NLO in the large charge sector of the critical $O(N)$ model at large N

Based on [2409.06781] with G. Sbervegliieri

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New directions in the large charge expansion
Les Diablerets, June 2026

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Motivation

In CFT$_{d>3}$ large Q-charge sectors with a non-trivial macroscopic limit exist:

$$E \sim \epsilon R^{d-1}, \quad Q \sim \rho R^{d-1}$$

$$\implies \Delta \sim Q^{\frac{d}{d-1}}$$

Bottom-up

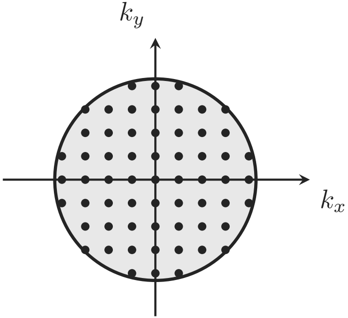
EFT-building for large-Q sectors: superfluids (w/ vortices), Fermi liquids, supersolids...

$\downarrow \uparrow$

Top-down

Find large-Q sectors in "UV-complete" CFTs: EFT matching.

More motivation



- In $d = 3$, large-N GN and NJL models have large-charge sectors described by filled Fermi spheres.
- The corresponding operators have the same LO scaling as in FREE_ψ. Difference appears at NLO.

$$\frac{\Delta}{2N} = \frac{2}{3}\left(\frac{Q}{2N}\right)^{\frac{3}{2}} + \frac{1}{12}\left(\frac{Q}{2N}\right)^{\frac{1}{2}} - \frac{1}{192}\left(\frac{Q}{2N}\right)^{-\frac{1}{2}} + \cdots$$

- Need a conformal Fermi liquid EFT matching.

[NAD, Hellerman, Kalogerakis, Moser, Orlando, Reffert '22]
[Delacretaz, Chowdhury, Metha '25]
[More on superfluid ground states in these models: see Jahamall's talk.]

Summary of results

We focused on the $O(N)$-CFT at $N, Q \to \infty$ with Q/N fixed, large-charge sector corresponding to the operator

$$\mathcal{O}_Q = \phi_{\{i_1} \cdots \phi_{i_Q\}},$$
$$\Delta(\hat{Q}) = N\Delta_{-1}(Q/N) + N^0\Delta_0(Q/N) + \ldots$$

- Determined *numerically* the function Δ_0 at $Q/N \ll 1$ and $Q/N \gg 1$.
- Superfluid EFT matching at NLO (Wilsonian coefficients, GB casimir...) for $Q/N \gg 1$.
- Matching with perturbation theory for $Q/N \ll 1$.

Outline

- Setup and leading order
- Next-to-leading order
- Results

Setup and leading order

[Hellerman, Orlando, Reffert, Watanabe ’15]
[Monin, Pirtskhalava, Rattazzi, Seibold ’17]
[Badel, Cuomo, Monin, Rattazzi ’19]
[Alvarez-Gaume, Orlando, Reffert ’19]
[...]
[ϵ-expansion: Antipin, Bersini, Sannino...]

Conformal superfluid EFT

$$S_{\text{EFT}} = \int \mathrm{d}^d x \sqrt{\widehat{g}} \left(-c_1 + c_2 \widehat{\mathcal{R}} - c_3 \widehat{\mathcal{R}}_{\mu\nu} \partial^\mu \chi \partial^\nu \chi + c_4 \widehat{\mathcal{R}}^{\,2} + \cdots \right),$$

$$\widehat{g}_{\mu\nu} = g_{\mu\nu} \left(-\partial_\rho \chi \, \partial^\rho \chi \right), \qquad \chi = -i\mu\tau + \pi.$$

We work strictly in $d = 3$, where

$$\Delta(Q) = \alpha_{\frac{3}{2}} Q^{\frac{3}{2}} + \alpha_{\frac{1}{2}} Q^{\frac{1}{2}} + \beta_0 + \alpha_{-\frac{1}{2}} Q^{-\frac{1}{2}} + \beta_{-1} Q^{-1} + \cdots .$$

- The Wilsonian paramters $c_1, c_2, c_3, c_4 ... \sim O(1)$ determine the ground-state contributions $\alpha_{3/2}, \alpha_{1/2}, \alpha_{-1/2} ....$
- GB loops generate the contribution $\beta_0 = -0.093725...$ (universal Casimir contribution) and the series $\beta_{-1} ...$ (non-universal, c_i-dependent).

$O(N)$-model, finite density formulation

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial \boldsymbol{\phi})^2 + \frac{1}{2} m^2 \boldsymbol{\phi}^2 + \frac{1}{2\sqrt{N}} \sigma \boldsymbol{\phi}^2 - \frac{\sigma^2}{4g} \right.$$
$$\left. + i\mu(\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) - \frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) \right].$$

- Chemical potential μ for the $\mathfrak{so}(N)$ cartan rotating ϕ_1, ϕ_2: equivalent to sourcing $\mathcal{O}_Q = \boldsymbol{\phi}_{\{i_1} \ldots \boldsymbol{\phi}_{i_Q\}}$.
- Renormalization is set up in a $1/N$ expansion (more convenient to work with bare fields and couplings).
- Critical at $g \to \infty$ and m^2 tuned appropiately.

Large-N saddle expansion

At $N \to \infty$ we expect a *homogeneous* saddle to dominate path integrals:

$$\sigma = \sqrt{N}\overline{\Sigma} + \hat{\sigma}, \qquad \boldsymbol{\phi} = \sqrt{N}\left(\overline{\Phi}, 0, ...0\right) + \left(\hat{\phi}_1, \hat{\phi}_2, \hat{\eta}_1, ..., \hat{\eta}_{N-2}\right),$$

$$\overline{\Sigma} = \Sigma + \sum_{n=1}^{\infty} \Sigma^{(n)} N^{-n}, \qquad \overline{\Phi} = \Phi + \sum_{n=1}^{\infty} \Phi^{(n)} N^{-n}.$$

The Lagrangian is also expanded as

$$\mathcal{L} = N\mathcal{L}_{\text{tree}} + \sqrt{N}\mathcal{L}_{\text{tad}} + \mathcal{L}_{\text{gauss}} + \frac{1}{\sqrt{N}}\mathcal{L}_{\text{int}},$$

The observables we will compute are partition functions in flat space and cylinder

$$\mathcal{Z}(\mathbb{R}^d) = e^{-\text{Vol}(\mathbb{R}^d) \times V_{\text{flat}}},$$

$$\mathcal{Z}(S^{d-1} \times S^1_\beta) \xrightarrow{\beta \to \infty} e^{-\beta \times V_{\text{cyl}}} \qquad V_{\text{flat/cyl}} = NV^{(-1)} + N^0 V^{(0)} + ...$$

An aside on regulators

$$V^{(-1)} = \frac{1}{2}\Phi^2(m^2 - \mu^2 + \Sigma) - \frac{\Sigma^2}{4g} + \underbrace{\frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \log\left(k^2 + m^2 + \Sigma\right)}_{I} .$$

At the conformal point $m^2 \to 0$, $g \to \infty$ we have multiple choice of regulators:

$$[I]_{\zeta,d} := -\frac{1}{2}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int \frac{\mathrm{d}^d k}{(2\pi)^d}(k^2 + \Sigma)^{-s} \xrightarrow[d\to 3]{=} -\frac{\Sigma^{\frac{3}{2}}}{12\pi}$$

$$[\mathcal{I}]_{\Lambda,d} := \left[I - I|_{\mu=0}\right]_{\Lambda,d} \xrightarrow[\Lambda\to\infty,\, d\to 3]{} \frac{\Lambda^3}{18\pi^2} - \frac{\Sigma^{\frac{3}{2}}}{12\pi}$$

$$[\mathcal{I}]_{\Lambda,3} := \left[I - I|_{\mu=0}\right]_{\Lambda,d=3} \xrightarrow[\Lambda\to\infty]{} \frac{\Lambda\Sigma}{4\pi^2} - \frac{\Sigma^{\frac{3}{2}}}{12\pi}$$

Leading order in flat space

At the critical point,

$$V^{(-1)} = \frac{1}{2}\Phi^2(\Sigma - \mu^2) - \frac{\Sigma^{3/2}}{12\pi}. \qquad \Longrightarrow \qquad \Sigma_* = \mu^2, \qquad \Phi_*^2 = \frac{\mu}{4\pi}$$

which is related to the equation of state via

$$\hat{\rho} = \frac{\partial}{\partial\mu} V^{(-1)}|_* \qquad \Longrightarrow \qquad \hat{\epsilon} = V^{(-1)}|_*(\mu) + \mu\hat{\rho} = \frac{4\sqrt{\pi}}{3}\,\widehat{\rho}^{3/2},$$

where $\hat{\rho} \coloneqq \rho/N$, $\hat{\epsilon} \coloneqq \epsilon/N$ are charge and energy densities per degree of freedom.

Leading order on the cylinder

Using a combined hard cutoff

$$
\begin{aligned}
[\mathcal{I}_{\text{cyl.}}]_{\Lambda,d=3} &= \frac{1}{2}\left[\sum_{\ell=0}\int\frac{\mathrm{d}\omega}{2\pi}\right]_{\Lambda}(2\ell+1)\log\left(1+\frac{\Sigma}{\omega^2+\frac{1}{R^2}\left(\ell+\frac{1}{2}\right)^2}\right)\\
&\xrightarrow{\Lambda\to\infty}\frac{R^2\Lambda\Sigma}{\pi}+\frac{1}{R}\left(-\frac{(\Sigma R^2)^{\frac{3}{2}}}{3}+\frac{(\Sigma R^2)^{\frac{1}{2}}}{24}-\frac{7}{1920(\Sigma R^2)^{\frac{1}{2}}}+\dots\right).
\end{aligned}
$$

- Result for scaling dimension of $\mathcal{O}_Q$:

$$
\Delta_{-1}(\hat{Q}) = \frac{2}{3}\hat{Q}^{\frac{3}{2}}+\frac{1}{6}\hat{Q}^{\frac{1}{2}}-\frac{7}{720}\hat{Q}^{-\frac{1}{2}}+O\left(\hat{Q}^{-\frac{3}{2}}\right),
$$

- LO EFT matching:

$$
c_1 = \frac{N}{12\pi}+O\left(N^0\right), \qquad c_2 = \frac{N}{48\pi}+O\left(N^0\right), \qquad c_4 = -\frac{N}{960\pi}+O\left(N^0\right).
$$

- The Wilsonian coefficient c_3 does not contribute to the ground state energy.

Next-to-leading order

NLO in flat space

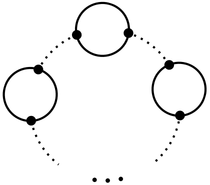
$$\mathcal{L} = N\mathcal{L}_{\text{tree}} + \sqrt{N}\mathcal{L}_{\text{tad}} + \mathcal{L}_{\text{gauss}} + \frac{1}{\sqrt{N}}\mathcal{L}_{\text{int}}\,,$$

$$D_{\hat\sigma\hat\sigma} = -\frac{1}{\Phi^2}\left\{\frac{4\mu^2k_0^2}{k^2+\Sigma-\mu^2} + \left(k^2+\Sigma-\mu^2\right)\right\}, \qquad D_{\hat\sigma\hat\phi_1} = D_{\hat\phi_1\hat\sigma} = \frac{1}{\Phi}\,,$$

$$D_{\hat\phi_2\hat\phi_2} = \frac{1}{k^2+\Sigma-\mu^2}, \qquad D_{\hat\sigma\hat\phi_2} = -D_{\hat\phi_2\hat\sigma} = \frac{1}{\Phi}\frac{2\mu k_0}{k^2+\Sigma-\mu^2}\,,$$

$$D_{\hat\eta_i\hat\eta_j} = \frac{\delta_{ij}}{k^2+\Sigma}\,, \qquad D_{\hat\phi_1\hat\phi_1} = D_{\hat\phi_1\hat\phi_2} = D_{\hat\phi_2\hat\phi_1} = 0\,.$$

$V^{(0)} \sim$

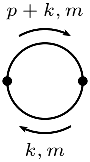


NLO in flat space

The (bare) result is found to be:

$$\mathcal{V}^{(0)} = \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \log\left(\frac{k^2\Phi_*^2 + \left(4\mu^2 k_0^2 + k^4\right)\Pi(k^2,\Sigma_*)}{k^4\Pi(k^2,0)}\right) - \int \frac{\mathrm{d}^d k}{(2\pi)^d}\log\left(1+\frac{\Sigma_*}{k^2}\right)$$

$$\Pi(p^2, m^2) =$$



$$= \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[k^2+m^2][(k+p)^2+m^2]}\,.$$

Flat-space NLO: result

This integral is computable:

$$[\mathcal{V}^{(0)}]_{\Lambda,3} = \frac{1}{4\pi}\left[-\frac{2}{3\pi}\Lambda\mu^2 + v^{(0)}\mu^3\right], \qquad v^{(0)} = -0.03530346\ldots.$$

The constant $v^{(0)}$ correspond to a particular definite integral and can be computed to arbitrary precision. The energy density becomes

$$\hat{\epsilon}(\widehat{\rho}) = \left[\frac{4\sqrt{\pi}}{3} + \frac{2\sqrt{\pi}\,v^{(0)}}{N} + O(N^{-2})\right]\widehat{\rho}^{3/2}.$$

Aside: Conformal Goldstone dispersion relation

- The flat-space determinant contains a conformal GB:

$$\omega^2 = \frac{p^2}{2} + \frac{p^4}{96\pi\widehat{\rho}} + O\left(\frac{p^6}{\widehat{\rho}^2}\right).$$

- The EFT parametrizes the cubic correction by

$$\omega = \frac{|p|}{\sqrt{2}} + \frac{\gamma}{8\pi N\widehat{\rho}}|p|^3 + \cdots, \qquad \gamma = \sqrt{8\pi^2}\,(c_2 + c_3).$$

- The matching gives

$$\gamma = \frac{N}{12\sqrt{2}} + O(N^0) \qquad \Longrightarrow \qquad c_3 = O(N^0).$$

- Similar to leading (tree-level) result of ϕ^6 in $d = 3 - \epsilon$.

[Badel, Cuomo, Monin, Rattazzi '20]

NLO on the cylinder

$$\left[\mathcal{V}^{(0)}_{\text{cyl.}}\right]_{\Lambda,d=3} = \left[\sum_{\ell=0}\int\frac{\mathrm{d}\omega}{2\pi}\right]_{\Lambda}\left(\ell+\frac{1}{2}\right)f(\omega,\ell;\mu)\,.$$

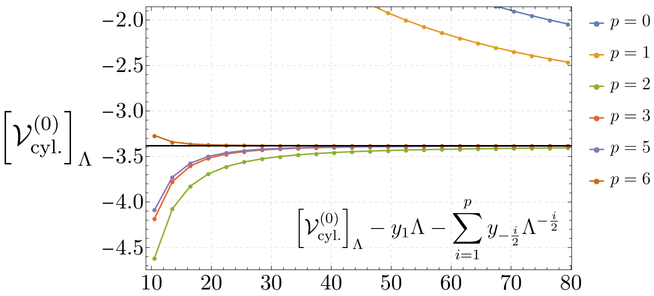
- Combined cutoff $\omega^2 + \left(\ell + \frac{d-2}{2}\right) < \Lambda$.
- $f(\omega,\ell;\mu)$ is a complicated expression involving constrained double sums over $\ell_1, \ell_2 \in \mathbb{Z}_{\geq 0}$.

$$\Pi^{n'\ell'm'}_{n\ell m} = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}\frac{1}{16\pi R^2}\sum_{\ell_1\ell_2}(2\ell_1+1)(2\ell_2+1)\begin{pmatrix}\ell_1 & \ell_2 & \ell\\ 0 & 0 & 0\end{pmatrix}^2\frac{E_{\ell_1}+E_{\ell_2}}{E_{\ell_1}E_{\ell_2}}\frac{1}{\omega_n^2+[E_{\ell_1}+E_{\ell_2}]^2}$$

$$E_\ell^2 = \frac{1}{R^2}\left(\ell+\frac{d-2}{2}\right)+\Sigma$$

Numerical implementation

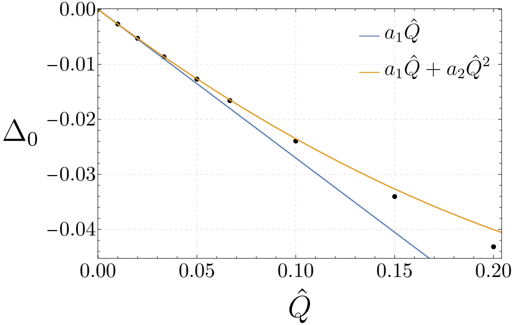
- We first deal with the double sums in Π_ℓ: at fixed ℓ, ℓ_1 the sum over ℓ_2 is finite and can be performed.
- The remaining ℓ_1 sum is convergent: we perform it by explicit summation up to $\bar{\ell}_1$, and for $\ell_1 > \bar{\ell}_1$ we use large-ℓ_1 asymtptotics (obtained analytically).
- The integral / sum over (ω, ℓ) is divergent: We can estimate semi-analytically the leading divergence and some tail terms.



[Pufu, Sadchev '13]
[Pufu '14]
[Chester, Iliesiu, Mezei, Pufu '18]

Results

Small-$\widehat{Q}$ regime



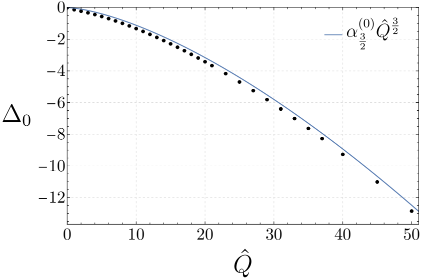
$$\Delta_0(\widehat{Q}) = a_1\widehat{Q} + a_2\widehat{Q}^2 + a_3\widehat{Q}^3 + a_4\widehat{Q}^4 + \cdots .$$

	Perturbation theory	Fit
a_1	−0.27018982304	−0.2701898230(1)
a_2	0.34992965	0.34992968(8)
a_3	–	−0.474333(4)
a_4	–	0.4845(9)

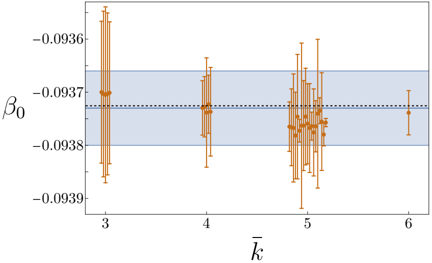
[Lang, Ruhl '93]
[Derkachov, Manashov '98]

Large-$\widehat{Q}$ regime

$$\Delta_0(\widehat{Q}) = \alpha^{(0)}_{\frac{3}{2}}\widehat{Q}^{3/2} + \alpha^{(0)}_{\frac{1}{2}}\widehat{Q}^{1/2} + \beta_0 + \alpha^{(0)}_{-\frac{1}{2}}\widehat{Q}^{-1/2} + \beta^{(0)}_{-1}\widehat{Q}^{-1} + \cdots .$$



$$\alpha^{(0)}_{3/2} = -0.03530346\ldots$$



$$\beta_0^{\text{fit}} = -0.09373(7)$$

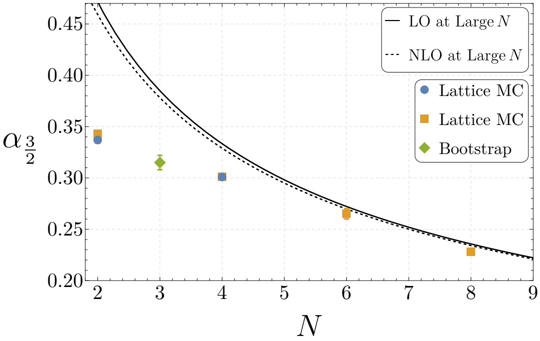
NLO EFT matching

	$\alpha^{(0)}_{\frac{1}{2}}$	β_0	$\alpha^{(0)}_{-\frac{1}{2}}$	$\beta^{(0)}_{-1}$
EFT	–	$-0.093725\dots$	–	$0.0081\dots$
This work	$-0.03938216(9)$	$-0.09373(7)$	$0.02364(2)$	$0.008(2)$

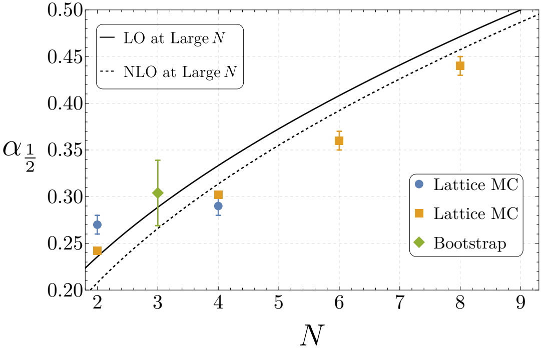
- The fitted constant term reproduces the GB Casimir energy β_0
- The first non-trivial correction $\beta^{(0)}_{-1}$ is compatible with the prediction controlled by c_2, c_3
- our best fit for the NLO Wilsonian coefficients is

$$\boxed{\begin{aligned} c_1 &= \frac{N}{12\pi} + 0.00280936\ldots + O(N^{-1}), \\ c_2 &= \frac{N}{48\pi} - 0.001215796(3) + O(N^{-1}), \\ c_4 &= -\frac{N}{960\pi} + 0.0005459(4) + O(N^{-1}). \end{aligned}}$$

Comparison with finite N



$\alpha_{3/2}$



$\alpha_{1/2}$

[Banerjee, Chandrasekharan, Orlando '18]
[Banerjee, Chandrasekharan '22]
[Singh '22]
[Rong, Su '23]

Conclusions and outlook

- We computed the full NLO function $\Delta_0(\widehat{Q})$ for the critical $O(N)$ model in $d = 3$.
- At small $\widehat{Q}$, it reproduces known large-N perturbation theory and predicts higher coefficients.
- At large $\widehat{Q}$, it reproduces the conformal-superfluid EFT, including β_0 and β_{-1}.
- We extracted the NLO corrections to c_1, c_2, c_4, while the leading $O(N)$ contribution to c_3 vanishes.

Thank you!