Effect of the trapping potential on the phonon excitation spectrum

S. R. Beane, A. Le Borgne, D. Orlando & S. Reffert arXiv:2510.26876v2

Plan

Introduction

- Non-relativistic conformal field theory (NRCFT)
- Example: Unitary fermi gas
- EFT framework

Collective excitations of the superfluid in a trapping potential

- Spectrum
- Dynamic structure factor
- Comparison with experiment

NRCFT

Centrally extended Schrödinger group

- Phase rotation N
- Space and time translation P_i, H
- Rotation M_{ij}
- Galilean boosts K_i
- Dilatation D
- Special conformal transformation C

Galilean subgroup

Y. Nishida and Dam T. Son arXiv:0706.3746v2

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Galilean subgroup

Some differences with the relativistic case

Central charge

The number of generators is different

Only one analogue to SCT

Allowed to have dimensionful parameters

Y. Nishida and Dam T. Son arXiv:0706.3746v2

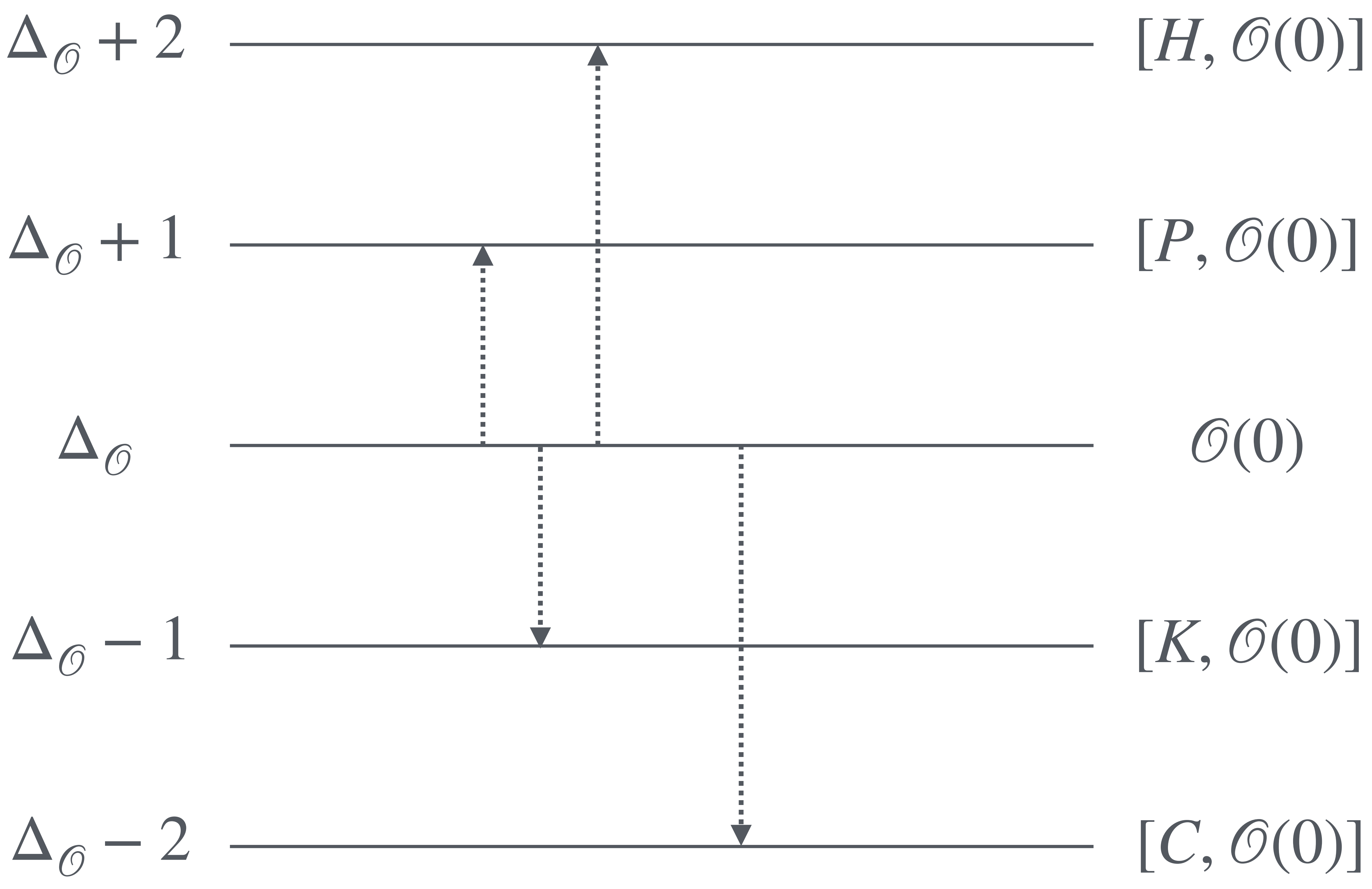
NRCFT

Local operators and representations of the Schödinger algebra

Operators with well-defined particle number and scaling dimension:

$$[D, \mathcal{O}(0)] = i\Delta_{\mathcal{O}}\mathcal{O}(0)$$

$$[N, \mathcal{O}(0)] = Q_{\mathcal{O}}\mathcal{O}(0)$$



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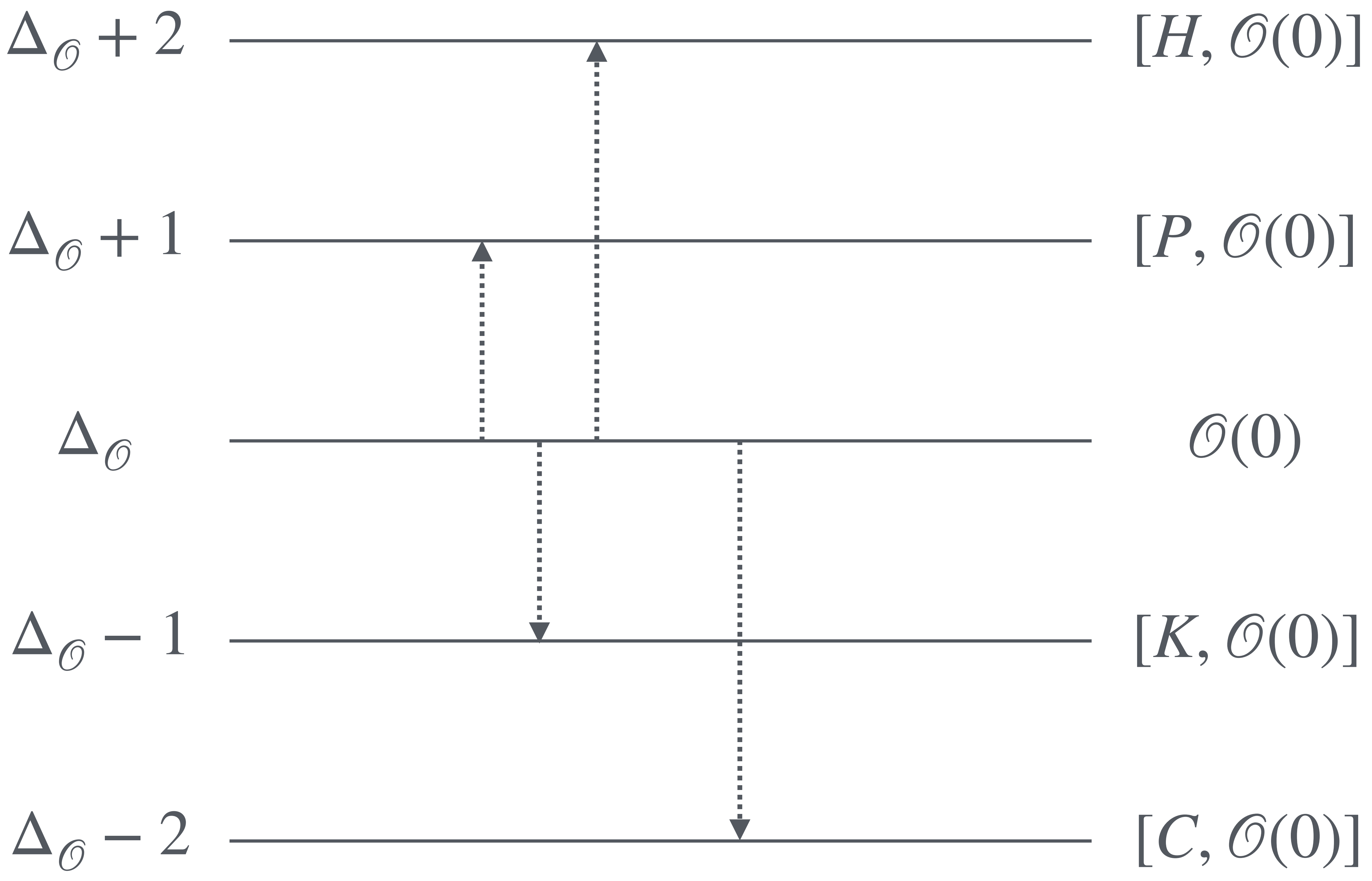

Primary operators

$$[K, \mathcal{O}(0)] = [C, \mathcal{O}(0)] = 0$$

Y. Nishida and Dam T. Son arXiv:0706.3746v2

S.M. Kravec & Sridip Pal arXiv:1809.08188v2
Y. Nishida and Dam T. Son arXiv:0706.3746v2

NRCFT

State-operator correspondance

The state-operator correspondence for an NRCFT is based on the following definition:

Primary operator of charge $Q_{\mathcal{O}^\dagger}$

$$|\mathcal{O}\rangle \equiv e^{-\frac{H}{\omega}}\mathcal{O}^\dagger(0)|0\rangle$$

By Schrödinger algebra, this state satisfies:

$$N|\mathcal{O}\rangle = Q_{\mathcal{O}^\dagger}|\mathcal{O}\rangle, \qquad H_\omega|\mathcal{O}\rangle = \omega\Delta_{\mathcal{O}}|\mathcal{O}\rangle$$

Scaling operator D

$$\Delta_{\mathcal{O}}$$

Hamiltonian in a harmonic trap $H_\omega \equiv H + \omega^2 C$

$$\frac{E}{\omega}$$

S.M. Kravec & Sridip Pal arXiv:1809.08188v2
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NRCFT

State-operator correspondance

Scaling operator D

$\Delta_{\mathcal{O}}$

↔

Hamiltonian in a harmonic trap $H_\omega \equiv H + \omega^2 C$

$\frac{E}{\omega}$

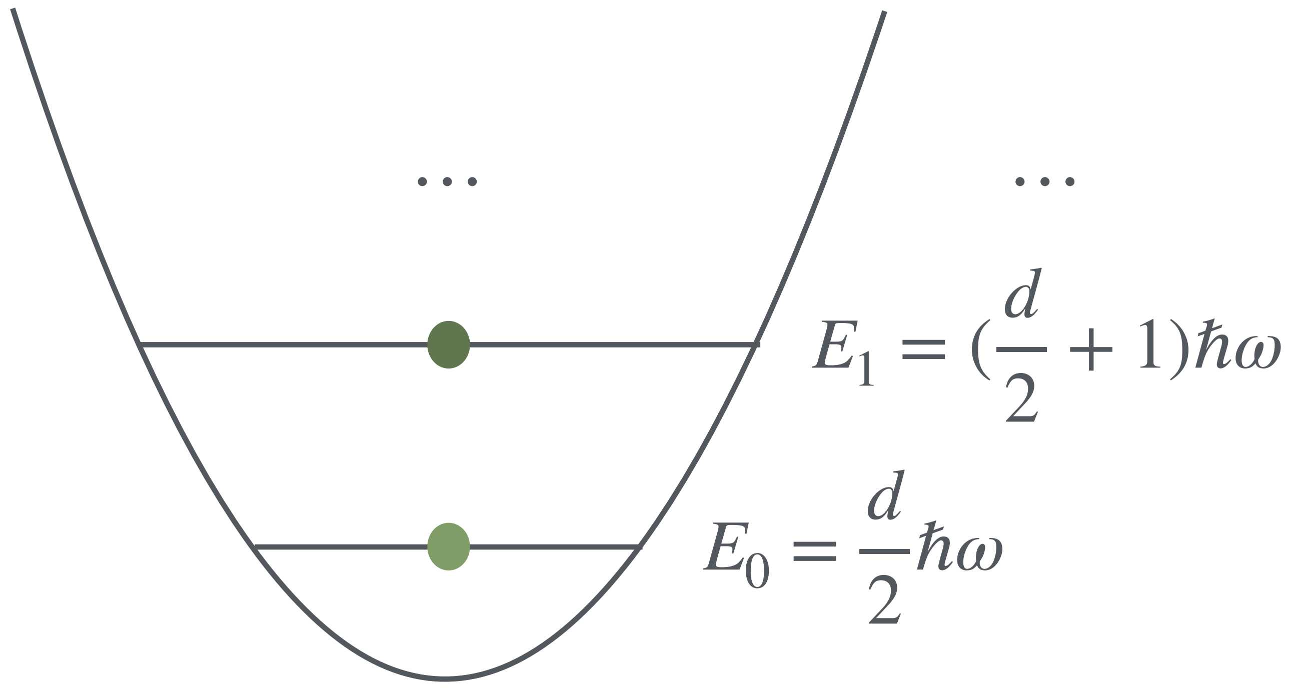
Example: free theory in d-dimensions

$$S = \int dt d^d x \phi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \phi$$

One particle

Lowest operator $\mathcal{O} = \phi$ $\Delta_{\mathcal{O}} = \frac{d}{2}$

Second lowest $\mathcal{O} = \nabla \phi$ $\Delta_{\mathcal{O}} = \frac{d}{2} + 1$

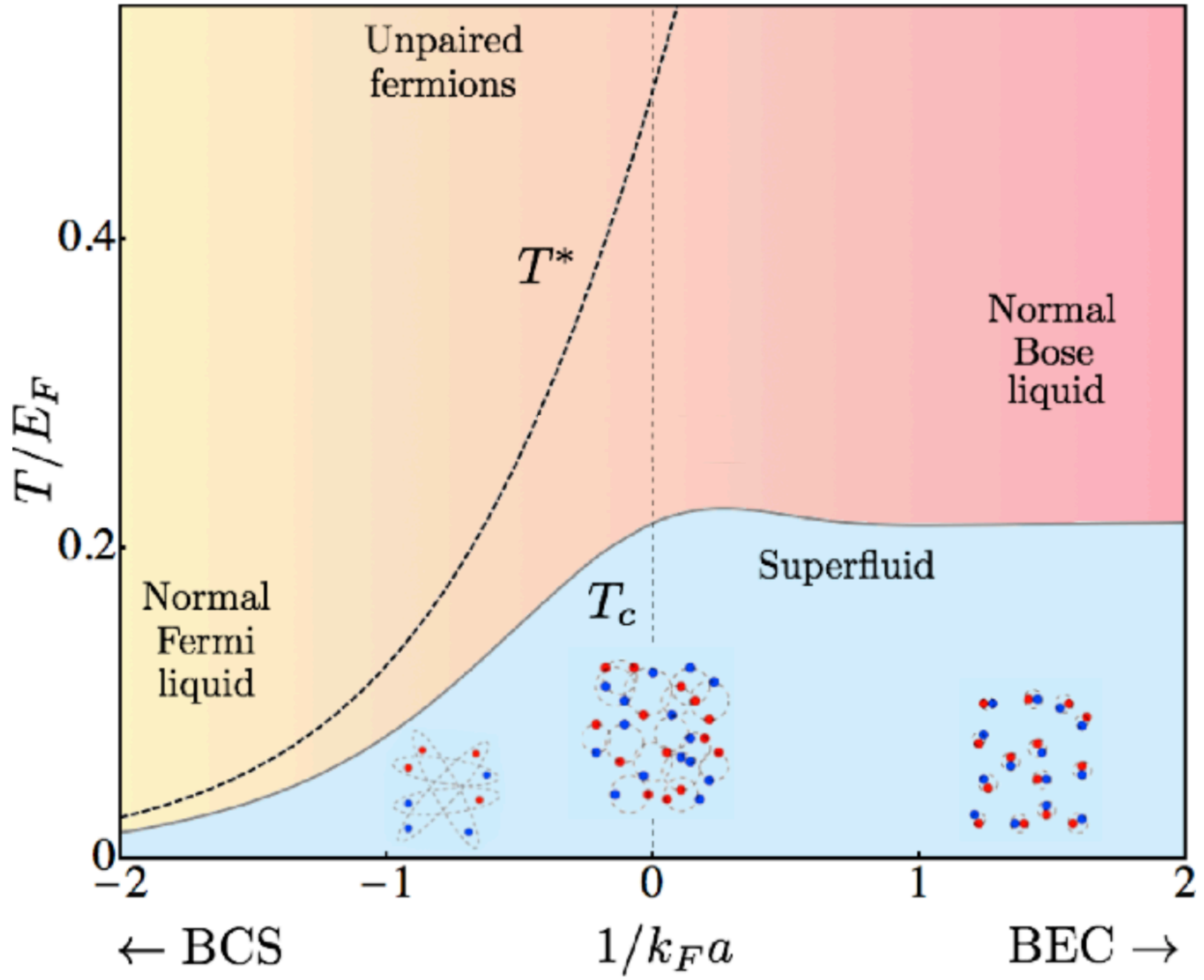


Not only theoretically interesting but can be realized in the lab !

Unitarity Fermi gas

Cold atom in a trap

- Tunability of the interaction - Feschbach resonance
- Strongly interacting: $|a| \to \infty$
- Unitarity: Cross section saturates the unitarity bound
- Scale invariant
- $\xi = \frac{H_{int.}}{H_{free}}$ characteristic number of the interacting critical point



NRCFT in nature:

Neutron stars $r_0 \sim 1.4$ fm $\ll a_{nn} \sim 18.5$ fm

How to describe the unitary Fermi gas at 0T?

Thomas-Fermi theory and superfluid hydrodynamics 1927 & 1938

↓

go beyond the leading order, include systematic corrections

EFT description of the large-charge sector

Conformally invariant Lagrangian in the nonrelativistic case, describing the quantum critical points in interacting fermion systems at unitarity.

The large-charge sector admits an EFT for a single conformal Goldstone mode, with a controlled derivative expansion in which higher-order terms are parametrically suppressed at large Q.

D. T. Son and M. Wingate arXiv:cond-mat/0509786v2

Juan L. Mañes, Manuel A. Valle arXiv:0810.3797v2

EFT in a trap (spherically symmetric)

Effective theory for the massless Goldstone $\theta(t, \mathbf{x}) = \mu t + \pi(t, \mathbf{x})$

Phonon field

Galilean invariant building block $X = \partial_t \theta - \frac{1}{2}(\nabla\theta)^2 - V(t, \mathbf{x})$

$$\boxed{\mathscr{L} = c_0 X^{5/2} + c_1 X^{-1/2}(\nabla X)^2 + c_2 X^{1/2}\left((\Delta\theta)^2 - 3(\nabla \otimes \nabla\theta)^2\right) + \ldots}$$

D. T. Son and M. Wingate arXiv:cond-mat/0509786v2
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S. Hellerman, I. Swanson arXiv:2010.07967v1

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Validity

$$\frac{\Lambda_{UV}}{\Lambda_{IR}} = \frac{\mu}{\varpi} \sim Q^{1/3} \gg 1$$

$$\Lambda_{IR} = \varpi = \frac{\sqrt{2\mu}}{R_{cl}}$$

$$\Lambda_{UV} = \mu$$

Charge density vanishes
$\mu - V(R_{cl}) = 0$

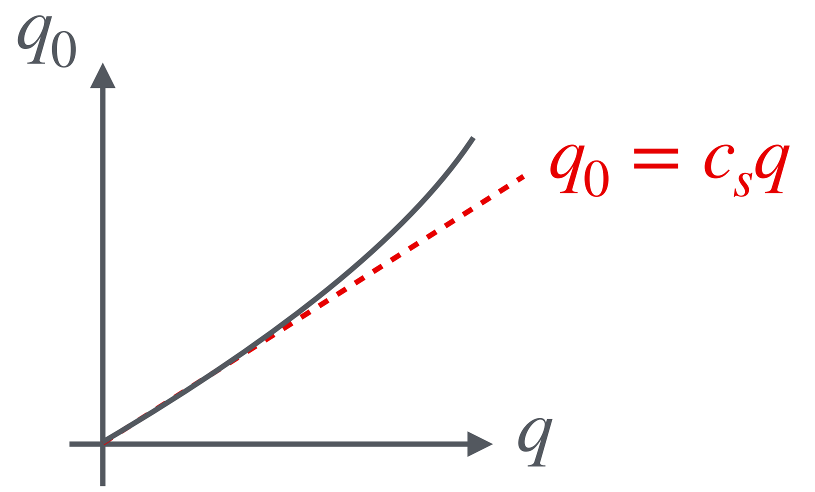
Relaxation mechanisms of superfluids

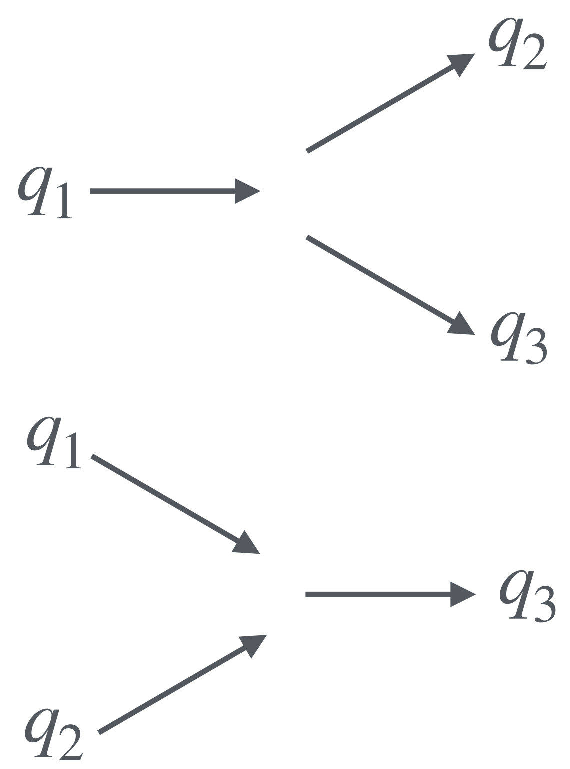
Acoustic branch

Sign of the cubic term?

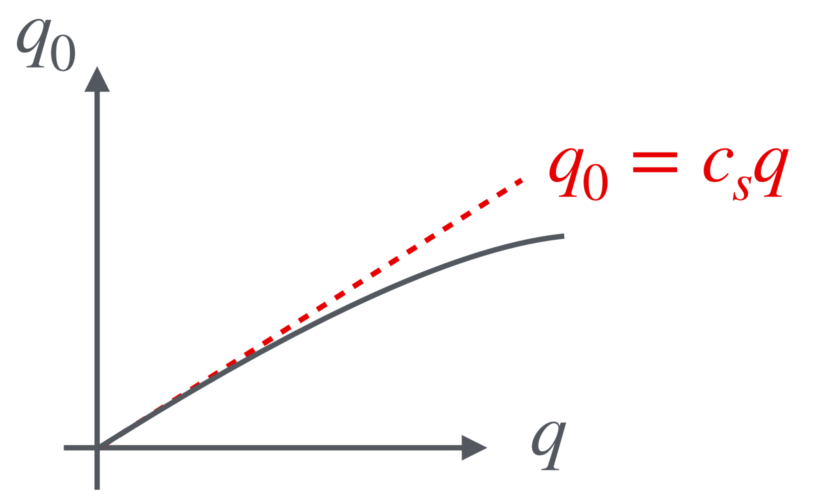
$$q_0(q) = c_s q \left(1 + \frac{\gamma}{8c_s^2} q^2 + \mathcal{O}(q^4 \log(q))\right) \qquad (V=0)$$

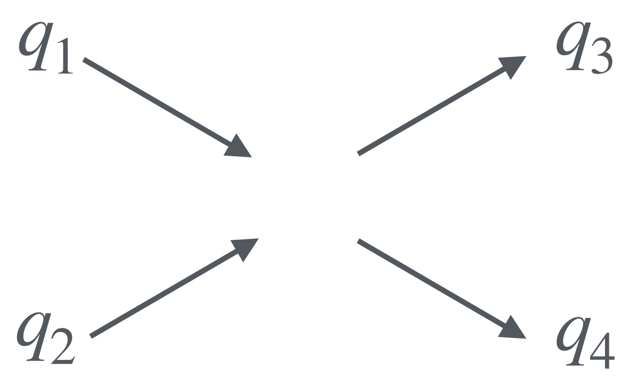
- $\gamma > 0$, Beliaev and Landau 3-phonon processes are dominant





- $\gamma < 0$, 3-phonon processes forbidden by energy conservation, Landau-Khalatnikov 4-phonon processes are dominant





Study the effect of the *trapping potential* on the phonon fluctuations

Y. Castin arXiv:2410.20866v2

EFT in a trap (spherically symmetric) + Fluctuations

$$\mathscr{L} = c_0 X^{5/2} + c_1 X^{-1/2}(\nabla X)^2 + c_2 X^{1/2}\left((\Delta\theta)^2 - 3(\nabla \otimes \nabla\theta)^2\right) + \ldots$$

Energy of the fluctuations q_0

Expand at quadratic order in π

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Energy of the fluctuations q_0

Expand at quadratic order in π

$$u = \frac{r}{R_{cl}}$$

$$\bar{t} = q_0 t$$

$q_0 \sim \sqrt{\mu}/R_{cl}$

$$L^{(2)}[\pi] = -\frac{5}{8}c_0 \; \mu^{5/2}\eta^2\sqrt{1-\bar{V}}\left(2(1-\bar{V})\left(\nabla_u\pi\right)^2 - 3\frac{q_0^2R_{cl}^2}{\mu}\left(\frac{\partial\pi}{\partial\bar{t}}\right)^2\right) + c_2 \; \mu^{5/2}\eta^4\sqrt{1-\bar{V}}\left[\left(\Delta_u\pi\right)^2 - 3\left(\nabla_u \otimes \nabla_u\pi\right)^2\right]$$

$$+c_1 \; \mu^{5/2}\eta^4\left[\frac{\left(\nabla_u\bar{V}\right)^2}{4(1-\bar{V})^{3/2}}\left(\nabla_u\pi\right)^2 + \frac{q_0^2R_{cl}^2}{\mu}\left(\frac{\left(\nabla_u\partial\pi/\partial\bar{t}\right)^2}{\sqrt{1-\bar{V}}} + \frac{\nabla_u\bar{V}\cdot\nabla_u(\partial\pi/\partial\bar{t})^2}{2(1-\bar{V})^{3/2}} + \frac{3\left(\partial\pi/\partial\bar{t}\,\nabla_u\bar{V}\right)^2}{8(1-\bar{V})^{5/2}}\right)\right]$$

EFT is controlled by

$$\eta = \frac{q_0}{\mu}$$

EFT in a trap (spherically symmetric) + Fluctuations

$$L^{(2)}[\pi] = -\frac{5}{8}c_0\mu^{5/2}\eta^2\sqrt{1-\bar{V}}\left(2(1-\bar{V})\left(\nabla_u\pi\right)^2 - 3\left(\frac{\partial\pi}{\partial\bar{t}}\right)^2\right)$$

EOM at leading order:

$$\frac{2\mu}{(q_0R_{cl})^2}\nabla_u\left((1-\bar{V})^{3/2}\nabla_u\pi(\mathbf{u})\right) + 3\pi(\mathbf{u})\sqrt{1-\bar{V}} = 0$$

WKB expansion is controlled by

$$\delta = \frac{\sqrt{2\mu}}{q_0R_{cl}} = \frac{\varpi}{q_0}$$

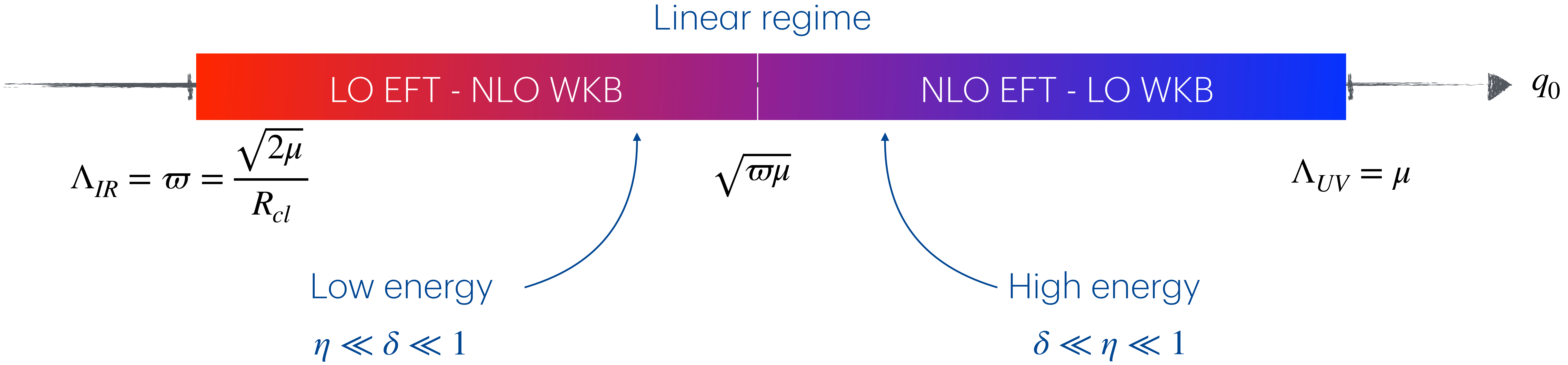
WKB is equivalent to a gradient expansion $\pi(u) = e^{\frac{i}{\delta}S_0(u)+S_1(u)+i\delta S_2(u)+\delta^2S_3(u)}$

EFT in a trap (spherically symmetric) + Fluctuations

Energy of the fluctuations q_0

$$\eta = \frac{q_0}{\mu}$$ controlling the EFT

$$\delta = \frac{\varpi}{q_0}$$ controlling the WKB expansion



Linear regime

LO EFT - NLO WKB

NLO EFT - LO WKB

q_0

$$\Lambda_{IR} = \varpi = \frac{\sqrt{2\mu}}{R_{cl}}$$

$$\sqrt{\varpi\mu}$$

$$\Lambda_{UV} = \mu$$

Low energy

$$\eta \ll \delta \ll 1$$

High energy

$$\delta \ll \eta \ll 1$$

Spectrum vs dispersion relation

$V(t, \mathbf{r}) = V(r)$

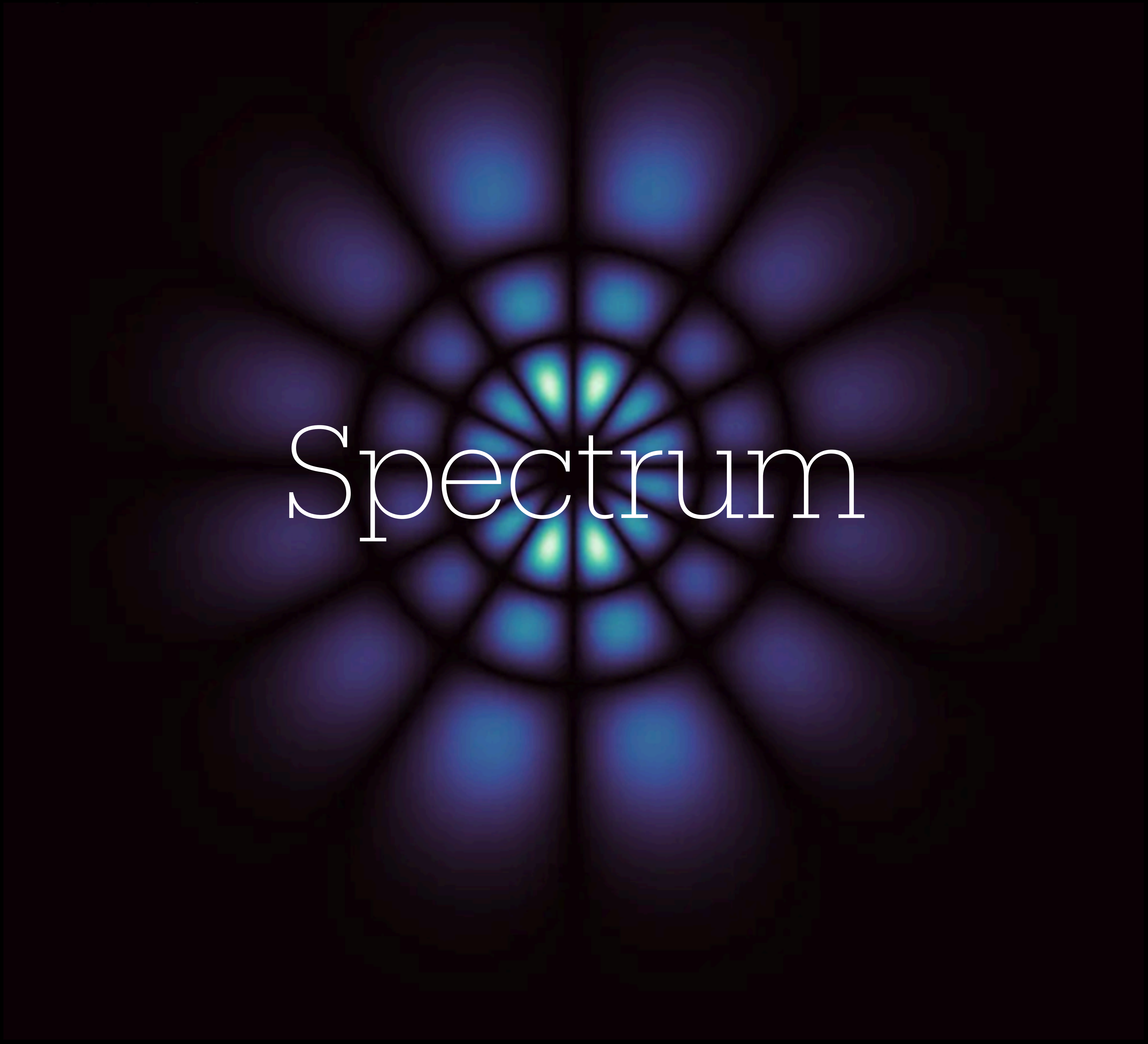
$$\pi(t, \mathbf{r}) \sim e^{iq_0(n,l)t} f_{nl}(r) Y_{lm}(\theta, \phi)$$

Time translation invariance, the spectrum is well-defined and discrete $q_0(n, l)$

The potential breaks spatial translation invariance. The right quantity to study is the dynamic structure factor.

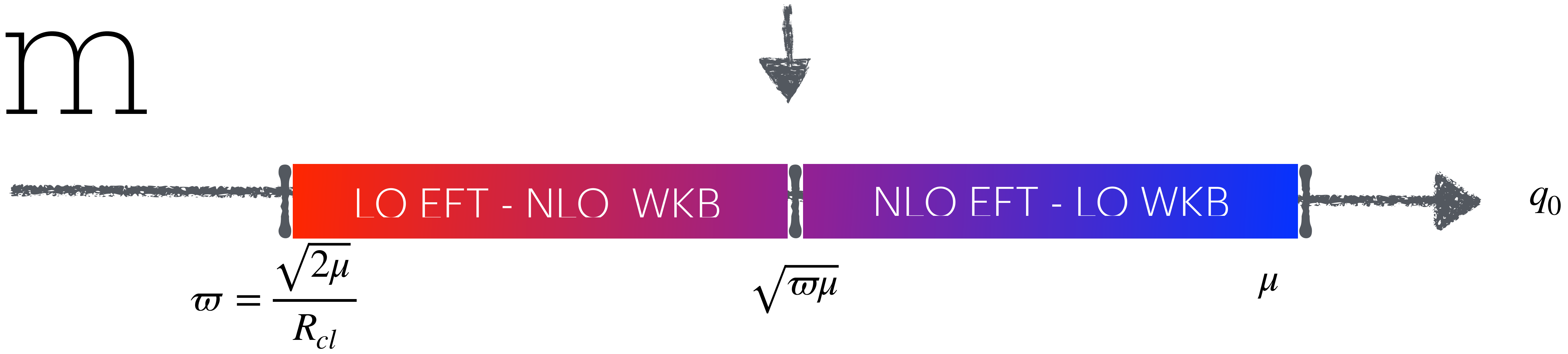
The probability to excite the many-body system from its ground state by transferring momentum q and energy q_0 is proportional to $S(q_0, \mathbf{q})$.

$$S(q_0, \mathbf{q}) = \sum_{\mathbf{n}} |\langle 0 | \delta\rho^\dagger(\mathbf{q}) | \mathbf{n} \rangle|^2 \delta\left(q_0 - q_0(\mathbf{n})\right)$$



Spectrum

Fluctuation spectrum



Linear regime

Equation of motion

$$\delta^2 \nabla_u \left((1-\overline{V})^{3/2} \nabla_u \pi \right) = 3\sqrt{1-\overline{V}} \frac{\partial^2 \pi}{\partial \bar{t}^2}$$

$u = \frac{r}{R_{cl}}$

$\bar{t} = q_0 t$

Ansatz from symmetries

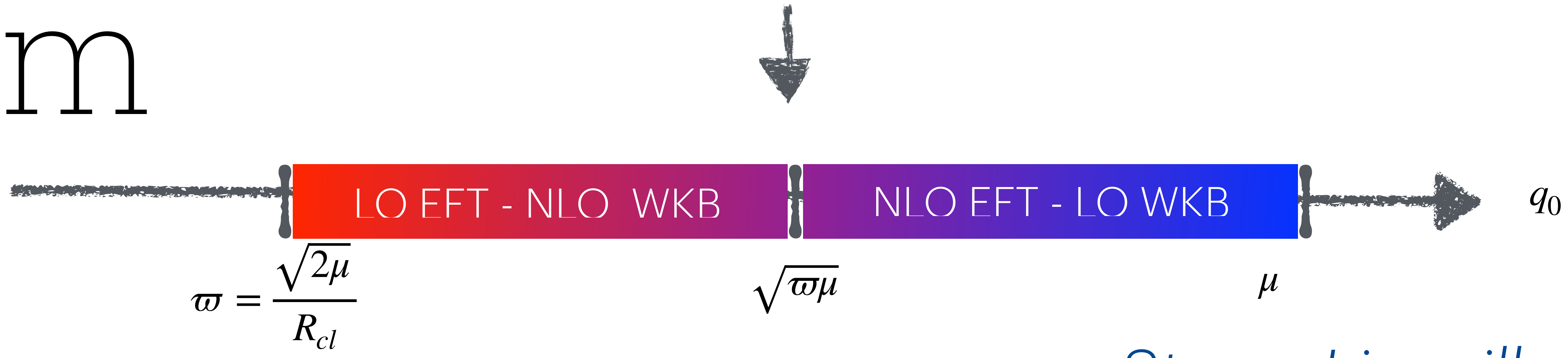
$$\pi(\bar{t}, \mathbf{u}) = e^{i\bar{t}} \pi(u) Y_{lm}(\theta, \phi)$$

$$\delta^2 \frac{d}{du} \left(u^2 (1-\overline{V})^{3/2} \frac{d}{du} \pi(u) \right) - \delta^2 l(l+1)(1-\overline{V})^{3/2} \pi(u) + 3u^2 \sqrt{1-\overline{V}} \pi(u) = 0$$

Sturm-Liouville problem

Fluctuation spectrum

Linear regime



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Sturm-Liouville problem

Sturm-Liouville problem

$$\frac{d}{dr}\left[p(r)\frac{d\pi}{dr}\right] + q(r)\pi = -q_0^2 w(r)\pi \qquad 0 < r < R_{cl}$$

The solutions q_0^2 and π are eigenvalues and eigenvectors of the self adjoint operator L

$$L\pi = -\frac{1}{w(r)}\left[\frac{d}{dr}\left[p(r)\frac{d\pi}{dr}\right] + q(r)\pi\right] = q_0^2\pi$$

With the following weighted inner product

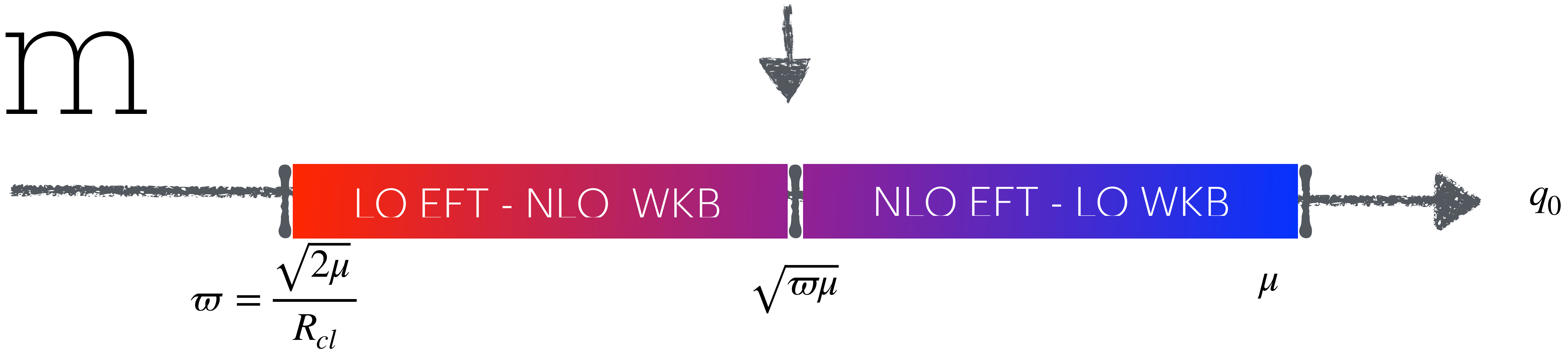
$$\langle n \mid m\rangle = \int_0^{R_{cl}} w(r)\pi_n(r)\pi_m(r)dr = \delta_{nm}$$

$$L[X] = c_0 X^{5/2}$$

$$\frac{d}{dr}\left(r^2 L'[\mu - V]\frac{d}{dr}\pi\right) - L'[\mu - V]l(l+1)\pi = -r^2 L''[\mu - V]q_0^2\pi$$

$$w(r) = r^2\frac{\rho(r)}{c_s^2(r)}$$

Fluctuation spectrum


LO EFT - NLO WKB

NLO EFT - LO WKB

q_0

$\varpi = \frac{\sqrt{2\mu}}{R_{cl}}$

$\sqrt{\varpi\mu}$

μ

Linear regime

Equation of motion

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$$\delta^2 \frac{d}{du}\left(u^2 (1-\overline{V})^{3/2} \frac{d}{du}\pi(u) \right) - \delta^2 l(l+1)(1-\overline{V})^{3/2}\pi(u) + 3u^2\sqrt{1-\overline{V}}\pi(u) = 0$$

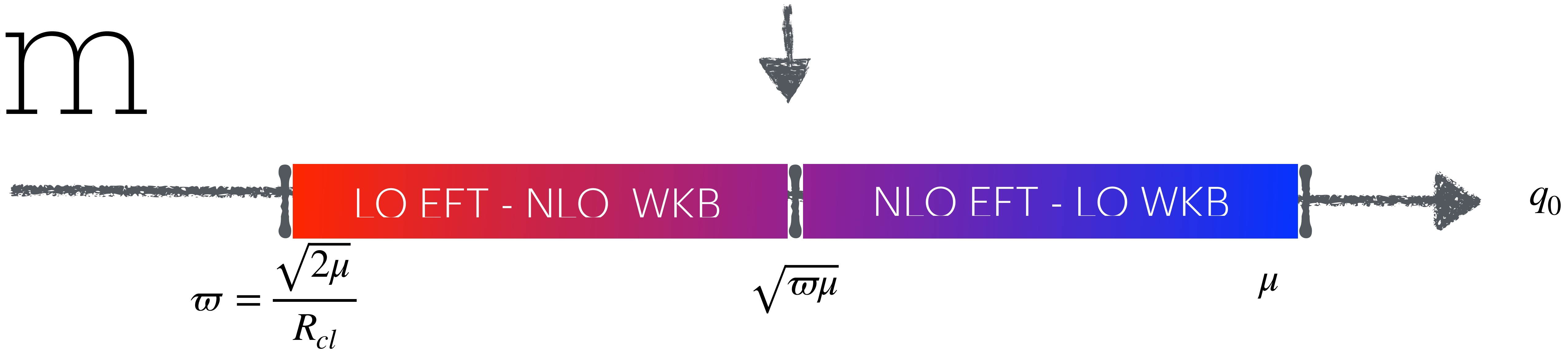
Sturm-Liouville problem

WKB ansatz for the radial mode

$$\pi(u) = e^{\frac{i}{\delta}S_0(u) + S_1(u)}$$

Eikonal and transport equations

Fluctuation spectrum



$$V(r) = \mu \left(\frac{r}{R_{cl}}\right)^{2k}$$

Linear regime

$$\pi(u) = \frac{D_+ \exp\left[i\frac{\sqrt{3}}{\delta}\int_{u_0}^{u}\frac{dw}{\sqrt{1-\bar{V}(w)}}\right] + D_- \exp\left[-i\frac{\sqrt{3}}{\delta}\int_{u_0}^{u}\frac{dw}{\sqrt{1-\bar{V}(w)}}\right]}{u\sqrt{1-\bar{V}(u)}},$$

Singularity in 0

Set $u_0 = 0$ and $D_+ = -D_- = \dfrac{D}{2i}$

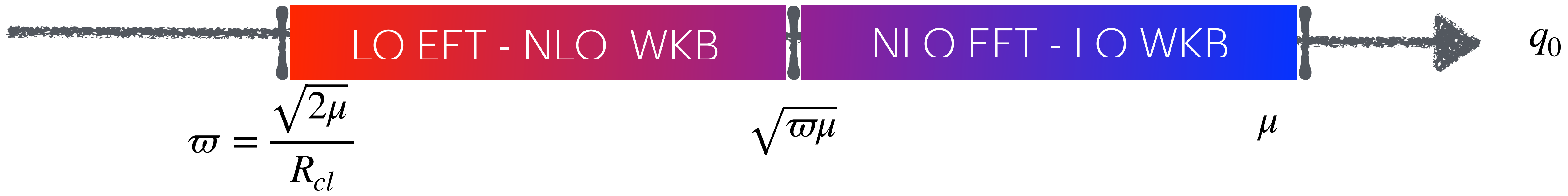
Singularity in 1

$$q_0 = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}}\varpi n$$

Quantization condition !

Fluctuation spectrum

$$V(r) = \mu \left(\frac{r}{R_{cl}} \right)^{2k}$$



LO EFT - NLO WKB

NLO EFT - LO WKB

q_0

$\varpi = \frac{\sqrt{2\mu}}{R_{cl}}$

$\sqrt{\varpi\mu}$

μ

In all three regimes, the radial mode satisfies a Sturm-Liouville problem.

High energy

$$\frac{q_0}{\varpi} = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}} n + \frac{4(5c_1 - 51c_2)}{3^{7/4}\pi k (5c_0)^{1/4}(c_1 - 9c_2)^{3/4}\binom{1/(2k)}{1/2}^{5/2}} \sqrt{\frac{\varpi}{\mu}} n^{3/2} + O(\frac{\varpi}{\mu} n^2)$$

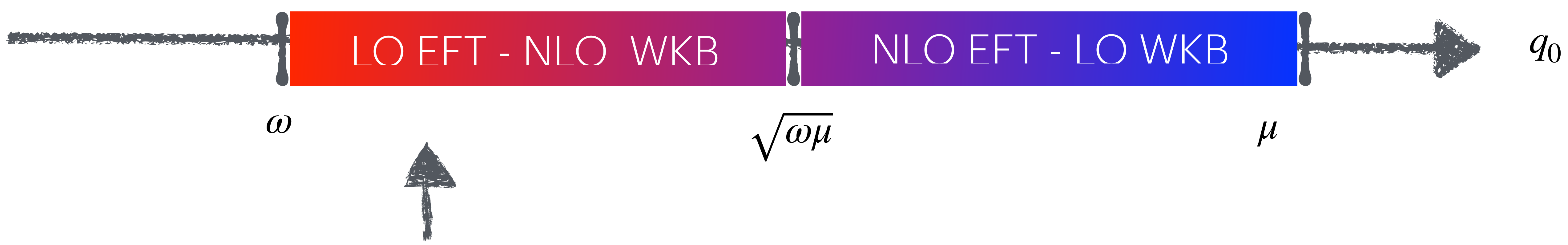
Linear

$$q_0 = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}} \varpi n$$

Low energy

$$\frac{q_0}{\varpi} = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}} n - \frac{2(k+1)}{2\sqrt{3}\pi\binom{(k-1)/2k}{1/2}} \frac{1}{n} + \dots$$

Fluctuation spectrum


LO EFT - NLO WKB

NLO EFT - LO WKB

q_0

ω $\sqrt{\omega\mu}$ μ

Low energy $V(r) = \dfrac{\omega^2 r^2}{2}$

For the harmonic potential, the radial equation of motion is solvable *exactly*, giving a hypergeometric function:

$$\pi(u) = u^l {}_2F_1(-n, n + l + 2, 3/2 + l; u^2)$$

With spectrum:

$$q_0(n, l) = \omega\sqrt{\frac{4n}{3}(n + l + 2) + l}, \quad n, l = 0,1,\ldots$$

First excited states:

$$q_0(0,1) = \omega$$

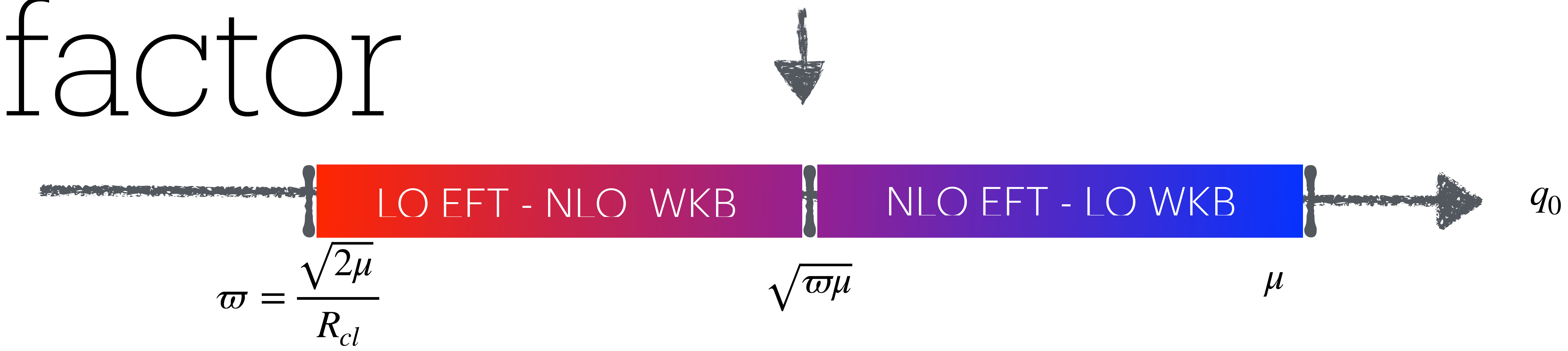
$$q_0(1,0) = 2\omega$$

Descendant operators

S. M. Kravec and S. Pal arXiv:1809.08188

Dynamic structure factor

Dynamic structure factor



LO EFT - NLO WKB

NLO EFT - LO WKB

q_0

$\varpi = \frac{\sqrt{2\mu}}{R_{cl}}$

$\sqrt{\varpi\mu}$

μ

$$S(q_0, \mathbf{q}) = \sum_{\mathbf{n}} |\langle 0 | \delta\rho^\dagger(\mathbf{q}) | \mathbf{n} \rangle|^2 \delta\left(q_0 - q_0(\mathbf{n})\right)$$

Linear regime

$$S\left(q_0, q\right) \approx \mu^{1/2} \frac{c_0}{\omega q^2} \sum_n \frac{q_0^2}{\left|S_0''(\bar{u})\right|} \sin^2\left(qR_{cl}\bar{u}\right) \delta\left(q_0 - \frac{n\pi\varpi}{S_0(1)}\right)\Bigg|_{q=\frac{q_0}{\sqrt{2\mu}}S_0'(\bar{u})}$$

The curve on which the Fourier integral localizes $q_0 = \sqrt{\frac{2(\mu - V(r))}{3}}q$

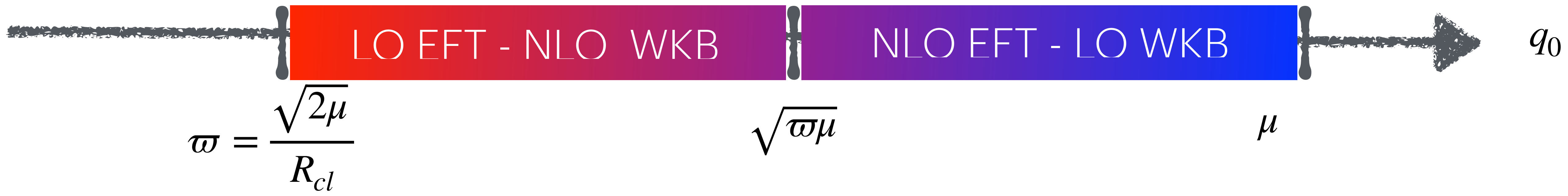
Local density approximation

The dynamic structure factor peaks where $S_0''(u) = 0$

$$q_0 = \sqrt{\frac{2\mu}{3}}q$$

Type-I Goldstone boson

Dynamic structure factor



Curve along which the dynamic structure factor peaks

High energy

$$q_0 = \sqrt{\frac{2\mu}{3}}q\left(1 - \frac{4}{15}\frac{c_1 - 3c_2}{c_0}\frac{q^2}{\mu}\right)$$

It doesn't depend on the potential!

Linear

$$q_0 = \sqrt{\frac{2\mu}{3}}q = c_s q$$

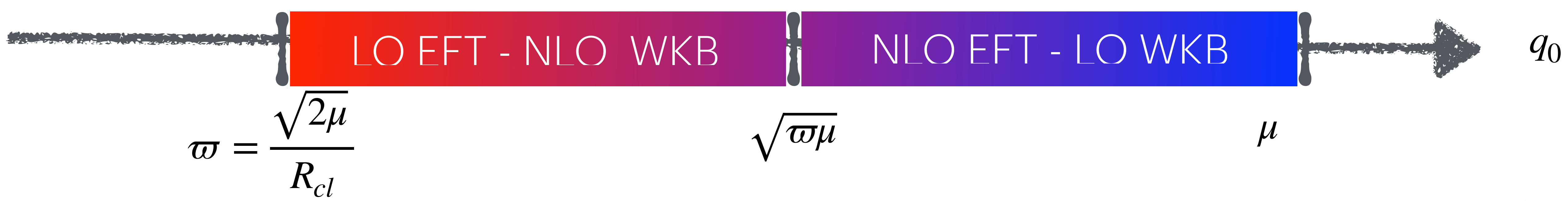
Low energy

$$q_0 = \sqrt{\frac{2\mu}{3}}q\left(1 - O(\frac{\omega^{2k}}{q^{2k}\mu^k})\right)$$

From exact phonon field in a harmonic potential

$$q_0 = \sqrt{\frac{2\mu}{3}\left(q^2 - \frac{2\omega^2}{\mu}\right)}$$

Dynamic structure factor



Curve along which the dynamic structure factor peaks

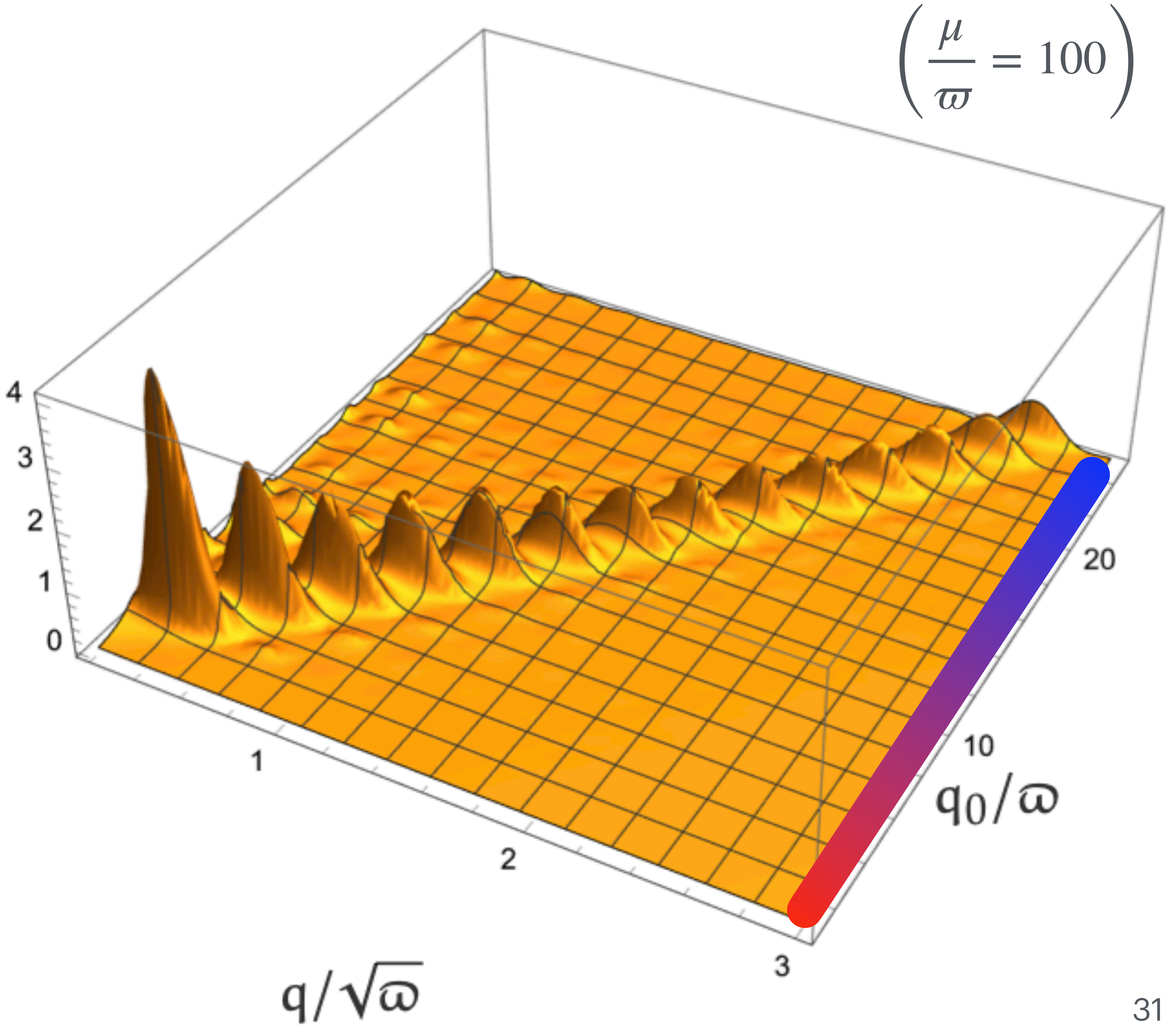
High energy $$q_0 = \sqrt{\frac{2\mu}{3}}\,q\left(1 - \frac{4}{15}\,\frac{c_1 - 3c_2}{c_0}\,\frac{q^2}{\mu}\right)$$

Linear $$q_0 = \sqrt{\frac{2\mu}{3}}\,q = c_s q$$

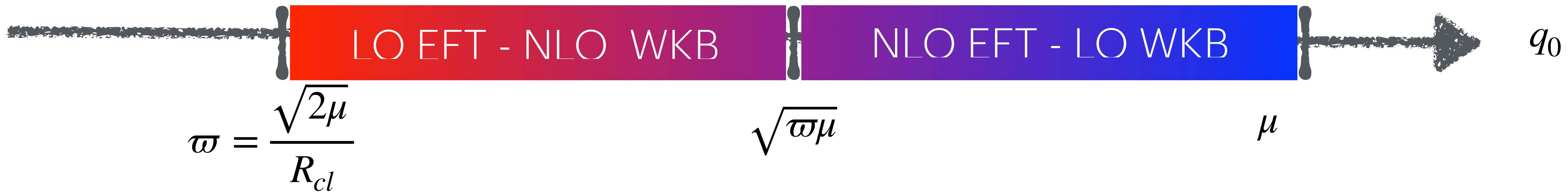
Low energy $$q_0 = \sqrt{\frac{2\mu}{3}}\,q\left(1 - O(\frac{\omega^{2k}}{q^{2k}\mu^k})\right)$$

$$V(u) = u^{16}$$

$$\left(\frac{\mu}{\varpi} = 100\right)$$



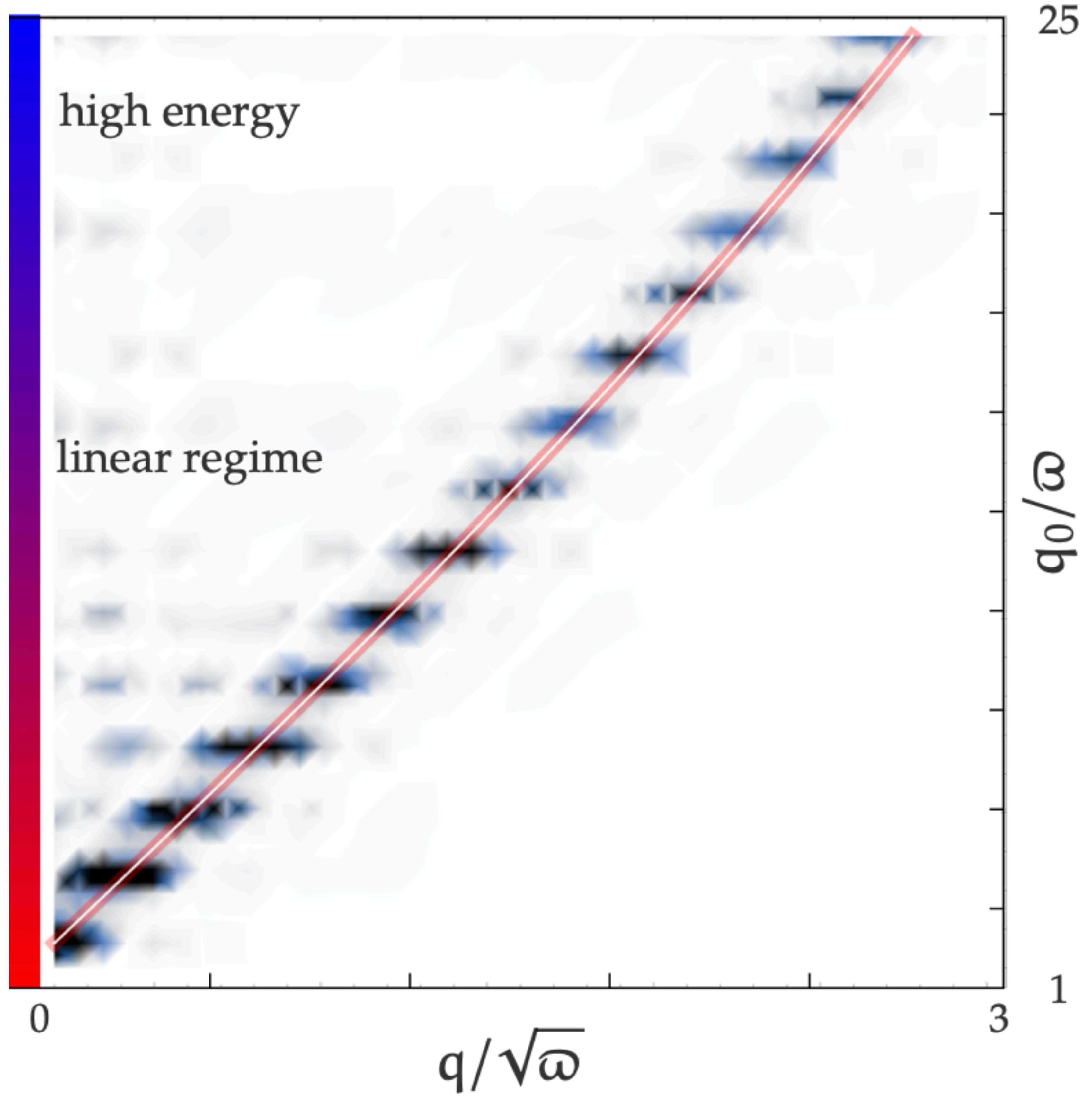
Dynamic structure factor

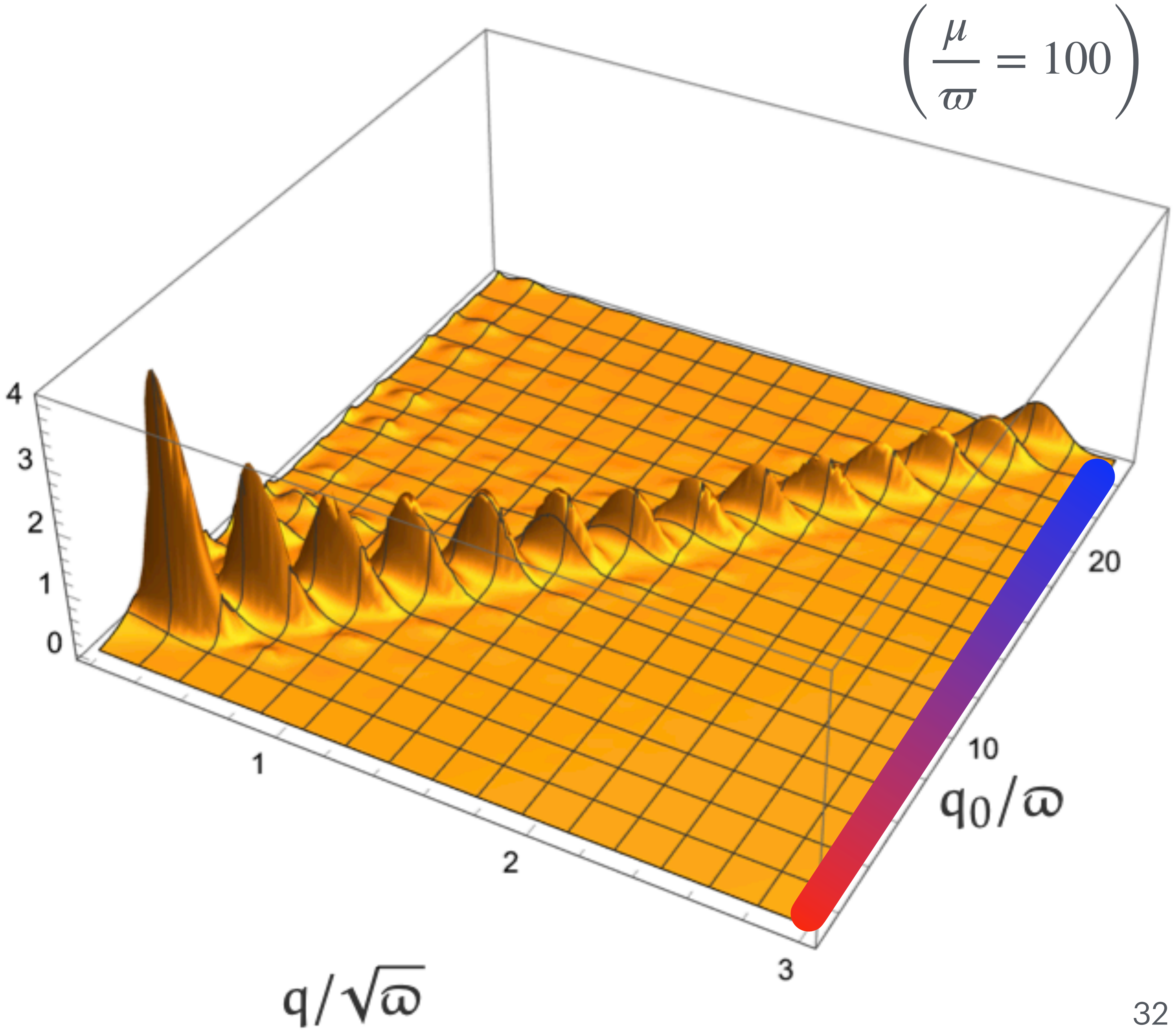


$V(u) = u^{16}$

$\left(\frac{\mu}{\varpi} = 100\right)$

Curve along which the dynamic structure factor peaks

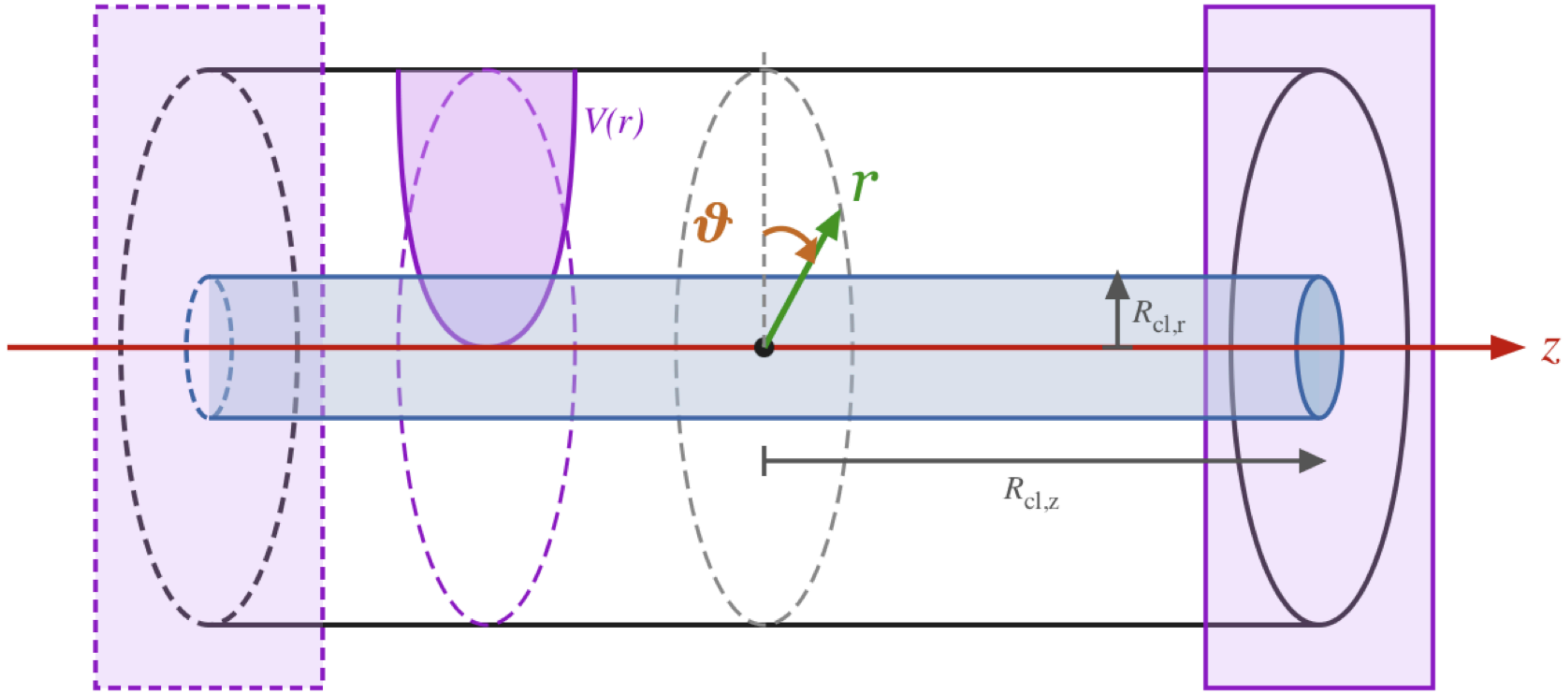




Closer to experiment

Cylinder geometry

$$\pi(t, \boldsymbol{r}) = e^{iq_0 t} \sin\left(k_z(z + \frac{L}{2})\right) e^{in_\vartheta \vartheta} \pi(r), \quad k_z = \frac{\pi n_z}{L}$$



2 scales $R_{cl,z}, R_{cl,r}$

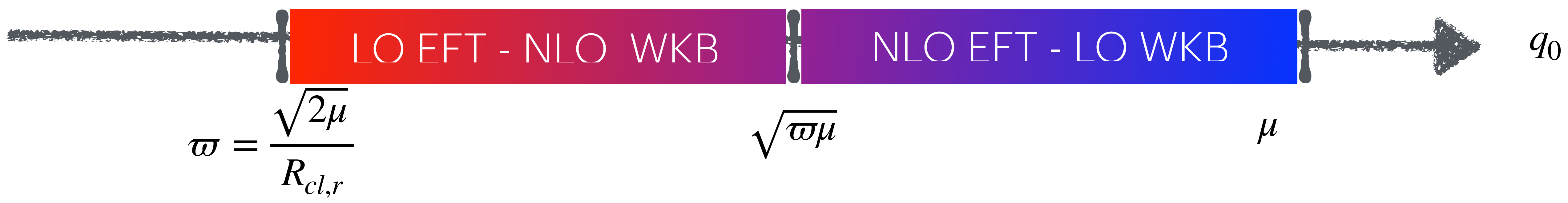
$$R_{cl} \to R_{cl,r}$$

We break isotropy $S(q_0, \mathbf{q})$

A. Le Borgne arXiv:2605.26336v1

Fluctuation spectrum

$$V(r) = \mu\left(\frac{r}{R_{cl,r}}\right)^{2k}$$



In all three regimes, the radial mode satisfies a Sturm-Liouville problem.

High energy

$$\frac{q_0}{\varpi} = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}}n + \frac{4(5c_1 - 51c_2)}{3^{7/4}\pi k(5c_0)^{1/4}(c_1 - 9c_2)^{3/4}\binom{1/(2k)}{1/2}^{5/2}}\sqrt{\frac{\varpi}{\mu}}n^{3/2} + O(\frac{\varpi}{\mu}n^2)$$

Linear

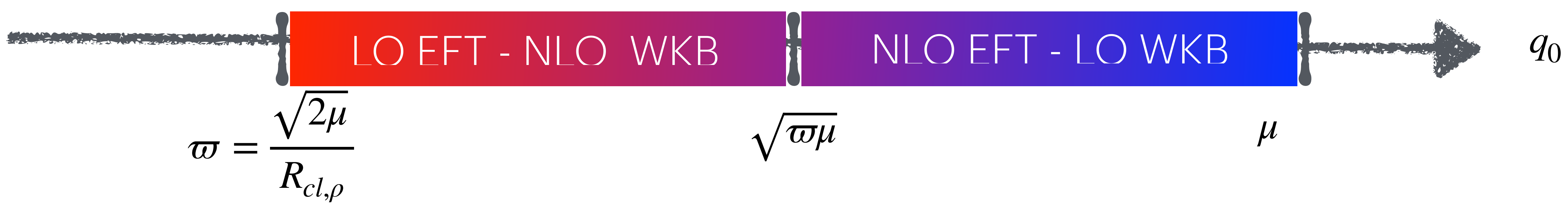
$$q_0 = \frac{2}{\sqrt{3}\binom{1/(2k)}{1/2}}\varpi n$$

$\epsilon = \frac{R_{cl,\rho}}{R_{cl,z}}$

Low energy

$$\frac{q_0}{\varpi} = \sqrt{\frac{\pi}{3}}\frac{\Gamma\left(\frac{1+k}{2k}\right)}{\Gamma\left(1+\frac{1}{2k}\right)}n - \frac{1}{16\sqrt{3\pi}\,n}\left[\frac{2(1+4k)\Gamma\left(1-\frac{1}{2k}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{2k}\right)} - \frac{n_z^2\pi^2\epsilon^2\Gamma\left(1+\frac{1}{2k}\right)}{\Gamma\left(\frac{1}{2}\left(3+\frac{1}{k}\right)\right)}\right] + \cdots$$

Dynamic structure factor



Curve along which the dynamic structure factor peaks

High energy $$q_0 = \sqrt{\frac{2\mu}{3}} q_\perp \left(1 - \frac{4}{15} \frac{c_1 - 3c_2}{c_0} \frac{q_\perp^2}{\mu}\right)$$

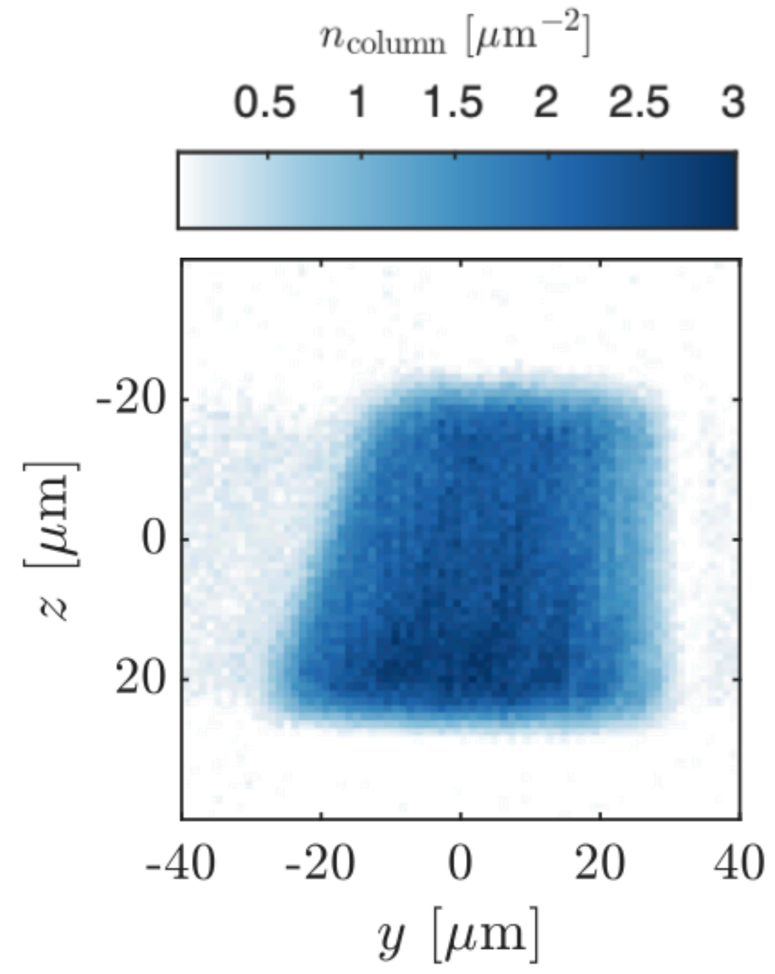
Linear $$q_0 = \sqrt{\frac{2\mu}{3}} q_\perp = c_s q_\perp$$

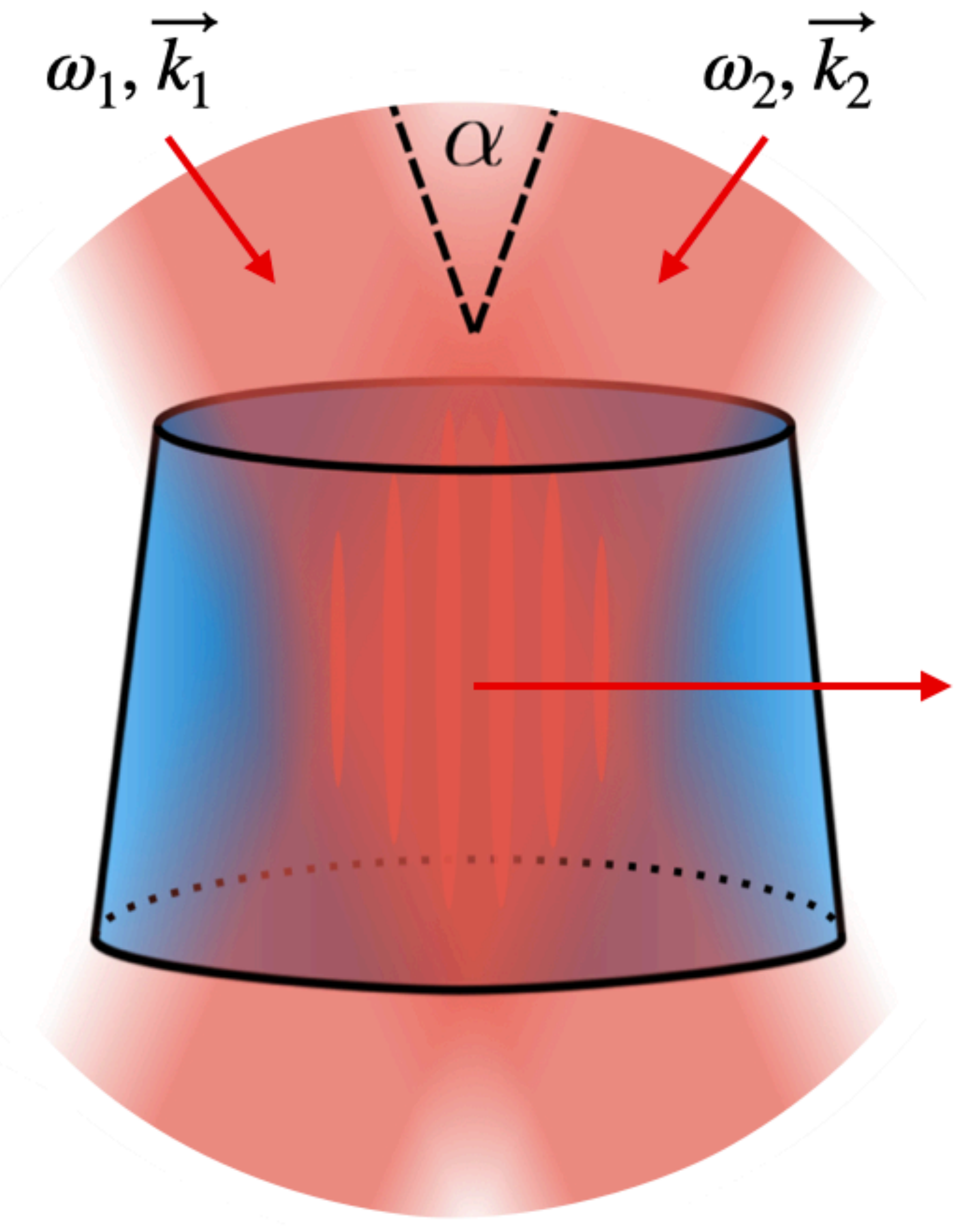
Low energy $$q_0 = \sqrt{\frac{2\mu}{3}} q_\perp \left(1 + \frac{1}{2}\left(\frac{q_z^2}{q_\perp^2}\right) - \frac{5}{8} O(\frac{\omega^{2k}}{q_\perp^{2k} \mu^k})\right)$$

Correction always convex

Except when $n_z = 0$

Comparison with experiment

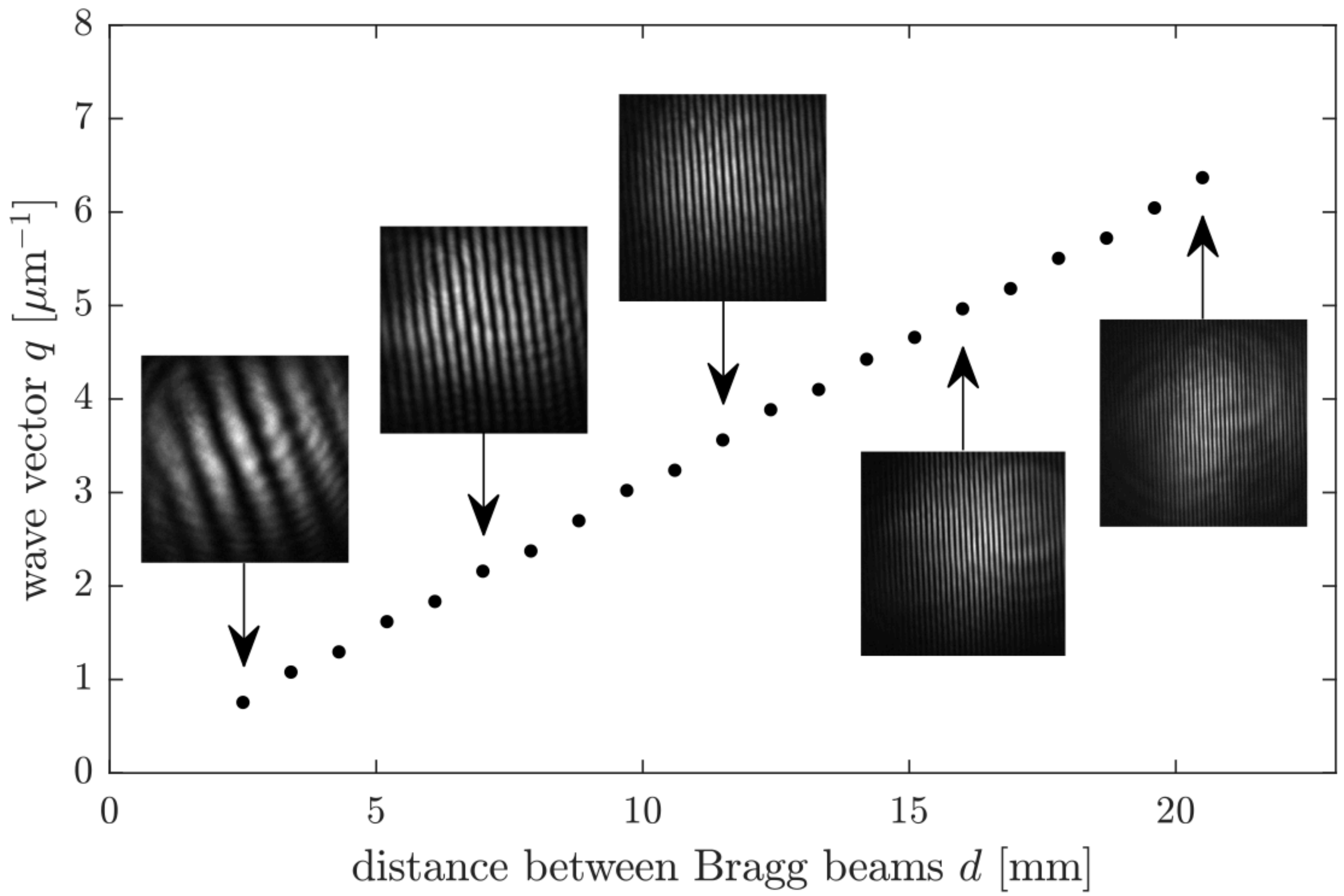




$$\hbar\omega = \hbar\omega_1 - \hbar\omega_2$$

$$\hbar\vec{q} = \hbar\vec{k_1} - \hbar\vec{k_2}$$

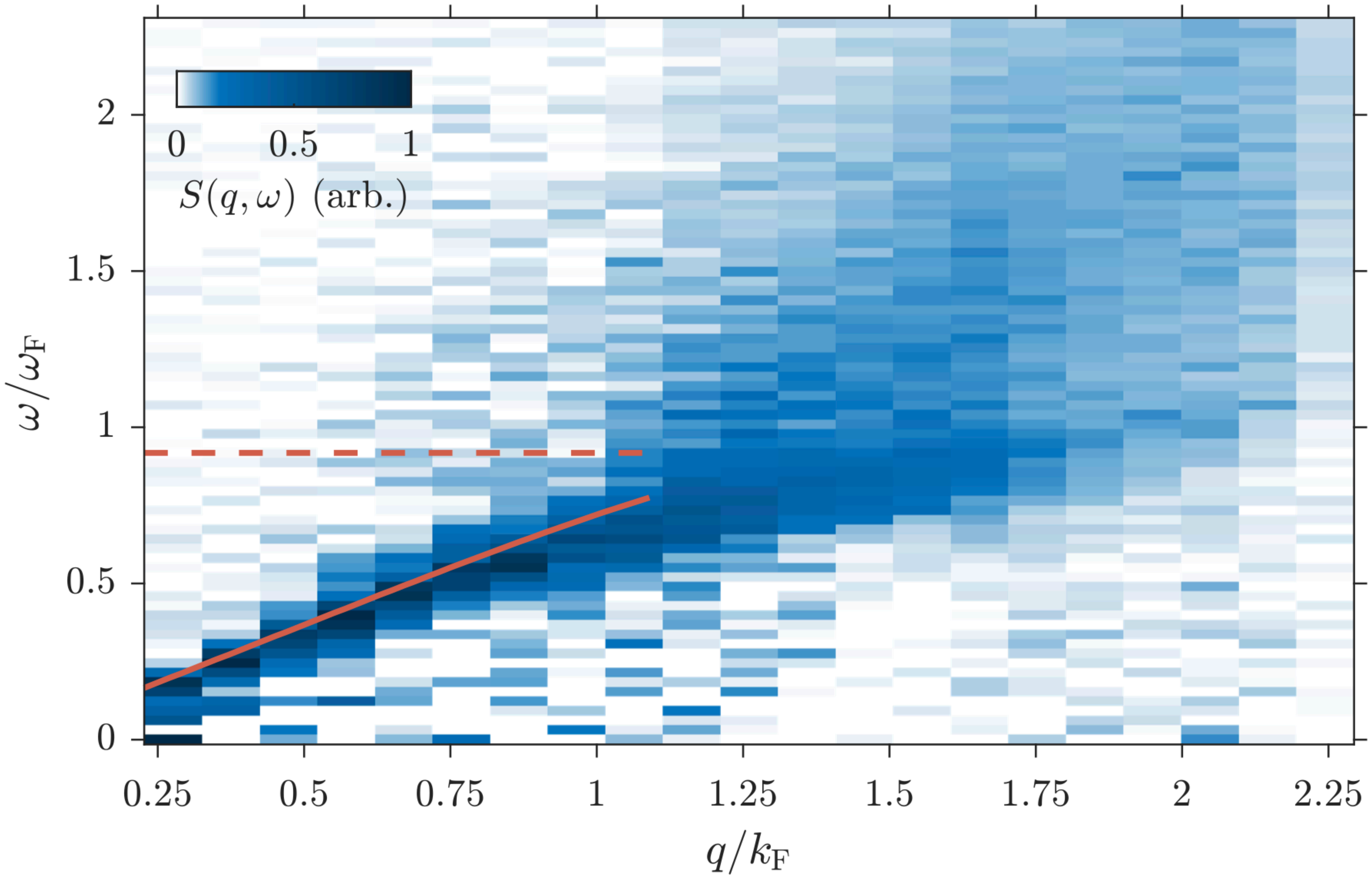
Bragg spectroscopy



H. Biss, L. Sobirey, N. Luick, M. Bohlen, J.J. Kinnunen, G.M. Bruun, T. Lompe & H. Moritz Phys. Rev. Lett. 128 (2022) 100401

Comparison with experiment

Dynamic structure factor



$$\gamma_{\text{exp}} = -0.085(8)$$

High energy $q_0 = \sqrt{\frac{2\mu}{3}} q_\perp \left(1 - \frac{4}{15} \frac{c_1 - 3c_2}{c_0} \frac{q_\perp^2}{\mu}\right)$

Linear $q_0 = \sqrt{\frac{2\mu}{3}} q_\perp = c_s q_\perp$

Low energy $q_0 = \sqrt{\frac{2\mu}{3}} q_\perp \left(1 - O(\frac{\omega^{16}}{q_\perp^{16} \mu^8})\right)$

Conclusion

Scale study of Schrödinger invariant superfluid $\varpi \ll q_0 \ll \mu$

Define 3 distinct regimes to compute the spectrum and dynamics structure factor

The most significant trap-induced modifications appear in the low-energy regime in both spherical and cylindrical geometries; the strongest effect occurs for a harmonic potential.

Future direction

Finite temperature