Resurgence of the NJL model at large charge

Jahmall Bersini

Swiss National Science Foundation

UNIVERSITÄT BERN
AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

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Based on: [JB, S. Hellerman, D. Orlando, S. Reffert, arXiv:2505.21631 [hep-th]]

Large charge expansion

$$\Delta_Q = Q^{\frac{d}{d-1}}\left[\alpha_1 + \alpha_2 Q^{\frac{-2}{d-1}} + \alpha_3 Q^{\frac{-4}{d-1}} + \dots\right] + Q^0\left[\beta_0 + \beta_1 Q^{\frac{-2}{d-1}} + \dots\right] + \mathcal{O}\left(Q^{-\frac{d}{d-1}}\right)$$

I am interested in understanding the general properties of the expansion. In strongly coupled theories (large charge EFT), this is challenging.

Progress is possible by considering weakly coupled theories in the "double scaling limit" approach:

- Large N - large Q with q = Q/N
- Small ε – large Q with q = Qε

Semiclassical calculations yield results valid for arbitrary values of q. One can then consider

1) a small q expansion that reproduces the usual diagrammatic expansion (1/N, ε-expansion).

2) a large q expansion that matches the predictions of the large charge EFT.

Questions

- Can we determine the small/large q expansion to all orders?
- Do the small/large q expansions converge?
- Are there nonperturbative corrections?
- Can we understand the properties of the large charge expansion beyond the double scaling limit?

Resurgence

[J. Ecalle (1981-1985)]

To answer these questions, we resort to

Resurgence theory

"*mathematical framework that links perturbative series and nonperturbative effects*"

Nonperturbative corrections are encoded as singularities in the Borel plane.

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BOREL SUMMATION:

$$F(z) = \sum_{n=1}^{\infty} a_n z^n$$

$$a_n \underset{n\to\infty}{\sim} n!$$

$$\mathcal{B}[F](t) = \sum_{n=1}^{\infty} \frac{a_n}{n!} t^n$$

$$F_{\rm sum} = \int_0^{\infty} e^{-\frac{t}{z}} \mathcal{B}[F](t)\, dt$$

Poles on the integration path produce ambiguities scaling as $e^{-1/z}$

These are fixed by adding nonperturbative terms of the same order.

Resurgence: an example

Consider the ODE defining the Dawson function F(z):

$$\begin{cases} \dfrac{dF}{dz} = +2zF = 1 \\ F(0) = 0 \end{cases} \Longrightarrow \quad F(z) = e^{-z^2} \int_0^z du\; e^{-u^2}$$

The series solution is asymptotic: $F \sim \displaystyle\sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^{k+1}} \frac{1}{z^{2k+1}}$

The Borel transform of the series has a pole at t = 1 and Borel summation leads to ambiguities that are resolved by adding a nonperturbative term:

$$Ce^{-z^2}$$

The perturbative series is upgraded to a transseries:

~ perturbative series + exponential corrections.

$C = i\sqrt{\pi}/2$ is the integration constant of the ODE which disappeared in the perturbative expansion.

Resurgence meets large charge

Over the years, there has been significant progress in understanding resurgence in quantum mechanics and 2d QFTs.

However, quantum field theories in d>2 pose formidable challenges. While it is generally expected that QFTs lead to asymptotic series (Dyson's argument), usually one knows only a handful of perturbative coefficients.

The large charge expansion offers the opportunity to study resurgence relations as one has access to complete and nontrivial series expansions in the double scaled parameter q.

These ideas were first investigated in [N. Dondi, I. Kalogerakis, D. Orlando, S. Reffert (2021)] and further explored in [O. Antipin, JB, F. Sannino, M. Torres (2022)] for bosonic CFTs.

Our goal: study the small/large q expansion in the fermionic Nambu--Jona-Lasinio model in the large N – large Q double scaling limit (q=Q/N).

The NJL model in d=3

$$\mathcal{L} = \bar{\Psi}_i \Gamma^\mu \partial_\mu \Psi_i - \frac{g}{N}\left[\left(\bar{\Psi}_i \Psi_i\right)^2 - \left(\bar{\Psi}_i \Gamma_5 \Psi_i\right)^2\right] \qquad i = 1, \ldots N$$

[Y. Nambu, G. Jona-Lasinio (1961)]

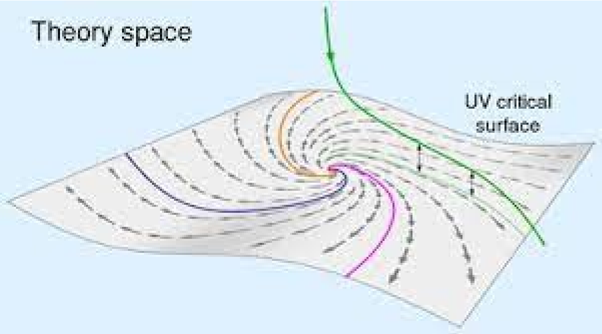
In d =3 the four-fermion interaction is non-renormalizable.

The model can be studied via the 1/N expansion.

At large N the model flows to an interacting fixed point in the UV.

<u>Nonperturbative renormalizability</u>

Asymptotic safety



The NJL model in d=3

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To study the model at large N, one employs the Hubbard–Stratonovich trick and introduces an auxiliary complex scalar Φ.

$$\mathcal{L} = \bar{\Psi}\left(\Gamma_\mu \partial^\mu + \Phi \frac{1+\Gamma_5}{2} + \bar{\Phi}\frac{1-\Gamma_5}{2}\right)\Psi + \frac{N}{4g}\bar{\Phi}\Phi$$

- Integrating out Φ one recovers the NJL Lagrangian.
- Integrating out the fermion one generates the 1/N expansion.
- The model has a U(N) flavor symmetry plus a $U(1)_A$ chiral symmetry

$$\boxed{\Psi_i \to e^{i\alpha\Gamma_5}\Psi_i \qquad \Phi \to e^{-2i\alpha}\Phi}$$

We study the large $U(1)_A$ charge Q sector of the model.

The grand potential

To understand the dynamics at fixed $U(1)_A$ charge we compute the grand potential, which at the leading order in 1/N reads (functional determinant)

$$\Omega = -N\,\mathrm{Tr}\log(\Gamma_\mu\partial^\mu - \mu\Gamma_3\Gamma_5 + \Phi_0)$$

and can be expressed as a sum over the one-particle energies ω_+, ω_-

$$\frac{\Omega}{N} = \frac{\Phi_0^2}{4g} - \int^\Lambda \frac{d^2p}{(2\pi)^2}\left[\omega_+ + \omega_- + \frac{2}{\beta}\log\left(1+e^{-\beta\omega_+}\right) + \frac{2}{\beta}\log\left(1+e^{-\beta\omega_-}\right)\right]$$

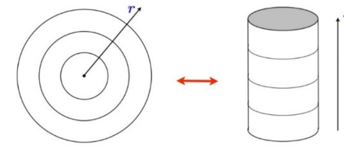
where $\omega_\pm = \sqrt{\Phi_0^2 + (p\pm\mu)^2}$ with μ the chemical potential.

The gap equation $\frac{\partial\Omega}{\partial\Phi_0} = 0$ yields $\Phi_0 = \mu\sqrt{\kappa_0^2 - 1}$ $\qquad \kappa_0 \tanh\kappa_0 = 1$

The one-particle energies are always positive: no Fermi surface. Due to Yukawa interactions the charge leaks from the fermions into the scalars.

The ground state breaks a linear combination of time translations and $U(1)_A$: conformal superfluid phase.

The scaling dimension



The scaling dimension Δ_Q of the lowest-lying operator with charge Q is obtained by evaluating the grand potential Ω on the cylinder $\mathbb{R}\times S^2$ (state-operator correspondence)

$$\Delta_Q = r(\Omega(\mu) + \mu Q)\big|_{\mu=\mu(Q)} \qquad Q = -\frac{\partial\Omega}{\partial\mu}$$

$$\frac{\Omega}{N} = -\sum_{\ell=0}^{\infty} n_\ell \left(\omega_+(\ell) + \omega_-(\ell)\right) \qquad \omega_\pm = \sqrt{\Phi_0^2 + (\lambda_\ell \pm \mu)^2}$$

λ_l are the eigenvalues of the Dirac operator on the sphere and have degeneracy n_l.

RESULTS:

$$\frac{\Delta_Q}{2N} = \frac{2}{3}(q/\kappa_0)^{3/2} + \frac{1}{6}(q/\kappa_0)^{1/2} + \mathcal{O}\left(q^{-1/2}\right) \qquad q \gg 1$$

$$\frac{\Delta_Q}{2N} = \frac{q}{2} + \frac{2q^2}{\pi^2} + \mathcal{O}\left(q^3\right) \qquad q \ll 1$$

$$\lambda_\ell = \frac{\ell+1}{r}$$

$$n_\ell = 2(\ell+1)$$

$$q = \frac{Q}{2N}$$

$$\kappa_0 \tanh\kappa_0 = 1$$

[N. Dondi, S. Hellerman, I. Kalogerakis, R. Moser, D. Orlando, S. Reffert (2022)]

Questions

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Zeta function

Our central quantity is the grand potential on the cylinder

$$\Omega = -N\ \mathrm{Tr}\log(\Gamma_\mu \partial^\mu - \mu\Gamma_3\Gamma_5 + \Phi_0)$$

To regularize the functional determinant, we adopt a Zeta-function scheme

$$\begin{aligned}\zeta(s|\not{\partial},\mu,\Phi_0) &= \sum_{\ell=0}^{\infty} n_\ell\Big(\omega_+(\ell)^{-2s} + \omega_-(\ell)^{-2s}\Big)\\ &= 2r^{2s}\sum_{\ell=1}^{\infty}\ell\Big[\Big((\ell+r\mu)^2+(r\Phi_0)^2\Big)^{-s} + \Big((\ell-r\mu)^2+(r\Phi_0)^2\Big)^{-s}\Big]\end{aligned}$$

$$\boxed{\begin{aligned}&\Omega = -N\zeta(-1/2|\not{\partial},\mu,\Phi_0)\\ &\Phi_0\zeta(1/2|\not{\partial},\mu,\Phi_0) = 0\end{aligned}}$$

Large q expansion

We start from the zeta-regularized functional determinant

$$\boxed{\zeta(s|\partial,\mu,\Phi_0) = 2r^s \sum_{\ell=1}^{\infty} \ell \left[\left((\ell + r\mu)^2 + (r\Phi_0)^2 \right)^{-s} + \left((\ell - r\mu)^2 + (r\Phi_0)^2 \right)^{-s} \right]}$$

Next, we consider the Mellin transform of the zeta function:

$$\zeta(s|\partial,\mu,\Phi_0) = \frac{1}{r^s\Gamma(s)} \int_0^{\infty} \frac{dt}{t} t^s e^{-\Phi_0^2 t} K(t|\mu)$$

$$K(t|\mu) = \sum_{\ell=1}^{\infty} \ell \left(e^{-(\ell/r+\mu)^2 t} + e^{-(\ell/r-\mu)^2 t} \right)$$

The large q limit corresponds to the large Φ_0 limit. For $\Phi_0 >> 1$ the integral localizes around t = 0.

Hence, determining the large q expansion boils down to the calculation of the expansion of the heat kernel K(t|μ) around t = 0.

The Dawson function strikes back

Using the Poisson summation formula: $\sum_{l\in\mathbb{Z}} f(l) = \sum_{k\in\mathbb{Z}} \int d\rho\, f(\rho) e^{2\pi i k\rho}$, we obtain

$$K(t|\mu) = \frac{r^2}{t}\left(e^{-\mu^2 t} + \mu\sqrt{\pi t}\,\mathrm{erf}(\mu\sqrt{t})\right) + \frac{r^2 e^{-\mu^2 t}}{t^{3/2}} \sum_k{}' \left(\sqrt{t} - (\pi r k + i\mu t)\, F\!\left(\frac{\pi r k + i\mu t}{\sqrt{t}}\right) - (\pi r k - i\mu t)\, F\!\left(\frac{\pi r k - i\mu t}{\sqrt{t}}\right)\right)$$

where F denotes the Dawson function. The small t expansion of the heat kernel is determined by the asymptotic expansion of the Dawson function

$$K(t|\mu) \sim \sqrt{\frac{\pi}{t}}\mu r^2\,\mathrm{erf}(\mu\sqrt{t}) + e^{-\mu^2 t} r^2\left[\frac{1}{t} - \frac{1}{t}\sum_{m=1}^{\infty}\sum_{l=0}^{m-1}\frac{(-4)^l |B_{2m}|}{m\Gamma(m-l)\Gamma(2l+1)}(\mu^2 t)^l \frac{t^m}{r^{2m}}\right]$$

Borel transform

We rewrite the double sum as a series expansion around t = 0

$$K(t|\mu) \sim \sqrt{\frac{\pi}{t}}\mu r^2 \operatorname{erf}(\mu\sqrt{t}) + e^{-\mu^2 t} r^2 \left[\frac{1}{t} - \frac{1}{t}\sum_{n=1}^{\infty} a_n \left(\frac{t}{r^2}\right)^n\right]$$

$$a_n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^n 4^l B_{2n-2l}}{(2l)!(l-n)\Gamma(n-2l)} \mu^{2l}$$

The large order behavior of the coefficients is

$$a_n \sim a(\mu)\, \pi^{-2n} \Gamma\left(n + \frac{1}{2}\right)$$

The coefficients grow factorially the series is asymptotic.

The Borel transform has a finite radius of convergence determined by a singularity (z = Borel parameter) at .

$$z^* = (\pi r)^2$$

The Mellin integral increases the order of divergence to (2n)!.

Some thoughts...

The Mellin integral increases the order of divergence to (2n)!.

- Same factorial growth observed in the 3d O(N) CFT at large N.
  [N. Dondi, I. Kalogerakis, D. Orlando, S. Reffert (2021)]

- Typical QFT series diverge as n! (e.g., instantons, renormalons,...). A (2n)! growth is observed in string theory/matrix models.

- The optimal truncation order of the series is $n^* \sim \pi\sqrt{q}$, leading to an error that scales as $\exp(-\pi\sqrt{q})$.

While this is the expected size of the nonperturbative corrections, as we will shortly see, this is not the full story.

Exponential corrections

By including the exponential corrections to the perturbative expansion of the Dawson function we obtain the nonperturbative contribution to the heat kernel K(t|μ)

$$F(x) \underset{x\to\infty}{=} \sum_{n=0} \frac{(2n-1)!!}{2^{n+1}} \left(\frac{1}{x}\right)^{2n+1} \boxed{\mp \frac{i\sqrt{\pi}}{2} e^{-x^2}.}$$

$$K(t|\mu) \ni i\frac{\sqrt{\pi}r^2}{2t^{3/2}} e^{-\mu^2 t} \sum_k{}' \left((\pi k r + i\mu t) e^{-\frac{(\pi k r + i\mu t)^2}{t}} + (\pi k r - i\mu t) e^{-\frac{(\pi k r - i\mu t)^2}{t}} \right)$$

$$= i\frac{\sqrt{\pi}r^2}{t^{3/2}} \sum_k{}' \operatorname{sign}(k)(k\pi r \cos(2\pi k \mu r) + t\mu \sin(2\pi k\mu r)) \underline{e^{-k^2\pi^2 r^2/t}}.$$

We have multiple nonperturbative sectors labelled by the integer k. In the exponents, we recognize the position of the leading singularity (k = 1) of the Borel transform of the heat kernel expansion

$$z^* = (\pi r)^2$$

The resulting transseries (i.e., perturbative + exponential corrections) in the parameter t is rather peculiar: the series around each exponential truncates after only two terms ("two-loop exact"). NB: in the bosonic O(N) CFT, it is one-loop exact.

[N. Dondi, I. Kalogerakis, D. Orlando, S. Reffert (2021)]

A different limit

However, the Mellin integral localizes around t ~ r/μ, and t does not act as an independent perturbative parameter. Accordingly, the t → 0 limit should instead be considered at fixed "μt/r".

So let's rewrite the perturbative expansion of the heat kernel as follows:

$$\boxed{K(t|\mu) = e^{-\mu^2 t}\sum_{n=1}^{\infty} c_n(\mu t/r)\left(\frac{t}{r^2}\right)^n \qquad c_n(\mu t/r) = \sum_{l=0}^{\infty}\frac{(-1)^{n+1}4^l B_{2(l+n)}}{(2l)!(l+n)\Gamma(n)}\left(\frac{\mu t}{r}\right)^{2l}}$$

Large order behavior: $c_n \sim \frac{\Gamma(n)}{\sqrt{\pi}}\left((\pi - i\mu t/r)^{1-2n} + (\pi + i\mu t/r)^{1-2n}\right)$

It follows that the Borel transform of the series has poles at $\boxed{z^* = (\pi r \pm i\mu t)}$

In fact, we saw that (but now looking at the first line)

$$K(t|\mu) \ni i\frac{\sqrt{\pi}r^2}{2t^{3/2}}e^{-\mu^2 t}\sum_k{}' \left((\pi k r + i\mu t)e^{-\frac{(\pi k r + i\mu t)^2}{t}} + (\pi k r - i\mu t)e^{-\frac{(\pi k r - i\mu t)^2}{t}}\right)$$
$$= i\frac{\sqrt{\pi}r^2}{t^{3/2}}\sum_k{}' \operatorname{sign}(k)(k\pi r\cos(2\pi k\mu r) + t\mu\sin(2\pi k\mu r))e^{-k^2\pi^2 r^2/t}.$$

In this limit (small t and fixed "μt/r") the transseries is one-loop exact.

The scaling dimension

The exponential corrections to the heat kernel translate into the following nonperturbative contributions to the grand potential

$$\frac{\Omega}{2N} \ni \frac{i\Phi_0}{k\pi}\left[\Phi_0 \operatorname{sign}(k)\cos(2\pi k\mu r)K_2(2\pi\Phi_0|k|r) + \mu\sin(2\pi k\mu r)K_1(2\pi\Phi_0|k|r)\right]$$

(K_1, $K_2 \rightarrow$ Bessel functions)

and to the scaling dimension of the lowest operator with charge Q:

$$\boxed{\frac{\Delta(Q)}{2N} \ni -i\frac{\left(\kappa_0^2-1\right)^{1/4} q^{3/4} e^{-2\pi\sqrt{\kappa_0^{-1}-\kappa_0^{-3}}|k|\sqrt{q}}}{2\pi\kappa_0^{9/4}\sqrt{|k|}}\left(\sqrt{\kappa_0^2-1}\cos\left(\frac{2\pi k}{\kappa_0^{3/2}}\sqrt{q}\right)+\sin\left(\frac{2\pi k}{\kappa_0^{3/2}}\sqrt{q}\right)\right)+\ldots}$$

$$q = \frac{Q}{2N}$$

$$\kappa_0 \tanh\kappa_0 = 1$$

- The nonperturbative corrections do not simply scale as $e^{-\sqrt{q}}$
- Instead there are additional cosine/sine factors arising from Borel singularities away from the real axis ("complex exponentials").
- These are peculiar to the NJL model and are ultimately related to $\Phi_0 \neq \mu$.

Geometric interpretation

The nonperturbative corrections to the scaling dimension in the bosonic O(N) model have a geometric interpretation as worldline instantons.

[N. Dondi, I. Kalogerakis, D. Orlando, S. Reffert (2021)]

A similar interpretation holds for the transseries structure of the heat kernel K(t|μ) in the NJL model. Let's start again from the grand potential

$$\Omega = -N \operatorname{Tr} \log(\Gamma_\mu \partial^\mu - \mu \Gamma_3 \Gamma_5 + \Phi_0)$$

and use Schwinger trick:

$$\frac{\Omega}{N} = \frac{1}{2} \int_0^\infty \frac{dt}{t} e^{-\Phi_0^2 t} \operatorname{Tr} e^{-\Sigma t}$$

$$\Sigma = -\Gamma^\mu \Gamma^\nu \nabla_\mu \nabla_\nu - 2\Gamma_5 \left(\Gamma^\mu \Gamma^\nu - g^{\mu\nu}\right) a_{5\nu} \nabla_\mu + g^{\mu\nu} a_{5\mu} a_{5\nu} \qquad a_{5\nu} = \mu \delta_{\nu 3}$$

Geometric interpretation

$$\frac{\Omega}{N} = \frac{1}{2}\int_0^\infty \frac{dt}{t} e^{-\Phi_0^2 t}\,\mathrm{Tr}\, e^{-\Sigma t}$$

$$\Sigma = -\Gamma^\mu\Gamma^\nu\nabla_\mu\nabla_\nu - 2\Gamma_5\left(\Gamma^\mu\Gamma^\nu - g^{\mu\nu}\right)a_{5\nu}\nabla_\mu + g^{\mu\nu}a_{5\mu}a_{5\nu} \qquad a_{5\nu} = \mu\delta_{\nu 3}$$

Comparing the above to the Mellin transform of the Zeta function we have

$$\mathrm{Tr}\, e^{-\Sigma t} = \frac{2}{\sqrt{\pi t}} K(t|\mu) = \frac{2r^2}{\sqrt{\pi} t^{3/2}}(1+\mathcal{O}(t)) \pm \frac{2\pi i r^3}{t^2}\sum_k{}' |k| e^{-\frac{\pi^2 k^2 r^2}{t}}\cos(2\pi k\mu r)(1+\mathcal{O}(t))$$

We now reproduce this result by means of the worldline formalism: Σ is interpreted as the Hamiltonian for a particle with proper time τ that moves on a manifold $\mathscr{M}$ with metric $g_{\mu\nu}$ in a given coordinate system x_μ.

The trace can then be expressed as a quantum mechanical path integral.

Worldline formalism: "a first-quantized alternative to Feynman diagrams".

[Z. Bern and D. A. Kosower (1992) - M. J. Strassler (1992)]

Worldline action

$$\mathrm{Tr}\, e^{-\Sigma t} = \int \mathcal{D}x^\mu \mathcal{D}\psi^\mu \mathcal{D}\psi^4 \, e^{-S_{WL}[x^\mu,\psi^\mu,\psi^4,x^\mu;g_{\mu\nu},a_{5\mu}]}$$

$$S_{WL} = \frac{1}{4t}\int_0^1 d\tau \left[\psi^4\dot\psi^4 + g_{\mu\nu}\left(\dot x^\mu \dot x^\nu + \psi^\mu\dot\psi^\nu + \psi^\mu\left(\Gamma^\nu_{\lambda\rho}\dot x^\lambda - 4g_{\rho\sigma}\dot x^\nu a_5^\sigma\right)\psi^\rho\right)\right]$$

ψ^μ and ψ^4 are Grassmann fields introduced to account for the spinor trace.

x^μ (ψ^μ and ψ^4) satisfies periodic (antiperiodic) boundary conditions on the time circle.

Parameterizing RxS^2 as (s,θ,φ) we arrive at the following worldline action:

$$S_{WL} = \frac{1}{4t}\int_0^1 d\tau \Big[\psi^4\dot\psi^4 + \psi^s\dot\psi^s + \dot s^2 + r^2\Big(\dot\phi^2\sin^2\theta + \dot\theta^2 + \psi^\theta\dot\psi^\theta + \sin^2\theta\,\psi^\phi\dot\psi^\phi$$
$$+\frac{1}{2}\sin(2\theta)\psi^\phi\dot\theta\psi^\phi - \sin(2\theta)\psi^\theta\dot\phi\psi^\phi - 4\mu\psi^\theta\dot\theta\psi^s - 4\mu\sin^2\theta\,\psi^\phi\dot\phi\psi^s\Big)\Big]$$

The path integral can be computed semiclassically in the $t \to 0$ limit.

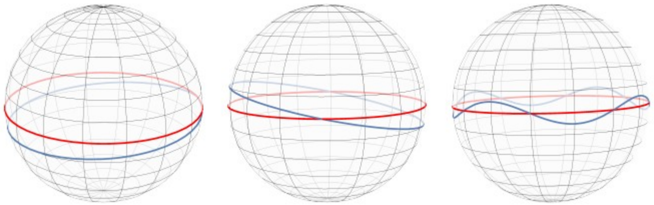
Saddle point

$$S_{WL} = \frac{1}{4t}\int_0^1 d\tau\Big[\psi^4\dot\psi^4 + \psi^s\dot\psi^s + \dot s^2 + r^2\Big(\dot\phi^2\sin^2\theta + \dot\theta^2 + \psi^\theta\dot\psi^\theta + \sin^2\theta\,\psi^\phi\dot\psi^\phi$$
$$+\frac{1}{2}\sin(2\theta)\psi^\phi\dot\theta\psi^\phi - \sin(2\theta)\psi^\theta\dot\phi\psi^\phi - 4\mu\psi^\theta\dot\theta\psi^s - 4\mu\sin^2\theta\,\psi^\phi\dot\phi\psi^s\Big)\Big]$$

We consider a vanishing classical profile for the fermions. The EOMs read

$$\theta(1) = \theta(0) = \pi/2 \qquad \phi(1) = \phi(0) = 0$$
$$2\ddot\theta - \sin(2\theta)\dot\phi^2 = 0 \qquad \ddot\phi + 2\dot\theta\dot\phi\cot\theta = 0$$

with solution $\boxed{\theta_0 = \pi/2 \qquad \phi_0 = 2\pi k\tau}$



Worldline instantons: geodesics wrapping k times around the great circle of the sphere. The action is

$$S_I = (\pi k r)^2$$

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The fluctuations around the saddle point yield

$$\mathrm{Tr}\, e^{-\Sigma t} = \frac{2}{\sqrt{\pi t}} K(t|\mu) = \frac{2r^2}{\sqrt{\pi} t^{3/2}} (1 + \mathcal{O}(t)) \pm \frac{2\pi i r^3}{t^2} \sum_k{}' |k| e^{-\frac{\pi^2 k^2 r^2}{t}} \underline{\cos(2\pi k \mu r)} (1 + \mathcal{O}(t))$$

FERMIONIC DETERMINANT

In agreement with the direct calculation. The exponentially suppressed terms are interpreted as multi-worldline instanton contribution.

It is reasonable to expect that these geometric effects survive beyond the large N limit in the strongly coupled regime.

However, …

As discussed, the Mellin integral localizes around t $\sim$r/μ, and the t $\to$ 0 limit should rather be considered at fixed “μt/r”. Let’s have another look at the heat kernel

$$K(t|\mu) \ni i\frac{\sqrt{\pi}r^2}{t^{3/2}}\sum_k{}' \operatorname{sign}(k)(k\pi r\cos(2\pi k\mu r) + t\mu\sin(2\pi k\mu r))e^{-k^2\pi^2 r^2/t}.$$

At fixed “μt/r”, the *cos(2πkμr)* and *sin(2πkμr)* terms are of the same order as the exponential. In the worldline formalism they may arise either from a different saddle point or from a resummation of the loop corrections around the worldline instantons.

Small q expansion

The conformal coupling to the Ricci scalar of the sphere generates a mass term 1/(2r) for Φ. As a consequence, both the gap and charge vanish for

$$\mu = 1/(2r)$$

so we have to expand around this point. Using the trinomial theorem, we have

$$\zeta(s|\not{\partial}, \mu, \Phi_0) = r^s \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{-s}{-s-m-n,\ m,\ n} \frac{1-3(-1)^m}{2} \left(2^{2\lceil m/2\rceil+2n+2s-1} - 1\right)$$

$$\zeta(2\lceil m/2\rceil + 2n + 2s - 1)(2\tilde{\mu})^m \left(\Phi_0^2 + \tilde{\mu}^2\right)^n r^{m+2n}$$

$$\mu = \frac{1}{2r} + \tilde{\mu}$$

Or more explicitly:

$$r\tilde{\mu} = \sum_{j=1} \mu_j (r\Phi_0)^{2j} = (r\Phi_0)^2 + \frac{\pi^2 - 8}{4}(r\Phi_0)^4 + \frac{192 - 8\pi^2 - \pi^4}{24}(r\Phi_0)^6 + \dots$$

$$\frac{\Omega}{2N} = \sum_{j=1}^{\infty} \Omega_j (r\Phi_0)^{2(j+1)} = -\frac{\pi^2}{8}(r\Phi_0)^4 - \frac{\pi^4}{48}(r\Phi_0)^6 + \dots$$

small q = small gap

Convergence

$$\zeta(s|\not{\partial},\mu,\Phi_0) = r^s \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \binom{-s}{-s-m-n,\ m,\ n} \frac{1-3(-1)^m}{2} \left(2^{2\lceil m/2\rceil+2n+2s-1}-1\right) \zeta(2\lceil m/2\rceil+2n+2s-1)(2\tilde{\mu})^m\left(\Phi_0^2+\tilde{\mu}^2\right)^n r^{m+2n}$$

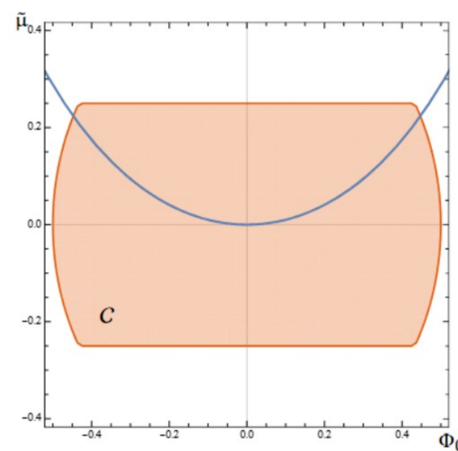
From the trinomial expansion, we see that the sum is absolutely convergent in the region

$$C = \{\, r|\tilde{\mu}| < 1/4 \cap r^2|\Phi_0^2+\tilde{\mu}^2| < 1/4 \,\}$$

The radius of convergence of the expansion is given by the intersection of the solution to the gap equation with the boundary of C

$$r\Phi_0 < 0.447$$

$$r\mu < 0.223$$



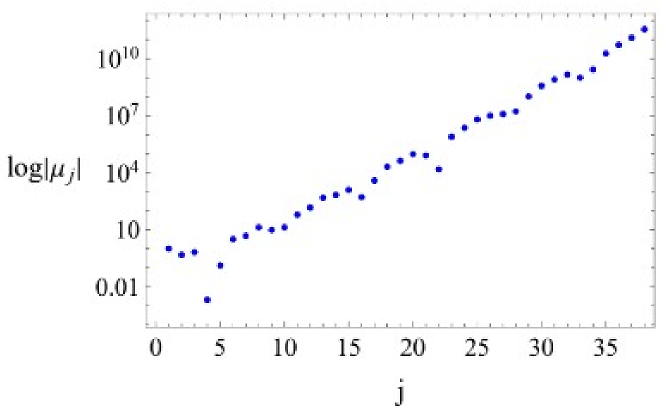
Further information can be obtained from the coefficients.

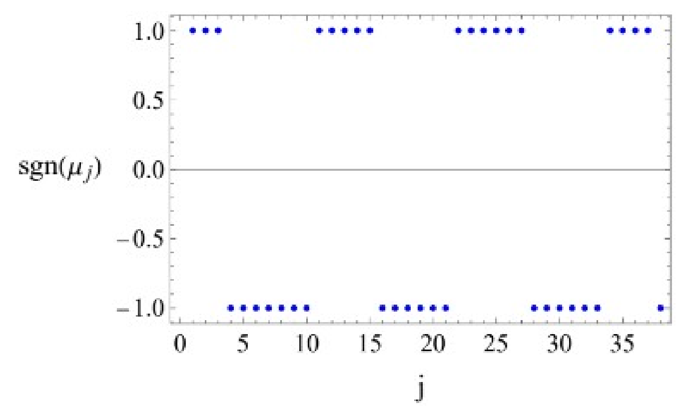
Grand canonical ensemble

$$r\tilde{\mu} = \sum_{j=1} \mu_j (r\Phi_0)^{2j} = (r\Phi_0)^2 + \frac{\pi^2 - 8}{4}(r\Phi_0)^4 + \frac{192 - 8\pi^2 - \pi^4}{24}(r\Phi_0)^6 + \dots$$

$$\frac{\Omega}{2N} = \sum_{j=1}^{\infty} \Omega_j (r\Phi_0)^{2(j+1)} = -\frac{\pi^2}{8}(r\Phi_0)^4 - \frac{\pi^4}{48}(r\Phi_0)^6 + \dots$$

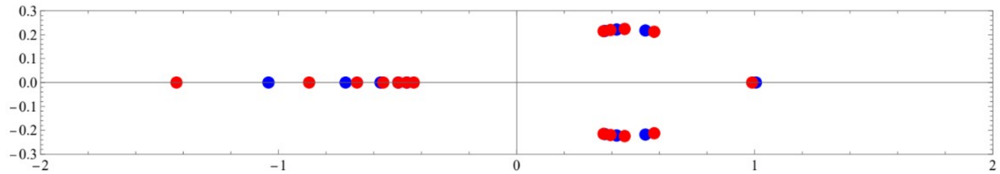
The sign of the coefficients fluctuates wildly while their magnitude oscillates.



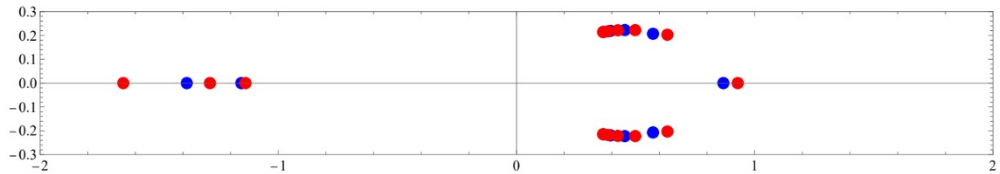


This is a typical sign of the presence of competing singularities in the complex Φ_0 plane. This can be further probed by looking at the distribution of poles in the Padé approximants.

Grand canonical ensemble



GAP EQUATION



GRAND POTENTIAL

Poles of the (14,15) (blue) and (18,20) (red) Padé approximants.

Gap equation: the accumulation of poles hints at a singularity around $r\Phi_0 = -0.45$ and two complex conjugate poles at roughly the same distance from the origin.

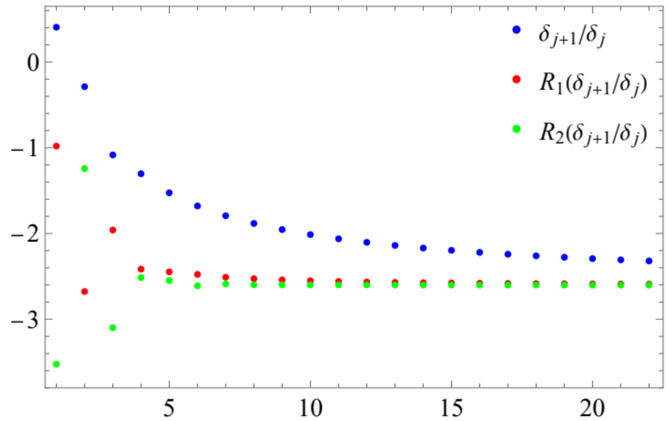
Grand potential: similar but the singularity on the negative real axis is further away from the origin.

Canonical ensemble

$$\frac{\Delta_Q}{2N} = \sum_{j=1}^{\infty} \delta_j \left(\frac{Q}{2N}\right)^j = \frac{1}{2}q + \frac{2}{\pi^2}q^2 + \ldots$$

The coefficients now exhibit a simple sign-alternating growing behavior!

RATIO TEST



$$\frac{\Delta_Q}{2N} = -0.060\ldots \left(1 - \frac{q}{q_0}\right)^{3/2} + \text{analytic}$$

$$q_0 = -0.384\ldots$$

- Singularity for negative q: possible analytic continuation for positive q.
- Same type of singularity as in various bosonic CFTs: universal behavior?

[O. Antipin, JB, F. Sannino, M. Torres (2022)]

- Conjectured physical interpretation: at q_0 the radial mode becomes gapless.

Questions

- Can we determine the small/large q expansion to all orders?
- Do the small/large q expansions converge?
- Are there nonperturbative corrections?
- Can we understand the properties of the large charge expansion beyond the double scaling limit?

Conclusions

- *We studied the resurgence properties of the large charge expansion in the 3d NJL model at large N.*
- *The large-q expansion is asymptotic and qualitatively resembles the O(N) CFT case studied in [N. Dondi, I. Kalogerakis, D. Orlando, S. Reffert (2021)].*
- *However, here the Borel singularities move away from the real axis, introducing additional sine/cosine modulations in the exponentially suppressed terms.*
- *As a result, the worldline instanton interpretation only partially carries over (OK for the heat kernel, less OK for the grand potential/scaling dimension).*
- *The small-q expansion is convergent. In the grand canonical ensemble, it exhibits a rich structure due to competing singularities in the complex μ plane. Remarkably, this simplifies in the canonical ensemble, where singularities appear only along the negative q-axis.*

Thank you!