The impact of θ-angle on the QCD phase diagram at non-zero chemical potentials

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Roadmap

- Motivation: QCD at finite density and θ-angle
- Our Setup: Chiral Lagrangian and vacuum structure
- Phase diagram and phase transitions
- $\theta = \pi$: CP symmetry and Dashen's phenomenon
- Novel Pion superfluid phase
- Condensates and low energy spectrum
- Summary and conclusions

The θ-angle in QCD

Yang-Mills theories possess a topological sector represented by a θ-term:

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \theta q(x) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \theta\frac{g_R^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a$$

where $q(x) =$ **topological charge density**.

- It is a volume term $tr(F_{\mu\nu}\tilde{F}^{\mu\nu}) = \partial_\mu K^\mu$,
- If $\theta \neq 0,\pi$ it introduces a CP violation → **Strong CP Problem** $(\bar{\theta}_{QCD} < 10^{-9})$

The effective Lagrangian at low energies with the inclusion of the θ-angle reads:

$$\mathcal{L}_{eff} = \frac{1}{2}Tr[\partial_\mu\Sigma\partial^\mu\Sigma^\dagger)] + \frac{F_\pi}{2\sqrt{2}}Tr[M(\Sigma+\Sigma^\dagger)] + \frac{i}{2}q(x)Tr\left[\log\frac{\Sigma}{\Sigma^\dagger}\right] + \frac{q(x)^2}{aF_\pi^2} - \theta q(x)$$

- The 3rd and the 4th terms reproduce the Adler-Bell-Jackiw anomaly:

$$U(N_f)_L \times U(N_f)_R \sim SU(N_f)_V \times U(1)_V \times SU(N_f)_A \times \cancel{U(1)_A}$$

Quantum Field Theory at fixed charge

Fixing a charge means to impose a constraint which breaks Lorentz invariance:

$$Q = \int d^{d-1}x\, j^0 = \bar{Q}\,, \qquad \hat{\mathscr{H}} = \mathscr{H} - \mu\bar{Q} : \quad \hat{\mathscr{H}}\,|0\rangle = 0$$

As an example, fixing a charge in a U(1) scalar field theory:

$$\mathscr{L} = \partial_\mu\phi^*\partial^\mu\phi - V(\phi^*\phi) = \partial_\mu\phi^*\partial^\mu\phi - \frac{\lambda}{4}(\phi^*\phi)^2\,, \quad \phi = \frac{1}{\sqrt{2}}\rho\, e^{i\chi}$$

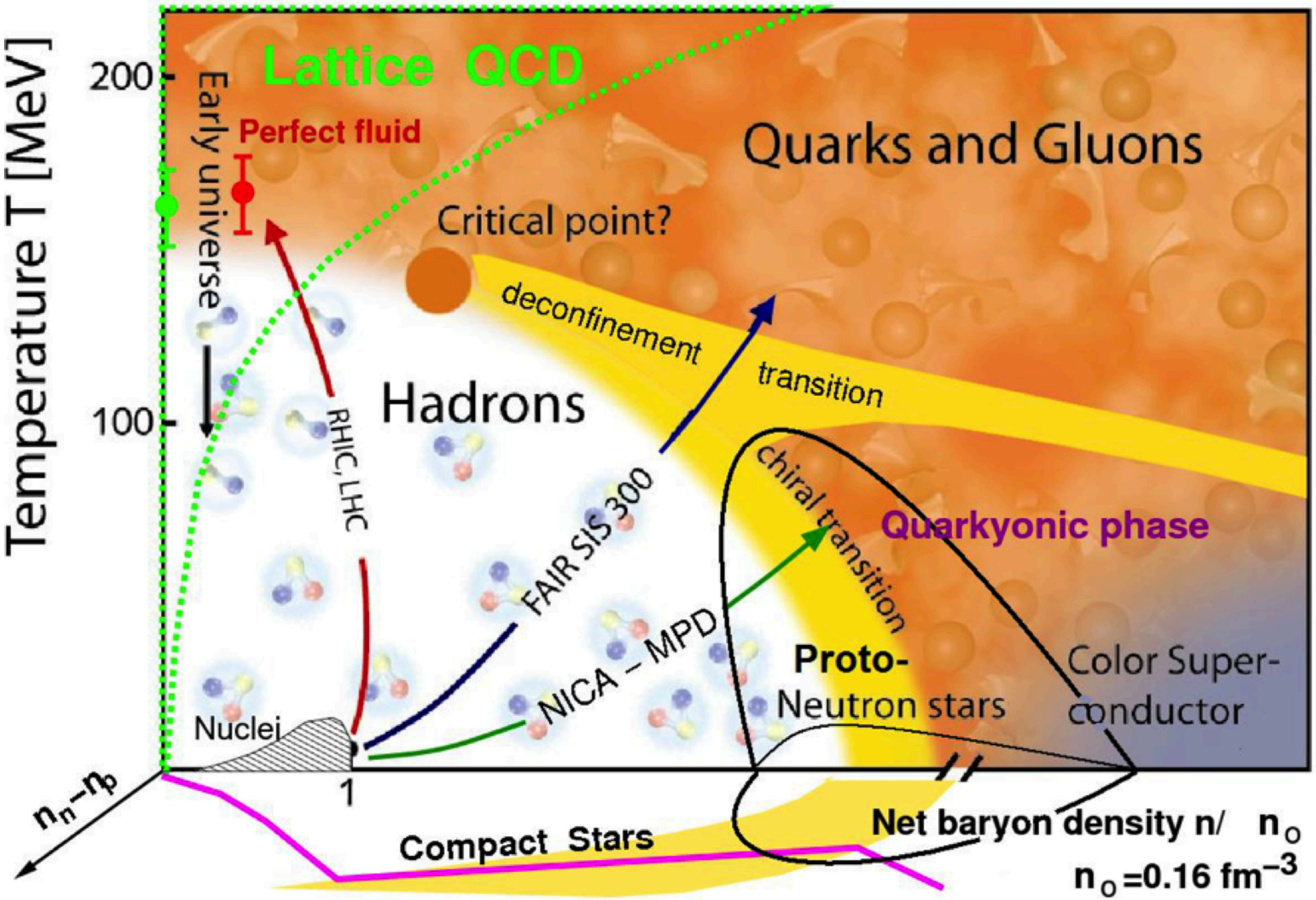
produces a centrifugal barrier resulting in a double-well effective potential

$$V_{eff}(\phi^*\phi) = V(\phi^*\phi) - \mu^2\phi^*\phi = \frac{\lambda}{4}(\phi^*\phi)^2 - \mu^2\phi^*\phi = \frac{\lambda}{16}\rho^4 - \frac{1}{2}\mu^2\rho^2 = V_{eff}(\rho)$$

Through a spontaneous breaking of the U(1) symmetry (**Superfluid Transition**):

- the radial mode ρ acquires a vev $\neq 0 \to m_\rho \propto \mu$,
- the angular field χ is the expected massless GB called **Superfluid Phonon**.

QCD Phase Diagram at non-zero μ/T



The Chiral Lagrangian with the θ-angle

The low-energy dynamics of $N_f = 3$ QCD at finite isospin and strangeness chemical potentials, in the presence of a nonzero θ-angle, is given by the Chiral Lagrangian

$$\mathscr{L} = \frac{F_\pi^2}{4} Tr\{\nabla_\mu \Sigma \nabla^\mu \Sigma^\dagger\} + \frac{F_\pi^2}{2} G Tr\{M\Sigma + M^\dagger \Sigma^\dagger\}\,, \qquad M = e^{-i\theta/3}\, diag(m_u, m_d, m_s)$$

describing the pseudo-Goldstone bosons arising from chiral symmetry breaking:

$$\Sigma = U\Sigma_0 U\,, \qquad U = exp\left(\frac{i\Phi}{\sqrt{2}F_\pi}\right), \qquad \Phi = \Pi^a T^a\,, \qquad \Pi^a T^a = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix},$$

Meson Octet

The chemical potentials are included through

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - i[B_\nu, \Sigma]\,, \qquad B_\nu = \delta_{\nu 0} B\,, \qquad B = \frac{1}{2}\begin{pmatrix} \mu_I & 0 & 0 \\ 0 & \mu_I & 0 \\ 0 & 0 & 2\mu_s \end{pmatrix}.$$

The Vacuum Σ_0

[J.B.Kogut, D.Toublan (2001)]

At zero θ-angle, the vacuum configuration can be written as:

$$\boxed{\Sigma_c = R^- \bar{\Sigma} R^+}$$

$$\bar{\Sigma} = \begin{pmatrix} \cos\varphi & 0 & 0 \\ 0 & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} + i \sin\varphi \left(\underbrace{\cos\xi \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sin\xi \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\text{residual } U(1)_{I,S}} \right)$$

$$R^{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & 0 \\ 0 & 0 & \cos\beta \end{pmatrix} \pm i \sin\beta \left(\underbrace{\cos\rho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \sin\rho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}}_{\text{residual } U(1)_{I,S}} \right)$$

$$\Sigma_c \Big|_{\varphi=0} = \Sigma_M = \mathbf{1}_3 \qquad \Sigma_c \Big|_{\beta=0} = \Sigma_I = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Sigma_c \Big|_{\beta=\frac{\pi}{2}} = \Sigma_S = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

Normal Phase **Pion Phase** **Kaon Phase**

The θ-angle is included through the Witten variables $\{\alpha_1, \alpha_2, \alpha_3\}$

$$\Sigma_0 = W \Sigma_c \,, \qquad W = \mathrm{diag}\{e^{-i\alpha_1}, e^{-i\alpha_2}, e^{-i\alpha_3}\}$$

Symmetry Breaking Pattern

$$\begin{tikzcd}
 & SU(3)_L \times SU(3)_R \times U(1)_V \arrow[r, "m_u = m_d = m_s \neq 0"] & SU(3)_V \times U(1)_V \arrow[d, "\mu_I, \mu_s \neq 0"] \\
\boxed{\text{Pion Phase}} & U(1)_S \times U(1)_{V=B} & U(1)_I \times U(1)_S \times U(1)_{V=B} \arrow[l, dashed, "\Sigma_I \neq 0"'] \arrow[d, dashed, "\Sigma_S \neq 0"] \\
 & \boxed{\text{Kaon Phase}} & U(1)_I \times U(1)_{V=B}
\end{tikzcd}$$

Phase Diagram - Normal phase

The Eoms for the Witten variables can be solved in the different phases:

$$
\begin{aligned}
2Gm_u \sin\alpha_1 \cos\varphi &= a\bar{\theta}\,,\\
2Gm_d \sin\alpha_2 \left(\cos^2\beta\cos\varphi + \sin^2\beta\right) &= a\bar{\theta}\,,\\
2Gm_s \sin\alpha_3 \left(\sin^2\beta\cos\varphi + \cos^2\beta\right) &= a\bar{\theta}
\end{aligned}
\qquad
\begin{cases}
\cos\varphi = 1\,, \quad \beta \in (0,\pi/2)\,, & \text{Normal phase}\,,\\
\cos\varphi = \dfrac{G(m_d\cos\alpha_2 + m_u\cos\alpha_1)}{\mu_I^2}\,, \quad \beta = 0\,, & \text{Pion phase}\,,\\
\cos\varphi = \dfrac{4G(m_s\cos\alpha_3 + m_u\cos\alpha_1)}{(\mu_I + 2\mu_s)^2}\,, \quad \beta = \frac{\pi}{2}\,, & \text{Kaon phase}\,.
\end{cases}
$$

In the case of non-degenerate masses $m_u = m_d = m \neq m_s$, we can define the quantity

$$\boxed{\gamma \equiv m/m_s = \frac{m_\pi^2}{2m_K^2 - m_\pi^2}\,.}$$

The solutions in the Normal Phase are given by

$$\alpha_1 = \alpha_2 = \alpha = \frac{\theta - \alpha_3}{2}\,,$$

$$\alpha_3^{(\pm)} = 2\arccos\left(\frac{1}{12}\left(-3\gamma\cos\left(\frac{\theta}{2}\right) \pm \sqrt{\frac{9}{2}\gamma^2(\cos\theta + 1) + \frac{3(-\gamma^2 + x + 4)^2}{x}}\right.\right.$$
$$\left.\left.\pm\sqrt{9\gamma^2(\cos\theta + 1) - 12(\gamma^2 - 4) \mp \frac{9\gamma\cos\left(\frac{\theta}{2}\right)(\gamma^2\cos\theta - \gamma^2 - 8)}{\sqrt{\frac{1}{2}\gamma^2(\cos\theta + 1) + \frac{(-\gamma^2 + x + 4)^2}{3x}}} - \frac{3(\gamma^2 - 4)^2}{x} - 3x}\right)\right)$$

$$x \equiv \left(3\sqrt{6}\gamma^2\sqrt{\sin^2\theta\left(2\gamma^6 - 27\gamma^4\cos(2\theta) + 3\gamma^4 + 96\gamma^2 - 128\right)}\right.$$
$$\left. +\gamma^6 + \gamma^4\left(15 - 27\cos(2\theta)\right) + 48\gamma^2 - 64\right)^{1/3}.$$

For $\gamma \leq 2$, the two solutions $\alpha_3^{(\pm)}$ minimise the energy in $\theta \in [0,\pi]$ and $\theta \in [\pi,2\pi]$,

While for $\gamma > 2$, the two solutions $\alpha_3^{(\pm)}$ can be glued together and the energy is analytic in θ.

$\theta = \pi$

For $\gamma < 2$, at $\theta = \pi$ they reduce to: $\alpha_3^{\pm}|_{\theta=\pi} = 2\sec^{-1}\left(\pm\frac{2\sqrt{2}}{\sqrt{4-\gamma^2}}\right)$

They are degenerate and related by CP ⟶ Dashen's phenomenon ✅

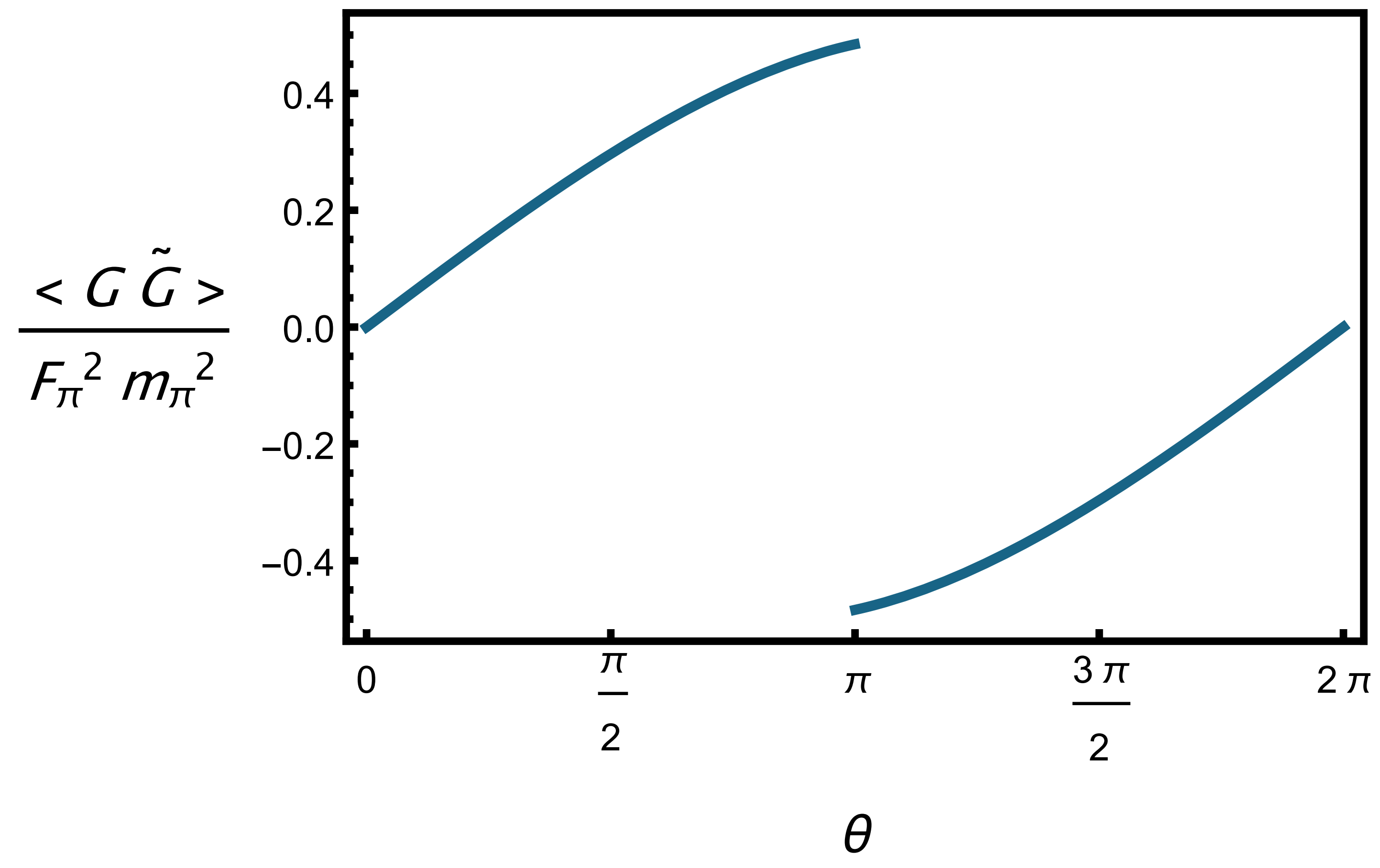
For $\gamma > 2$, at $\theta = \pi$ we have $\alpha_3 = \pi\,, \alpha_1 = \alpha_2 = 0$: the solution is unique ⟶ Dashen's phenomenon ❌

This can be seen easily the realistic limit $\gamma << 1$:

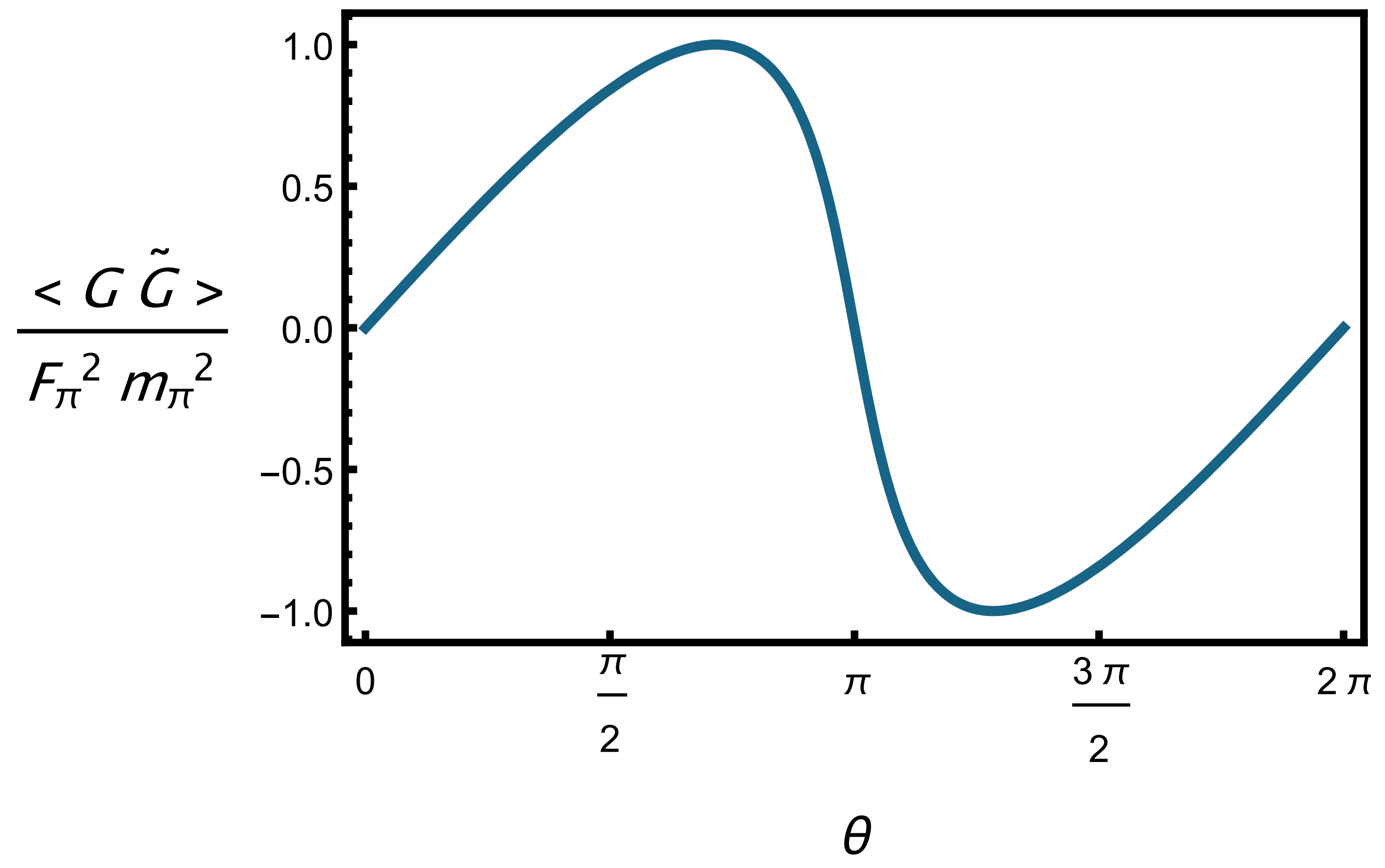
$$\alpha_1 = \alpha_2 \equiv \alpha = \frac{\theta - \alpha_3}{2}\,, \quad \alpha_3^{(\pm)} = \pm\sin\left(\frac{\theta}{2}\right)\gamma - \frac{1}{4}\sin\theta\,\gamma^2 + \mathcal{O}\left(\gamma^3\right) \quad \text{with} \quad \theta \in [0,\pi] \quad \text{or} \quad \theta \in [\pi, 2\pi]\,,$$

$$E_N^{(\pm)} = F_\pi^2(m_K^2 - m_\pi^2/2)\left(-1 \mp 2\cos(\theta/2)\,\gamma + \frac{1}{4}(\cos\theta - 1)\,\gamma^2 + \mathcal{O}\left(\gamma^3\right)\right)\,.$$

The even powers of γ are the same, while the odd ones appear with opposite coefficients that vanish at $\theta = \pi$!



The normalised CP order parameter in the Normal phase for $\gamma = 1/2$ (left) and $\gamma = 3$ (right).



Phase Diagram - Pion phase

In the Superfluid Pion Phase the solutions for the Witten variables are:

$$\alpha_1 = \alpha_2 \equiv \alpha = \frac{\theta - \alpha_3}{2}, \qquad \cos \alpha_3^{(\pm)} = \pm \frac{2\mu_I^2\left(2m_K^2 - m_\pi^2\right) + m_\pi^4 \cos\theta}{\sqrt{4\mu_I^4\left(2m_K^2 - m_\pi^2\right)^2 + 4\mu_I^2 m_\pi^4\left(2m_K^2 - m_\pi^2\right)\cos\theta + m_\pi^8}},$$

For $\mu_I > m_\pi\sqrt{\frac{\gamma}{2}(-\cos\theta)}$, the energy is minimised by α_3^+, while for $\mu_I < m_\pi\sqrt{\frac{\gamma}{2}(-\cos\theta)}$, it is minimised by α_3^-.

The energy value corresponds to

$$E_P/F_\pi^2 = -\frac{2\mu_I^4 + m_\pi^4 + \sqrt{m_\pi^8 + 4\mu_I^4(m_\pi^2 - 2m_K^2)^2 + 4\mu_I^2 m_\pi^4(2m_K^2 - m_\pi^2)\cos\theta}}{4\mu_I^2},$$

and in the $\gamma \ll 1$ it becomes:

$$E_P/F_\pi^2 = -\frac{m_\pi^2}{2\gamma} - \frac{2\mu_I^4 + m_\pi^4\cos\theta + m_\pi^4}{4\mu_I^2} - \frac{m_\pi^6 \sin^2\theta}{16\mu_I^4}\gamma + \mathcal{O}(\gamma^2)\,.$$

For generic values of μ_I it is an analytical function of θ, and the Dashen's phenomenon is absent ❌.

Novel Superfluid Pion Phase

$\theta = \pi$

At $\theta = \pi$, the solution for α_3 can be reduced to:

$$\cos \alpha_3^{(\pm)} = \pm \frac{\mu_I^2 - \mu_I^{*2}}{|\mu_I^2 - \mu_I^{*2}|} \,, \qquad \mu_I^* = m_\pi \sqrt{\frac{\gamma}{2}} = \frac{m_\pi^2}{\sqrt{2\left(2m_K^2 - m_\pi^2\right)}} \,.$$

For $\mu_I < \mu_I^*$ the vacuum is unique and given by $\alpha_3 = \pi, \alpha = 0 \longrightarrow$ Dashen's phenomenon ❌.

For $\mu_I > \mu_I^*$ there exist two degenerate minima $\alpha_3 = 0, \alpha = \pm\, \pi/2$, but the CP order parameter is 0!

$\longrightarrow$ Dashen's phenomenon ❌

At the special point $\mu_I = \mu_I^*$ the energy reduces to

$$E_P/F_\pi^2 = \frac{1}{4}\left(4\left(m_\pi^2 - 2m_K^2\right)\left|\cos\left(\frac{\theta}{2}\right)\right| + \frac{m_\pi^4}{m_\pi^2 - 2m_K^2} - 4m_K^2 + 2m_\pi^2\right),$$

and the Dashen's phenomenon occurs ✅.

How do we explain the vacuum degeneracy with no CP order parameter?

The θ-dependence of the critical value of μ_I for the onset of the Pion phase changes with γ. In fact,

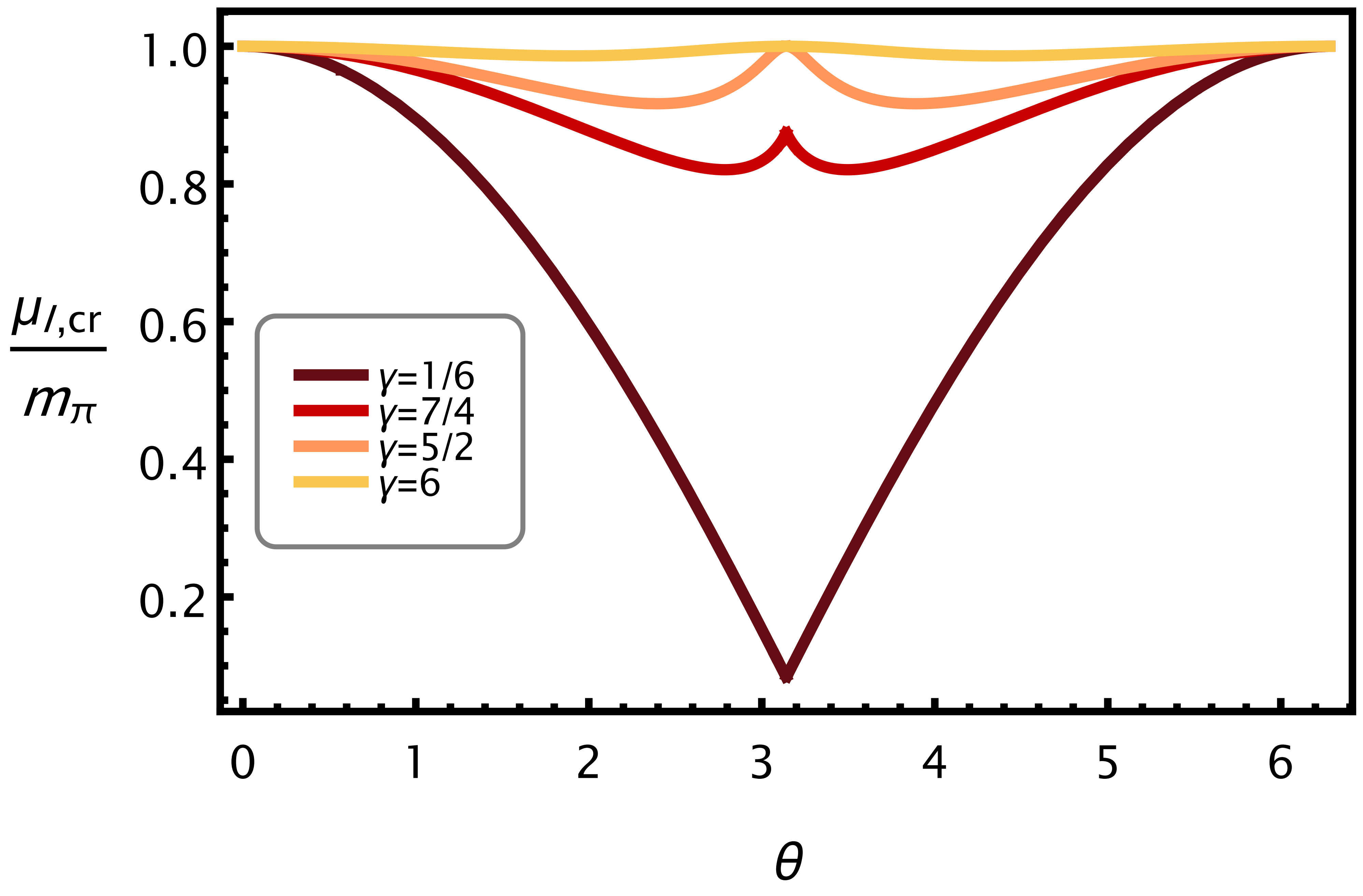
$$\mu_{I,cr}(\theta) = m_\pi \sqrt{\cos\left(\frac{\theta - \alpha_3^{(\pm)}}{2}\right)} \,,$$

where $\alpha_3^{\pm}$ are the solutions related to the Normal phase.

In the limit where the strange quark decouples from the dynamics, $\gamma \ll 1$ it simplifies to

$$\mu_{I,cr}(\theta) = m_\pi \sqrt{\cos\left(\frac{\theta}{2}\right)} \,,$$

that vanishes at $\theta = \pi$.



$\theta = \pi$

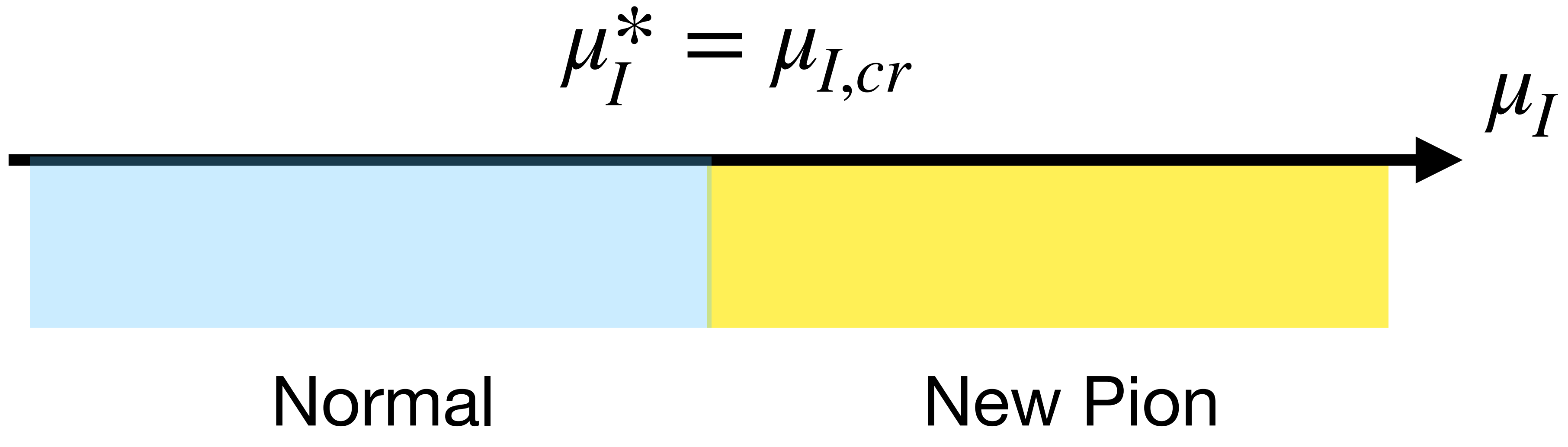
At $\theta = \pi$

$$\mu_{I,cr}(\pi) = m_\pi \sqrt{\frac{\gamma}{2}} = \mu_I^* \qquad \text{for } \gamma < 2\,, \qquad \text{and} \qquad \mu_{I,cr}(\pi) = m_\pi \qquad \text{for } \gamma > 2\,.$$

In particular, when $\gamma < \sqrt{2}$ this special value is a minimum, while for $\sqrt{2} < \gamma < 2$ is a local maximum.

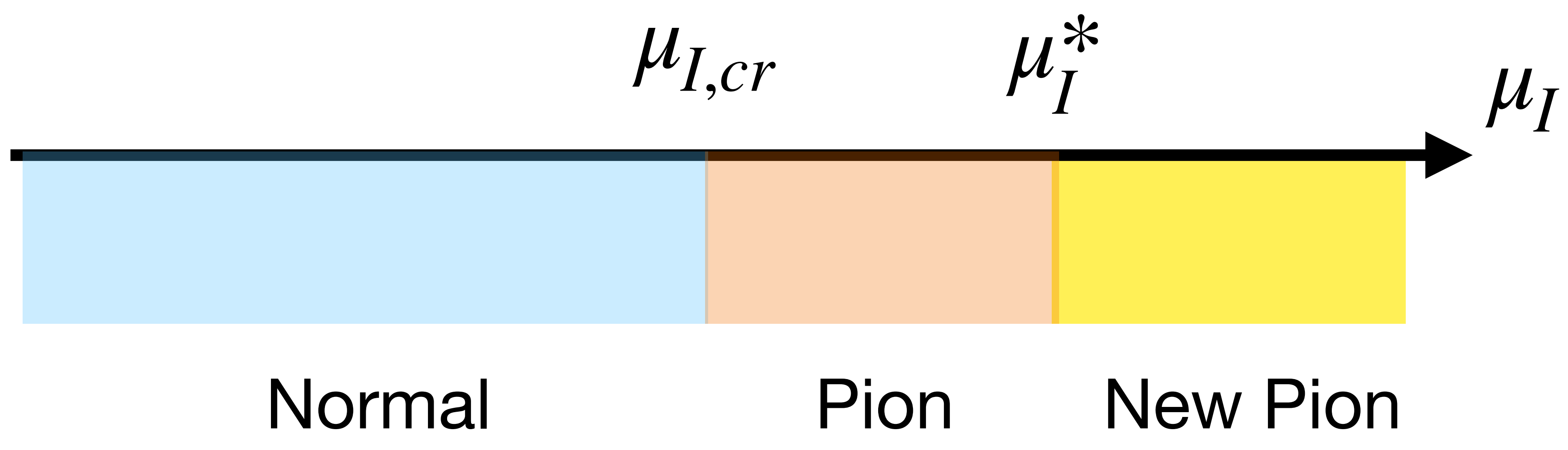
For larger values of γ the θ-dependence is progressively softer, as expected if $m_s = 0$.

We thus realise that:


$\mu_I^* = \mu_{I,cr}$ μ_I

Normal New Pion

$\gamma < 2$


$\mu_{I,cr}$ μ_I^* μ_I

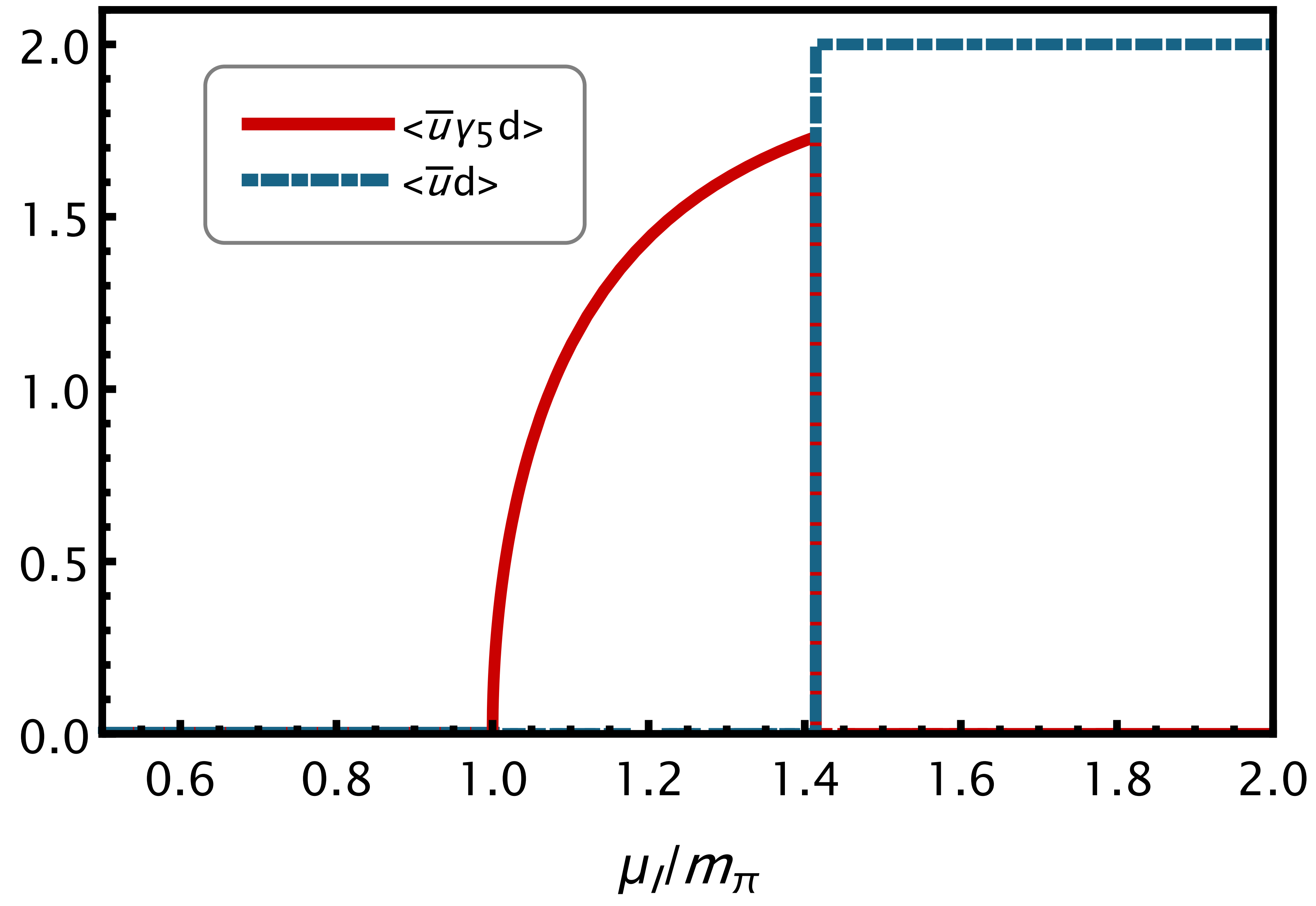
Normal Pion New Pion

$\gamma > 2$

The new pion phase is energetically favoured for $\mu_I > \mu_I^*$.
The solution for the Witten variables implies that

$$\langle \bar{u}\gamma_5 d\rangle = 2F_\pi^2 G \cos\left(\frac{\alpha_1+\alpha_2}{2}\right)\sin\varphi = 0$$

$$\langle \bar{u}d\rangle = 2F_\pi^2 G \sin\left(\frac{\alpha_1+\alpha_2}{2}\right)\sin\varphi \neq 0$$



There is a 1st order phase transition signalled by a different, parity-even order parameter!

Phase Diagram - Kaon phase

Lastly, in the Superfluid **Kaon Phase** the e.o.m.s for the Witten variables are solved by complicated expressions:

$$\cos\alpha_3 = \bar{A}_K\sqrt{A_K}\,, \qquad \cos\alpha_1 = -\sqrt{A_K/2}\,, \qquad A_K\,,\bar{A}_K = A_K\,,\bar{A}_K(\theta,\gamma,m_K,\mu_I,\mu_s)$$

The corresponding energy in the limit $\gamma << 1$ is:

$$E_K/F_\pi^2 = -\frac{m_\pi^4}{2(\mu_I+2\mu_s)^2}\frac{1}{\gamma^2} - \frac{m_\pi^4}{(\mu_I+2\mu_s)^2}\frac{1}{\gamma} - \frac{1}{8}\left(\frac{(\mu_I+2\mu_s)^4+4m_\pi^4}{(\mu_I+2\mu_s)^2}+4m_\pi^2\cos\theta\right)+$$

$$-\frac{1}{8}\sin^2\theta(\mu_I+2\mu_s)^2\gamma - \frac{1}{16}\sin^2\theta(\mu_I+2\mu_s)^2\left(\frac{\cos\theta(\mu_I+2\mu_s)^2}{m_\pi^2}+4\right)\gamma^2+\mathcal{O}\left(\gamma^3\right)$$

$\theta = \pi$

The energy is analytic in θ everywhere ⟶ Dashen ✗

Apparently, there is no kaonic counterpart to the novel pion phase.

Phase Transitions to Superfluid Phases

In the $\gamma \ll 1$ limit, the Normal to Superfluid phases transitions happen when the chemical potentials are greater than:

$$\mu_{I,cr}(\theta) = m_\pi \sqrt{\cos\left(\frac{\theta}{2}\right)}, \qquad \mu_{s,cr}^{(NK)}(\theta) = m_K - \frac{\mu_I}{2} + \frac{m_K}{2}\left(\left|\cos\left(\frac{\theta}{2}\right)\right| - 1\right)\gamma + \mathcal{O}(\gamma^2)\,.$$

2nd order PT

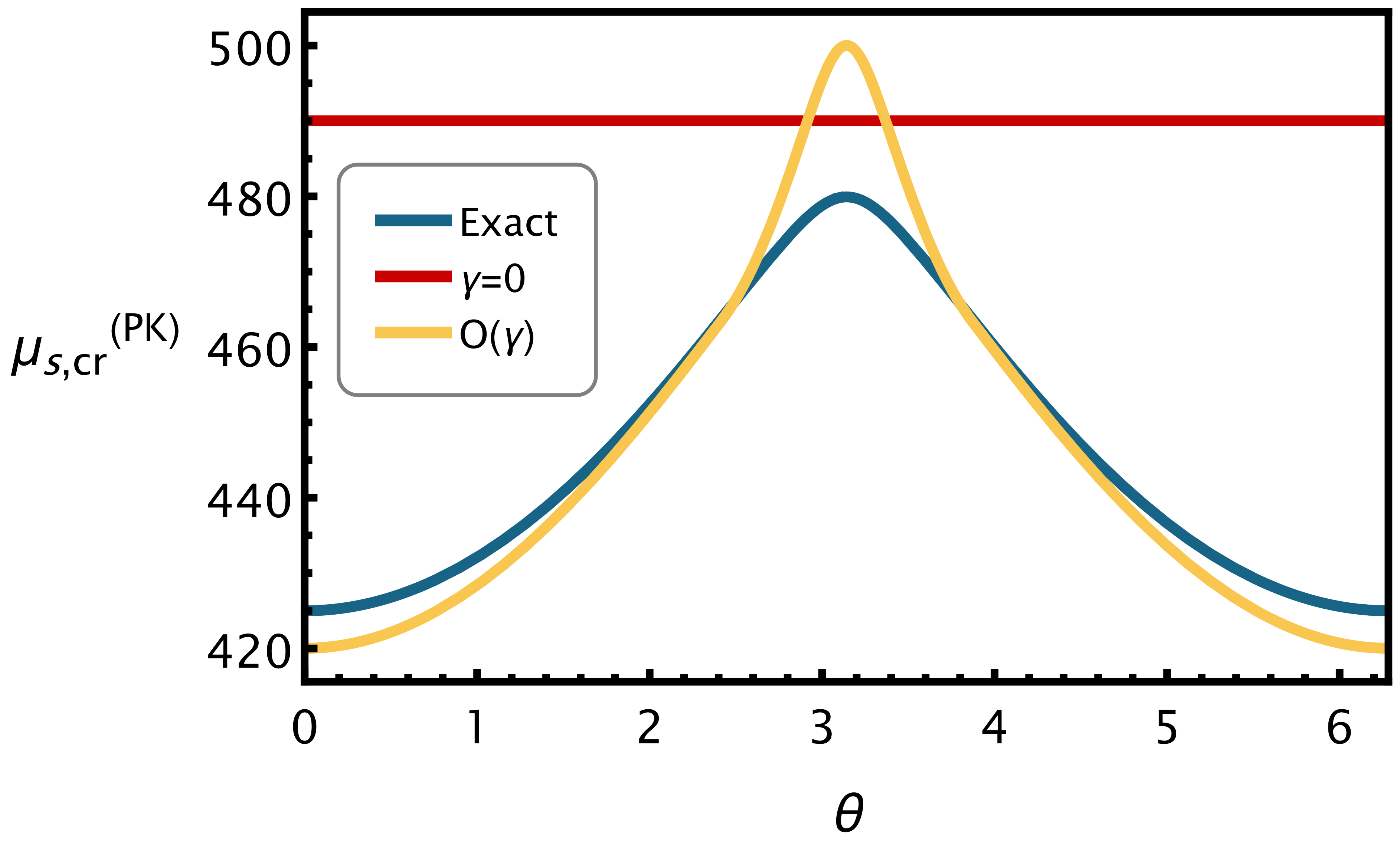
The Pion to Kaon is favoured for values greater than:

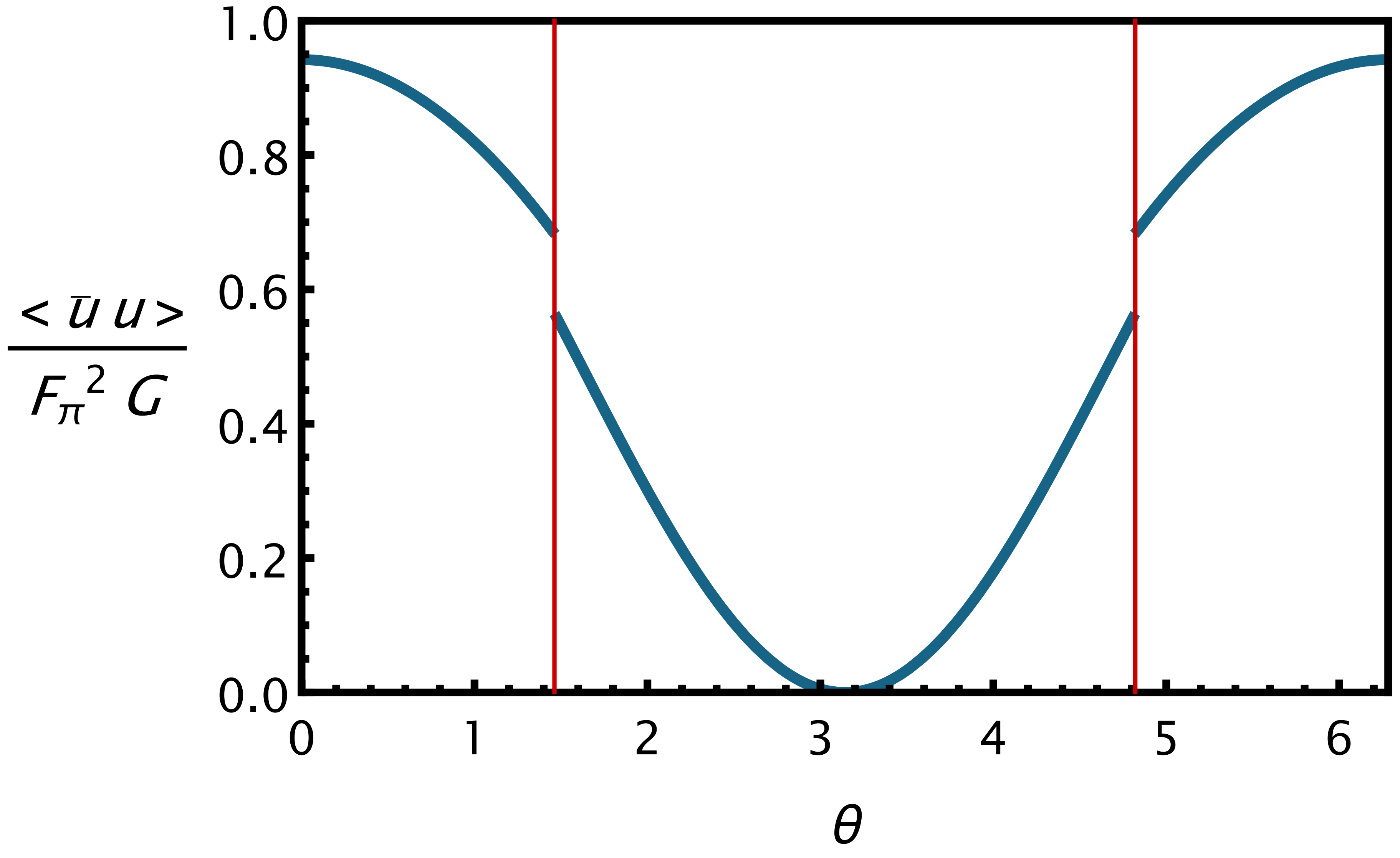
$$\mu_{s,\mathrm{cr}}^{(PK)} = \frac{c_2}{2} - \frac{8\left(2m_K^2 - m_\pi^2\right)\left(\left(c_2+\mu_I\right)^2 \sin\!\left(\operatorname{arcsec}\!\left(\sqrt{2c_1}\,\csc\theta\right) + \theta\right) + 2\sqrt{1-\frac{1}{2c_1}}\left(2m_K^2 - m_\pi^2\right)\right)}{32\mu_I\left[\mu_I(c_2+\mu_I) + 4m_K^2 - 2m_\pi^2\right]}\gamma + \mathcal{O}(\gamma^2)$$

$$c_1 = \frac{\mu_I^2 + 2(\cos\theta + 1)m_K^2 - m_\pi^2(\cos\theta + 1)}{\mu_I\left(c_2+\mu_I\right) + 2m_K^2 - m_\pi^2}, \quad c_2 = \sqrt{\mu_I^2 + 4m_K^2 - 2m_\pi^2}$$

1st order PT

Is it worth observing that in the degenerate quark mass case, this quantity does not depend on θ.





In general, the value of μ_S separating the two superfluid phases has to be determined numerically. For realistic values of the meson masses

$$m_\pi = 140\,MeV \quad \text{and} \quad m_K = 495\,MeV$$

the γ-corrections are quite large, regardless of its small value.

The behaviour of the condensate across the transition indicates that, in the case of non-degenerate masses, the transition is actually 1st order.

Condensates

Condensates	Normal phase	Pion phase	Kaon phase
$\langle \bar{u}u \rangle$	$F_\pi^2 G \cos\alpha_1$	$F_\pi^2 G \cos\alpha_1 \cos\varphi$	$F_\pi^2 G \cos\alpha_1 \cos\varphi$
$\left\langle \bar{d}d \right\rangle$	$F_\pi^2 G \cos\alpha_2$	$F_\pi^2 G \cos\alpha_2 \cos\varphi$	$F_\pi^2 G \cos\alpha_2$
$\langle \bar{s}s \rangle$	$F_\pi^2 G \cos\alpha_3$	$F_\pi^2 G \cos\alpha_3$	$F_\pi^2 G \cos\alpha_3 \cos\varphi$
$\langle \bar{u}d \rangle$	0	$2F_\pi^2 G \sin\left(\frac{\alpha_1+\alpha_2}{2}\right) \sin\varphi$	0
$\langle \bar{u}s \rangle$	0	0	$-2F_\pi^2 G \sin\left(\frac{\alpha_1+\alpha_3}{2}\right) \sin\varphi$
$\langle \bar{u}\gamma_5 u \rangle$	$-F_\pi^2 G \sin\alpha_1$	$-F_\pi^2 G \sin\alpha_1 \cos\varphi$	$-F_\pi^2 G \sin\alpha_1 \cos\varphi$
$\left\langle \bar{d}\gamma_5 d \right\rangle$	$-F_\pi^2 G \sin\alpha_2$	$-F_\pi^2 G \sin\alpha_2 \cos\varphi$	$-F_\pi^2 G \sin\alpha_2$
$\langle \bar{s}\gamma_5 s \rangle$	$-F_\pi^2 G \sin\alpha_3$	$-F_\pi^2 G \sin\alpha_3$	$-F_\pi^2 G \sin\alpha_3 \cos\varphi$
$\langle \bar{u}\gamma_5 d \rangle$	0	$2F_\pi^2 G \cos\left(\frac{\alpha_1+\alpha_2}{2}\right) \sin\varphi$	0
$\langle \bar{u}\gamma_5 s \rangle$	0	0	$-2F_\pi^2 G \cos\left(\frac{\alpha_1+\alpha_3}{2}\right) \sin\varphi$

Condensates	Pion phase	Kaon phase
$\langle \bar{u}\gamma_0 d \rangle$	$-F_\pi^2 \mu_I \sin\left(\frac{\alpha_1-\alpha_2}{2}\right) \sin(2\varphi)$	0
$\langle \bar{u}\gamma_0 s \rangle$	0	$-\frac{1}{2}F_\pi^2(\mu_I + 2\mu_s) \sin\left(\frac{\alpha_1-\alpha_3}{2}\right) \sin(2\varphi)$
$\langle \bar{u}\gamma_0\gamma_5 d \rangle$	$F_\pi^2 \mu_I \cos\left(\frac{\alpha_1-\alpha_2}{2}\right) \sin(2\varphi)$	0
$\langle \bar{u}\gamma_0\gamma_5 s \rangle$	0	$-\frac{1}{2}F_\pi^2(\mu_I + 2\mu_s) \cos\left(\frac{\alpha_1-\alpha_3}{2}\right) \sin(2\varphi)$

Normal Phase

$$\cos\varphi = 1$$

Pion Phase

$$\cos\varphi = \frac{G(m_u \cos\alpha_1 + m_d \cos\alpha_2)}{\mu_I^2}$$

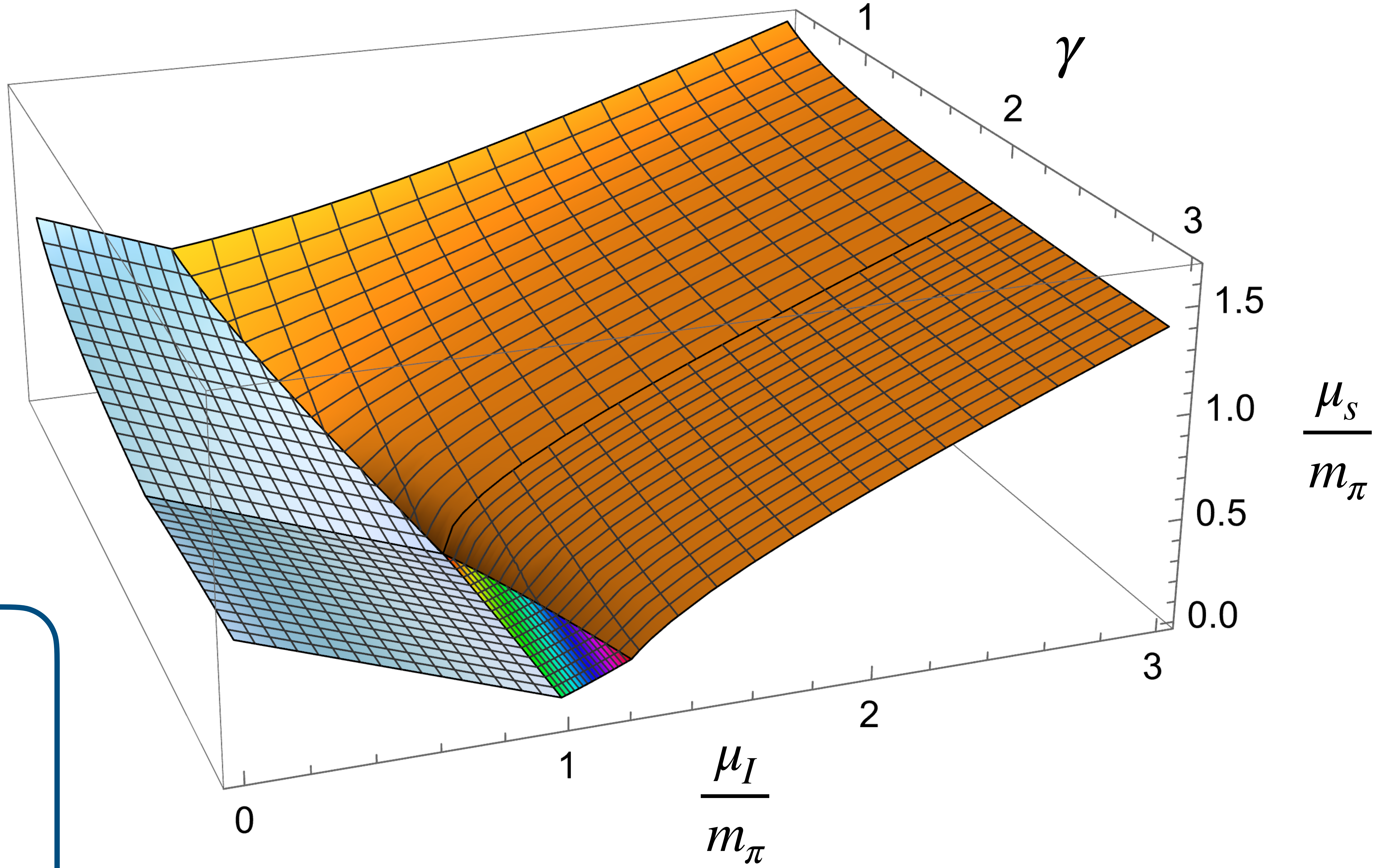
Kaon Phase

$$\cos\varphi = \frac{4G(m_u \cos\alpha_1 + m_s \cos\alpha_3)}{(\mu_I + 2\mu_s)^2}$$

Phase Diagram

Non-degenerate masses

$\theta = \pi$



Superfluid Pion Phase

- $\gamma < 2$

$\mu_I < \mu_{I,cr} = \mu_I{}^*$	Normal Phase
$\mu_I > \mu_{I,cr} = \mu_I{}^*$	Novel Pion Phase

- $\gamma > 2$

$\mu_I < \mu_{I,cr} < \mu_I{}^*$	Normal Phase
$\mu_{I,cr} < \mu_I < \mu_I{}^*$	Ordinary Pion Phase
$\mu_I > \mu_I{}^* > \mu_{I,cr}$	Novel Pion Phase

Quadruple line (NPK+Dashen)

- $\gamma < 2$ $\quad \mu_I = \dfrac{m_\pi^2}{\sqrt{4m_K^2 - 2m_\pi^2}}, \quad \mu_s = \sqrt{\dfrac{2}{2m_K^2 - m_\pi^2}}\left(m_K^2 - \dfrac{3m_\pi^2}{4}\right)$
- $\gamma \geq 2$ $\quad \mu_I = m_\pi, \quad \mu_s = \sqrt{m_\pi^2 - m_K^2} - \dfrac{m_\pi}{2},$

Spectrum

Here π_+ and K_+ are the Goldstone bosons stemming from the spontaneous breaking of $U(1)_{I/S}$!

Summary

- We determined the contribution of the topological sector to the QCD phase diagram with $N_f = 3$ at finite isospin and strangeness densities.
- We analysed the special case of $\theta = \pi$, searching for Dashen's phenomenon. For realistic values of γ, it is present in the Normal Phase, while it disappears in the Pion and Kaon phases.
- We discovered a novel pion superfluid phase in certain regions of the isospin phase diagram. Apparently, there is no counterpart of such a phase in the strangeness region.
- We completed our analysis by determining the condensates and the full low-energy spectrum.

Thank you!