, n$

Generalization

- Solutions are not sufficiently restricted by only considering scalars
- Existence of J^a and T^{ab} is necessary
- Additional universal constraints
- Technically more challenging
- We are interested in

$$G_{SB}(\tau, \cos\theta) = \left\langle Q - \frac{q}{2} \right| \phi_{-q}(x_3) B(x_2) \left| Q + \frac{q}{2} \right\rangle$$

and

$$G_{AB}(\tau, \cos\theta) = \langle Q | \, A(x_3) B(x_2) \, | Q \rangle$$

with

$$A, B = J^a, T^{ab}$$

Spinning 4-pt functions: primaries contibution

In this case the correlators have tensor structure. For example,

$$\left\langle Q-\frac{q}{2}\right|\phi_{-q}(x_3)J^a(x_2)\left|Q+\frac{q}{2}\right\rangle$$

$$=\sum_{\mathcal{O}} e^{\Delta(\tau_2-\tau_3)}\underbrace{\left\langle Q-\frac{q}{2}\right|\phi_{-q}(0,\vec{n}_3)\left|\mathcal{O}_{a_1\ldots a_\ell}\right\rangle}_{\lambda_S^*(n_3^{a_1}\ldots n_3^{a_\ell}-\text{traces})}\underbrace{\left\langle\mathcal{O}_{a_1\ldots a_\ell}\right|J^a(0,\vec{n}_2)\left|Q+\frac{q}{2}\right\rangle}_{\lambda_J^{(1)}(\ell+1)\oplus\lambda_J^{(2)}(\ell-1)}$$

Conservation of J^a implies that

$$\langle\mathcal{O}_{a_1\ldots a_\ell}|J^a(0,\vec{n}_2)|Q\rangle=\lambda_J^{(1)}T^{(1)}_{a,a_1,\ldots,a_\ell}+\lambda_J^{(2)}T^{(2)}_{a,a_1,\ldots,a_\ell},\quad \lambda_J^{(2)}\sim\lambda_J^{(1)}$$

Therefore

$$=\sum_{\mathcal{O}}\lambda^*_{S,\ell}\lambda_{J,\ell}\,e^{\Delta(\tau_2-\tau_3)}\Big\{n_2^a\left[A_\ell\,C_\ell^{(3/2)}(\cos\theta)+B_\ell\,C_{\ell-2}^{(1/2)}(\cos\theta)\right]$$

$$-n_3^a\,(A_\ell+B_\ell)\;C_{\ell-1}^{(3/2)}(\cos\theta)\Big\}$$

$$=G_{SJ,2}\,n_2^a+G_{SJ,3}\,n_3^a$$

Spinning 4-pt functions

A systematic way for computing the spinning conformal blocks:
Costa, Penedones, Poland, Rychkov JHEP 11 (2011) 154

- There is only one coupling constant (at fixed spin) for every channel
- All tensor structures are proportional to this coupling constant
- Example

$$\langle Q|J^a J^b|Q\rangle_{(\ell)} = |\lambda_{J,\ell}|^2 \left(f_{33} n_3^a n_3^b + f_{32} n_3^a n_2^b + f_{23} n_2^a n_3^b + f_{22} n_2^a n_2^b + f_\delta \delta^{ab}\right)$$

- ϕT^{ab}, $J^a T^{bc}$, and $T^{ab} T^{cd}$ have 3, 8, and 14 tensor structures

H-basis

- Crossing mixes these tensor structures, schematically

$$f_{ij}(-\tau, \cos\theta) = \sum_{k,l} M_{ij,kl}\, f_{kl}(\tau, \cos\theta)$$

Special basis for coefficient functions:

- Contain only one Gegenbauer polynomial for fixed spin, schematically

$$\sum_{\ell} C_{\ell-k}^{(1/2+k)}(\cos\theta)$$

- Are eigenfunctions of the crossing

$$h(-\tau, \cos\theta) = \pm h(\tau, \cos\theta)$$

H-basis illustration

- For the scalar-scalar channel

$$H_{SS}(\tau, \cos\theta) = G_{SS}(\tau, \cos\theta)$$

- For the scalar-vector channel

$$\begin{aligned} &\text{level 0}: \; G^a_{SV}(\tau, \cos\theta) && \supset n^a_3\, H_{SV,3}(\tau, \cos\theta) && \to \sum_\ell C^{(3/2)}_{\ell-1}(\cos\theta) \\ &\text{level 1}: \; n^a_2 G^a_{SV}(\tau, \cos\theta) && = H_{SV,0}(\tau, \cos\theta) && \to \sum_\ell C^{(1/2)}_{\ell}(\cos\theta) \end{aligned}$$

As a result

$$H_{SV,3}(\tau, \cos\theta) = G_{SV,3}(\tau, \cos\theta)$$

$$H_{SV,0}(\tau, \cos\theta) = G_{SV,2}(\tau, \cos\theta) + G_{SV,3}(\tau, \cos\theta)\cos\theta$$

Q-scaling

The Q-scaling of the NLO terms is fixed by the descendant:

$$\delta G_{SB} \sim \frac{1}{Q}, \quad \delta G_{AB} = \frac{1}{|Q|^{3/2}}, \quad A, B = J^a, T^{ab}$$

For H-functions

$$H_{AB,r} = H_{AB,0} \left[\delta_{0,r} + \kappa_{AB}\, h_{AB,r}\right].$$

with

$$\kappa_{SS} = \alpha\, \frac{3q^2}{8\sqrt{|Q|}}, \qquad \kappa_{SB} = \frac{q}{4Q}, \qquad \kappa_{AB} = \frac{1}{6\alpha|Q|^{3/2}}.$$

Asymptotic behavior

- The macroscopic limit fixes the leading singularities
- Example, for $\tau, \theta \to 0$

$$G^a_{SV} \underset{\theta\to 0}{=} n_2^a\, H_{SV,0} + (n_3^a - n_2^a)\, H_{SV,3} \underset{\tau,\theta\to 0}{\sim} \frac{1}{\tau^2+\theta^2}$$

For the NLO

$$|\vec{n}_3 - \vec{n}_2| = O(\theta)$$

therefore

$$h_{SV,0} \underset{\tau,\theta\to 0}{\sim} \frac{1}{\tau^2+\theta^2} \qquad h_{SV,3} \underset{\tau,\theta\to 0}{\sim} \frac{1}{(\tau^2+\theta^2)^{3/2}}$$

Bootstrap equations

- There is an equation for every h-function: $2+3+5+8+14=32$
- Not all are independent:

Consistency equations for the polynomials:

Divisibility by w

$$P_1^{(AB)}(w) = p_{AB} w, \quad n \geq 1, \quad A, B = \phi_{-q}, J^a, T^{ab}$$

Divisibility by w^2

$$P_n^{(AB)}(w) = w^2 \tilde{P}_{n-2}^{(AB)}(w), \quad n \geq 2, \quad A, B = J^a, T^{ab}$$

Spinning bootstrap equations: T-sector

The TT channel

$$h_{TT,\delta\delta} \supset G_{TT}^{abcd} \left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right), \qquad h_{TT,0} \supset G_{TT}^{abcd} n_3^a n_3^b n_2^c n_2^d$$

The expansion

$$h_{TT,\delta\delta} = \sum_i \sum_{\ell=2}^{\infty} (2\ell+1) |\lambda_{T,\ell,i}|^2 \frac{3\left(J_\ell^2 - \omega_{\ell,i}^2\right)^2}{2J_\ell^4 \left(J_\ell^2 - 1\right)^2} \, e^{\omega_{\ell,i}\tau} \, C_{\ell-2}^{(5/2)}(\cos\theta)$$

and

$$h_{TT,0} = 3(1+T_{1,1}) e^{\tau} \cos\theta + \sum_i \sum_{\ell=2}^{\infty} (2\ell+1) |\lambda_{T,\ell,i}|^2 \, e^{\omega_{\ell,i}\tau} \, C_{\ell}^{(1/2)}(\cos\theta)$$

Both functions scale as

$$h_{TT,\delta\delta}, \; h_{TT,0} \underset{\tau,\theta\to 0}{\sim} \frac{1}{\left(\tau^2 + \theta^2\right)^{3/2}} + \dots$$

Spinning bootstrap equations: T-sector

Let's compute the discontinuity integral for $\partial_\tau h$:

$$\partial_\tau \Big|_{\tau=\varepsilon} h_{TT,\delta\delta},\ \partial_\tau \Big|_{\tau=\varepsilon} h_{TT,0} \underset{\tau,\theta\to 0}{\sim} \frac{\varepsilon}{\left(\varepsilon^2+\theta^2\right)^{5/2}} + \dots = a\vec{\nabla}^2 \delta_{S^2}(\vec{n}) + \dots$$

As a result for $\ell \geq 2$

$$\sum_i \omega_{\ell,i} |\lambda_{T,\ell,i}|^2 \frac{\left(J_\ell^2 - \omega_{\ell,i}^2\right)^2}{J_\ell^4 \left(J_\ell^2 - 1\right)^2} = a \int d\theta \sin^5\theta \, C_{\ell-2}^{(5/2)}(\cos\theta) \vec{\nabla}^2 \delta_{S^2}(\vec{n}) + \dots = 0$$

and

$$\sum_i \omega_{\ell,i} |\lambda_{T,\ell,i}|^2 = a \int d\theta \sin\theta \, C_\ell^{(1/2)}(\cos\theta) \vec{\nabla}^2 \delta_{S^2}(\vec{n}) + \dots = P_1^{(TT,0)}(J_\ell^2)$$

For $\ell = 1$

$$1 + T_{1,1} = P_1^{(TT,0)}(J_\ell^2)$$

Spinning bootstrap equations: T-sector

All terms in the sum are non-negative

$$\sum_i \omega_{\ell,i} |\lambda_{T,\ell,i}|^2 \frac{\left(J_\ell^2 - \omega_{\ell,i}^2\right)^2}{J_\ell^4 \left(J_\ell^2 - 1\right)^2} = 0$$

Therefore

$$|\lambda_{T,\ell,i}|^2 = 0, \quad \text{or} \quad \omega_{\ell,i}^2 = J_\ell^2$$

It is inconsistent to have all $\lambda_{T,\ell,i} = 0$ because in this case

$$0 = P_1^{(TT,0)}(J_\ell^2) = 1 + T_{1,1} \geq 1$$

No degeneracy: there is one and only one Goldstone trajectory

$N = 1$: T-sector

For $w \geq 3$ and $n \geq 1$

$$P_n^{(TT)}(w) = T_1(w)\, w^n = p_{TT}\, w^n$$

$$P_{n-1}^{(ST)}(w) = \sqrt{S_1(w) T_1(w)}\, w^{n-1} = p_{ST}\, w^{n-1},$$

$$P_n^{(VT)}(w) = \sqrt{V_1(w) T_1(w)}\, w^n = p_{VT}\, w^{n-1}$$

For $w = 1$

$$P_n^{(TT)}(w) = p_{TT} = 1 + T_{1,1}$$

$$P_{n-1}^{(ST)}(1) = p_{ST} = 1 + \sqrt{S_{1,1} T_{1,1}} = 1,$$

$$P_n^{(VT)}(w) = p_{VT} = 1 + \sqrt{V_{1,1} T_{1,1}}$$

$N = 1$: SV-sector

For $w \geq 3$ and $n \geq 1$

$$P_{n-1}^{(SV)}(w) = \frac{(1+\sqrt{V_{1,1}T_{1,1}})}{1+T_{1,1}}\, w^{n-1}$$

$$P_{n}^{(VV)}(w) = \frac{(1+\sqrt{V_{1,1}T_{1,1}})^2}{1+T_{1,1}}\, w^{n}$$

For $w = 1$

$$P_{n-1}^{(SV)}(1) = \frac{(1+\sqrt{V_{1,1}T_{1,1}})}{1+T_{1,1}} = 1 + \sqrt{S_{1,1}V_{1,1}} = 1$$

$$P_{n}^{(VV)}(1) = \frac{(1+\sqrt{V_{1,1}T_{1,1}})^2}{1+T_{1,1}} = 1 + V_{1,1} \quad \Rightarrow \quad V_{1,1} = T_{1,1} = 0$$

Therefore

$$P_{n-1}^{(SS)}(w) = P_{n-1}^{(SV)}(w) = P_{n-1}^{(ST)}(w) = w^{n-1}$$

$$P_{n}^{(VV)}(w) = P_{n}^{(VT)}(w) = P_{n}^{(TT)}(w) = w^{n}$$

$N=2$: T-sector

In this case

$$x_1(w)=w, \qquad x_2(w)=c^2w+m^2$$

For $w\geq 3$ and $n\geq 1$

$$P_n^{(TT)}(w)=T_1(w)\,w^n=p_{TT}\,w^n$$

$$P_{n-1}^{(ST)}(w)=\sqrt{S_1(w)T_1(w)}\,w^{n-1}=p_{ST}\,w^{n-1},$$

$$P_n^{(VT)}(w)=\sqrt{V_1(w)T_1(w)}\,w^n=p_{VT}\,w^{n-1}$$

For $w=1$

$$P_n^{(TT)}(w)=p_{TT}=1+T_{1,1}$$

$$P_{n-1}^{(ST)}(1)=p_{ST}=1+\sqrt{S_{1,1}T_{1,1}},$$

$$P_n^{(VT)}(w)=p_{VT}=1+\sqrt{V_{1,1}T_{1,1}}$$

$N = 2$: SV-sector

For $w \geq 3$ and $n \geq 1$

$$P_{n-1}^{(SV)}(w) = \frac{(1+\sqrt{S_{1,1}T_{1,1}})(1+\sqrt{V_{1,1}T_{1,1}})}{1+T_{1,1}}\, w^{n-1}$$

$$+ \sqrt{S_2(w)V_2(w)}\,(c^2 w + m^2)^{n-1}$$

$$P_n^{(VV)}(w) = \frac{(1+\sqrt{V_{1,1}T_{1,1}})^2}{1+T_{1,1}}\, w^n + V_2(w)\,(c^2 w + m^2)^n \underset{n\geq 2}{=} w^2\, \tilde{P}_{n-2}^{(VV)}(w)$$

Divisibility for $m^2 \neq 0$ implies $V_2(w) = 0$

For arbitrary N the condition is that $x_i(0) \neq 0$ except for the Goldstone

$N = 2$

For $w = 1$, $n \geq 1$

$$P_{n-1}^{(SV)}(1) = \frac{(1 + \sqrt{S_{1,1}T_{1,1}})(1 + \sqrt{V_{1,1}T_{1,1}})}{1 + T_{1,1}}$$

$$= 1 + \sqrt{S_{1,1}V_{1,1}} + \sqrt{S_{1,2}V_{1,2}}\,(c^2 + m^2)^{n-1}$$

$$P_{n}^{(VV)}(1) = \frac{(1 + \sqrt{V_{1,1}T_{1,1}})^2}{1 + T_{1,1}} = 1 + V_{1,1} + V_{1,2}\,(c^2 + m^2)^{n}$$

Therefore

$$V_{1,1} = T_{1,1}, \qquad V_{1,2} = 0$$

$N = 2$

For $c^2 + m^2 \neq 1$

$$P_{n-1}^{(SS)}(w) = w^{n-1} + S_{0,2}\left(c^2 w + m^2\right)^{n-1},$$
$$P_{n-1}^{(SV)}(w) = P_{n-1}^{(ST)}(w) = w^{n-1},$$
$$P_{n}^{(VV)}(w) = P_{n}^{(VT)}(w) = P_{n}^{(TT)}(w) = w^{n}$$

For $c^2 + m^2 = 1$ the spin-1 primary coupling is not fixed

$$P_{n-1}^{(SS)}(w) = w^{n-1} + \frac{V_{1,1}}{\left(\sqrt{1+V_{1,1}}+1\right)^2}\left(c^2 w + m^2\right)^{n-1},$$
$$P_{n-1}^{(SV)}(w) = P_{n-1}^{(ST)}(w) = \sqrt{1+V_{1,1}}\, w^{n-1},$$
$$P_{n}^{(VV)}(w) = P_{n}^{(VT)}(w) = P_{n}^{(TT)}(w) = (1+V_{1,1})\, w^{n}$$

THANK YOU

Backup: $\lambda_{q,Q}$ corrections

Restoring the R-dependence for the three point function we get

$$\left\langle Q+q \left| \frac{\phi_q}{R^{\Delta_\phi}} \right| Q \right\rangle = \frac{\lambda_{q,Q}}{R^{\Delta_\phi}} = \text{finite}$$

We conclude that at LO

$$\lambda_{q,Q} = a_q |Q|^{\frac{\Delta_\phi}{2}}$$

Backup: Bootstrap equations an alternative derivation

The t-channel singularity is the only singularity

The existence of the macroscopic limit implies

$$h(\tau, \cos\theta) \underset{\tau,\theta\to 0}{=} \frac{a_0\left(\frac{\tau}{\theta}\right)}{\left(\tau^2+\theta^2\right)^{\frac{1}{2}}} + \text{less singular terms,}$$

After differentiating $2n+1$ times we get for $\tau, \theta \ll 1$

$$\partial_\tau^{2n+1}\Big|_{\tau=\varepsilon} h(\tau,\theta) = \frac{b_0\left(\frac{\tau}{\theta}\right)}{(\tau^2+\theta^2)^{n+1}} + \frac{b_1\left(\frac{\tau}{\theta}\right)}{\left(\tau^2+\theta^2\right)^{\frac{2n+1}{2}}} + \dots$$

We split the integral into 2 regions: $\theta \in [0, \varepsilon\delta)$ and $\theta \in [\varepsilon\delta, \pi]$, with

$$1 \ll \delta$$

The second integral is zero for any nonzero lower bound

Backup: Bootstrap equations an alternative derivation

As a result expanding

$$\sin\theta\, C_\ell^{(1/2)}(\cos\theta) = \theta\, C_\ell^{(1/2)}(1) \sum_{k=0}^{\infty} P_k(J_\ell^2)\, \theta^{2k}$$

and changing the variable of integration:

$$\theta = \frac{\varepsilon}{y}, \quad y \in \left[\frac{1}{\delta}, \infty\right)$$

the integral becomes

$$I_n = \lim_{\varepsilon\to 0} \sum_{p=0}^{n} \sum_{k=0}^{\infty} \varepsilon^{2k-2p}\, P_k(J_\ell^2) \int_{1/\delta}^{\infty} \frac{b_p(y)}{y^{2k+2}\,(1+y^2)^p} dy$$

Only terms with ε^0 contribute

$$I_n = 2P_n(J_\ell^2)$$

Backup: Bootstrap equations an alternative derivation