Semiclassics for composite operators

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Why semiclassical?

Bohr's correspondence principle:

Behaviour of systems described by quantum theory reproduces classical physics in the limit of **large quantum numbers**

Classical theory is easier to solve

Why composite operators?

To define a CFT data we need to specify:

1.Spectrum of ***primary operators***: $(\Delta_{O_{primary}}, \rho_{primary})$

Conformal dimension

Representation

This talk

2.OPE coefficients (structure constants)

Example: scalar primaries

1) Global U(1) symmetry with (electric) charge Q

$$L = \partial_\mu \bar{\phi} \partial^\mu \phi + \frac{\lambda}{4} \left(\bar{\phi} \phi \right)^2$$

The operators $\phi^Q(x)$ and $\bar{\phi}^Q(x)$ carry U(1) charge $+Q(-Q)$

Composite operators **charged** under the symmetry

$$\phi^Q(x) = \underbrace{\phi(x) \times \phi(x) \times \phi(x) \times \ldots.}_{Q \text{ times}}$$

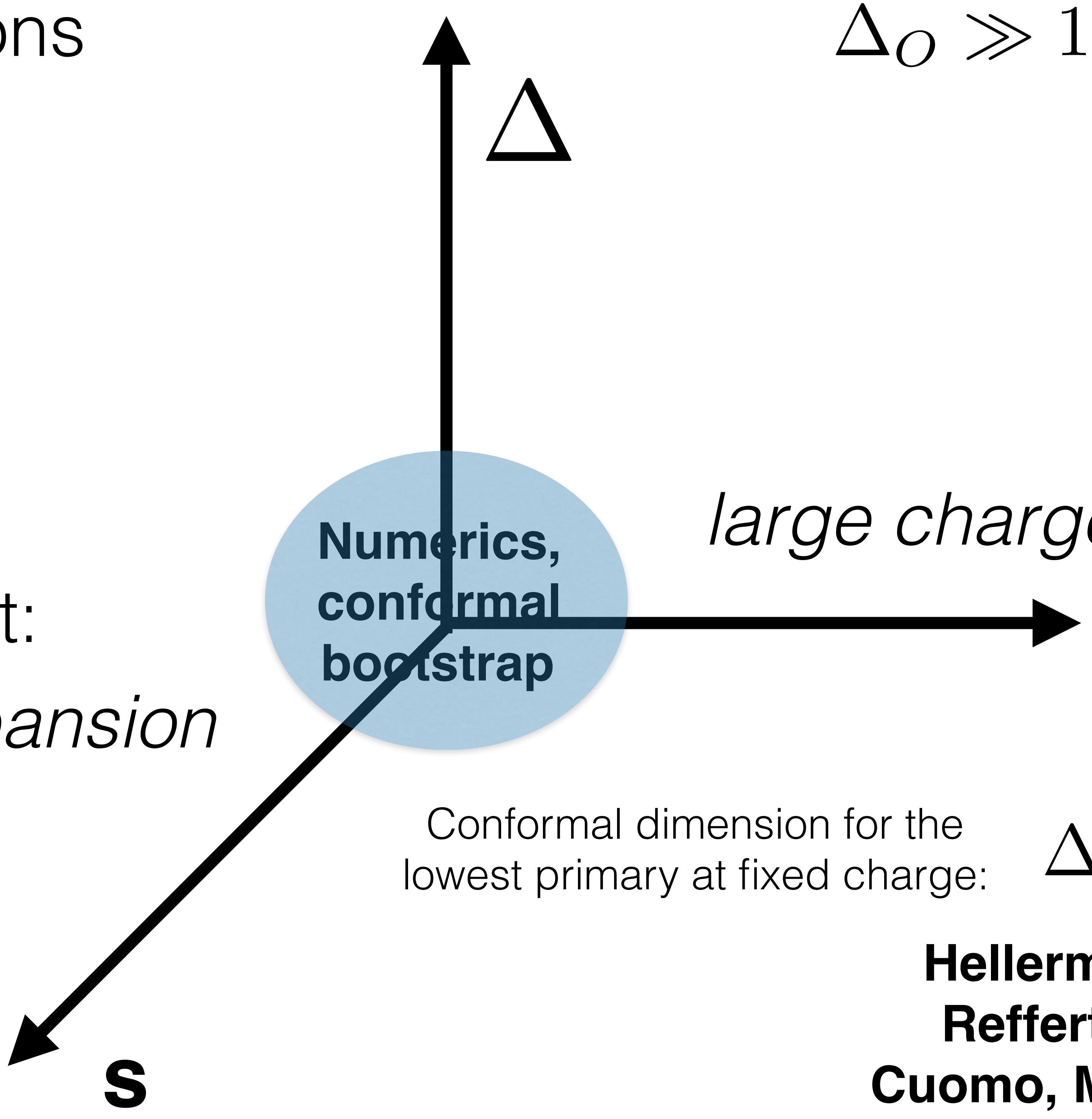
2) You can also construct **uncharged** (singlet) operators

$$\phi^n(x) = [\phi(x) \bar{\phi}(x)]^{n/2}$$

(Q,n) count number of fields in the operator and is *large*

Accessed via **1/Q** and **1/n** semiclassical expansions respectively

Large“quantum numbers“ expansions → <u>“Heavy” operators</u> $\Delta_O \gg 1$



Numerics, conformal bootstrap

large charge expansion

Regge limit:
large spin-s expansion

$$\Delta = s + \ldots$$

Komargodski, Zhiboedov, Fitzpatrick et al, …

Conformal dimension for the lowest primary at fixed charge: $\Delta \sim Q^{\frac{d}{d-1}}$

Hellerman, Orlando, Reffert, Watanabe, Cuomo, Monin, Rattazzi, Dondi, …

$$O = \partial^s \Box^p \phi^n \ :$$

This talk is about **1/n** semiclassical expansion. How it fits to the picture?

For a given model with action: $S = \int d^4x \; L(\phi, \partial\phi)$

Semiclassical expansion

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \ldots$$

Solves equation of motion
for a classical system

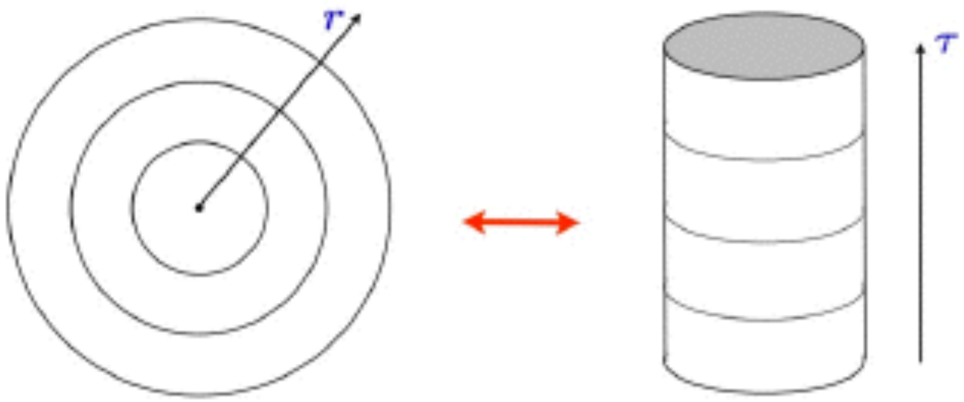
Quadratic fluctuations

• Weyl map and operator/state correspondence

Weyl map to the cylinder

Working at the fixed point we can map the theory to the cylinder

$$\mathbb{R}^d \to \mathbb{R} \times S^{d-1}, \qquad r = R e^{\tau/R}$$



The eigenvalues of the dilation charge, i.e. the scaling dimensions, become the energy spectrum on the cylinder.

$$\Delta = E\,R$$

State-operator correspondence:
States and operators are in 1-to-1 correspondence.

We need to compute energies of the states on the cylinder

Energy spectrum from poles of the resolvent

$$G(E) = Tr\ \frac{1}{H-E} = \sum_n \frac{1}{E_n - E} = i\int_0^\infty d\mathcal{T} e^{iE\mathcal{T}}\ Tr\ e^{-iH\mathcal{T}}$$

Trace of evolution kernel is a periodic path integral:

$$Tr\ e^{-iH\mathcal{T}} = \int \mathcal{D}\phi\ e^{i\mathcal{S}}$$

Semiclassical approximation: $\phi = v + \eta$

$$S[v+\eta] = S[v] + \frac{1}{2}\int d^dx d^dy\ \eta(x)\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}|_v\ \eta(y) + \mathcal{O}(\eta^3)$$

Taking Gaussian integral:

$$Tr\ e^{-iH\mathcal{T}} \approx \mathcal{N} e^{i\mathcal{S}_{\rm cl}} |\det{}' \mathcal{O}^{(2)}|^{-1/2} \qquad O^{(2)} \equiv \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}|_v$$

no zero modes!

Computing $\det'\mathcal{O}^{(2)}$:

$$\phi = v + \eta$$

Simplifying assumption: spatially homogeneous solution v(t):

$$\mathcal{O}^{(2)} = -\partial_t^2 + V(t) + \nabla^2_{S^{d-1}}$$

$$\eta(t,\Omega) = \sum_\ell \eta_\ell(t) Y_\ell(\Omega)$$

$$-\nabla^2_{S^{d-1}} Y_\ell = \Lambda(\ell) Y_\ell$$

Since the modes decouple, the determinant of the full operator factors as a product:

$$\det'\mathcal{O}^{(2)} = \prod_\ell (\det'\mathcal{O}^{(2)}_\ell)$$

where the **one-dimensional operator** for each mode is:

$$\mathcal{O}^{(2)}_\ell = -\partial_t^2 + V(t) - \Lambda(\ell)$$

Computing $\det{}'\mathcal{O}_\ell^2$

We can compute the det by considering two independent solutions of:

$$\left[-\partial_t^2 + V(t) - \Lambda(\ell)\right] \xi_{\ell,i}(t) = 0\,, \quad i = 1,2$$

Subject to boundary conditions:

$$\xi_{\ell,1}(0) = 1, \quad \xi'_{\ell,1}(0) = 0, \quad \xi_{\ell,2}(0) = 0, \quad \xi'_{\ell,2}(0) = 1\,.$$

The *stability* of the solution is described by *monodromy matrix*:

$$\begin{pmatrix} \xi_{\ell,1}(t+\mathcal{T}) \\ \xi_{\ell,2}(t+\mathcal{T}) \end{pmatrix} = \mathcal{M} \begin{pmatrix} \xi_{\ell,1}(t) \\ \xi_{\ell,2}(t) \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \xi_{\ell,1}(\mathcal{T}) & \xi'_{\ell,1}(\mathcal{T}) \\ \xi_{\ell,2}(\mathcal{T}) & \xi'_{\ell,2}(\mathcal{T}) \end{pmatrix}$$

with eigenvalues: $e^{\pm i\nu_\ell}$ known as *stability angles*

The solutions that diagonalise the monodromy matrix are *Bloch/Floquet solutions*:

$$\xi_{\ell,\pm}(t) = e^{\pm i\nu_\ell t/\mathcal{T}}\chi_{\ell,\pm}(t) \qquad \text{where } \chi_{\ell,\pm}(t+\mathcal{T}) = \chi_{\ell,\pm}(t)$$

Floquet/Bloch theorem: The eigenvalue equation for a second order differential operator for a potential with period T admits Bloch waves solution (plane wave modulated by periodic functions)

The stability angles measure the phase rotation acquired by fluctuations modes after one period. They need to be real as otherwise the solutions are growing exponentially

Charged operators: $\omega_\ell \mathcal{T} = \nu_\ell$

Bloch waves become plane waves: $\xi_{\ell,\pm}(t) = \text{const } e^{\pm i\omega_\ell t}$

Gelfand-Yaglom theorem:

$$\det{}'\mathcal{O}_\ell^{(2)} = \det(1-\mathcal{M})$$

$\mathcal{M}$ has eigenvalues $e^{\pm i\nu_\ell}$

$$\det(1-\mathcal{M}) = 1 - Tr\mathcal{M} + \det\mathcal{M} = 2 - Tr\mathcal{M}$$

$$Tr\mathcal{M} = 2cos\ \nu_\ell$$

$$\boxed{\det{}'\mathcal{O}^{(2)} = \det(1-\mathcal{M}) = 2 - 2cos\ \nu_\ell = 4\sin^2(\nu_\ell/2)}$$

$$\det{}'\mathcal{O}^{(2)} = 4\prod_{\nu_\ell>0}\sin^2(\nu_\ell/2) \qquad -4\sin^2\nu_\ell = \frac{\left(1-e^{-i\nu_\ell}\right)^2}{e^{-i\nu_\ell}}$$

$$Tr\ e^{-iH\mathcal{T}} \approx \mathcal{N}e^{i\mathcal{S}_{\rm cl}}\left|\det{}'\mathcal{O}^{(2)}\right|^{-1/2} \qquad \longrightarrow$$

$$\boxed{Tr\ e^{-iH\mathcal{T}} = \sum_{\text{periodic orbits}} e^{i\left(\mathcal{S}_{\rm cl}-\frac{1}{2}\sum_{\nu_\ell>0}\nu_\ell\right)}\Delta_1\Delta_2}$$

$$\Delta_2 = \prod_{\nu_\ell>0}(1-e^{-i\nu_\ell})^{-1} \qquad (1-e^{-i\nu_\ell})^{-1} = \sum_{q=0}e^{-iq\nu_\ell}$$

Δ_1 is contribution of zero modes

$$\boxed{G(E)\propto\int_0^\infty d\mathcal{T}\ \Delta_1\ e^{i\left(\mathcal{S}_{\rm cl}-\sum_{\nu_\ell>0}\left(\frac{1}{2}+q_\ell\right)\nu_\ell+E\,\mathcal{T}\right)}}$$

The saddle lies at: $$E = -\frac{d\mathcal{S}_{\rm cl}}{d\mathcal{T}} + \sum_{\nu_\ell>0}\left(\frac{1}{2}+q_\ell\right)\frac{d\nu_\ell}{d\mathcal{T}} \qquad (1)$$

Accounting that one can traverse the basic periodic orbit k-times we arrive at *Gutzwiller Trace Formula:*

$$G(E) \propto \sum_{k=1}^{\infty} e^{ik\left(I+\sum_{\nu_\ell>0}\left(q_\ell+\frac{1}{2}\right)\left(\mathcal{T}\frac{d\nu_\ell}{d\mathcal{T}}-\nu_\ell\right)\right)} = \frac{e^{i\left(I+\sum_{\nu_\ell>0}\left(q_\ell+\frac{1}{2}\right)\left(\mathcal{T}\frac{d\nu_\ell}{d\mathcal{T}}-\nu_\ell\right)\right)}}{1-e^{i\left(I+\sum_{\nu_\ell>0}\left(q_\ell+\frac{1}{2}\right)\left(\mathcal{T}\frac{d\nu_\ell}{d\mathcal{T}}-\nu_\ell\right)\right)}}$$

where $$I(E) = \mathcal{S}_{\rm cl} - \mathcal{T}\frac{d\mathcal{S}_{\rm cl}}{d\mathcal{T}}$$

I is the action variable.

↓ Poles occur at

$$\boxed{I(E) + \sum_{\nu_\ell>0}\left(q_\ell+\frac{1}{2}\right)\left(\mathcal{T}\frac{d\nu_\ell}{d\mathcal{T}}-\nu_\ell\right) = 2\pi n} \qquad (2)$$

This is generalisation of Bohr-Sommerfeld condition.

N.B: **Charged operators** $\omega_\ell \mathcal{T} = \nu_\ell \longrightarrow I(E) = 2\pi n$

charge-fixing condition

Renormalization:

$$G(E) \propto \int_0^\infty d\mathcal{T}\ \Delta_1\ e^{i\left(\mathcal{S}_{\rm cl}-\sum_{\nu_\ell>0}\left(\frac{1}{2}+q_\ell\right)\nu_\ell+E\,\mathcal{T}\right)}$$

$$\longrightarrow \qquad \mathcal{S}_{\rm cl}-\frac{1}{2}\sum_{\nu_\ell>0}\nu_\ell \qquad \text{has to be finite}$$

Stability angles are quantum, so solving (1) and (2) consistently to LO in them and renormalizing S_{cl} we arrive at:

$$\boxed{E_{n,q_\ell}=E_{cl}(n)+\delta E_1+\frac{1}{\mathcal{T}}\sum_{\nu_\ell>0}\left(q_\ell+\frac{1}{2}\right)\nu_\ell} \qquad \delta E_1=-\frac{\lambda_*\beta_0}{2}$$

$$I(E_{cl})=2\pi n$$

Charged operators:

$$\omega_\ell\mathcal{T}=\nu_\ell \qquad E_Q=E_{cl}(Q)+\frac{1}{2}\sum_\ell\omega_\ell$$

Template example: O(N) model

$$\mathcal{L}^{(\mathrm{cyl})} = \frac{1}{2}(\partial\phi_a)^2 - \frac{m^2}{2}\phi_a\phi_a - \frac{\lambda}{4}(\phi_a\phi_a)^2$$

$$m^2 = \left(\tfrac{d-2}{2R}\right)^2$$

stemming from the coupling to Ricci scalar

in 3D:

N=0: self-avoiding walks
N=2: superfluid helium
N=3: isotropic magnets
…

In d=4-ε there is an IR WF fixed point at :

$$\lambda_* \sim \frac{\epsilon}{N+8} + \cdots$$

Spatially homogeneous classical solution: $\phi_{1,cl} = \rho(t)$

$$\frac{d^2\rho}{dt^2} + m^2\rho + \lambda\rho^3 = 0$$

$$\rho(t) = \sqrt{n}\, x_0\, \mathrm{cn}(\omega t|\kappa)$$

time-dependent periodic solution with period: $\mathcal{T} = 4\mathcal{K}(\kappa)/\omega$

Apply the Bohr-Sommerfeld quantisation condition: $I(E) = 2\pi n$

$$I = \oint \Pi d\phi = \Omega_{d-1} R^{d-1} \int_0^{\mathcal{T}} \left(\frac{d\rho}{dt} \right)^2 dt$$

$$\Pi = \Omega_{d-1} R^{d-1} \dot{\rho} \qquad \rho(t) = \sqrt{n}\, x_0 \, \mathrm{cn}(\omega t | \kappa) \qquad \mathcal{T} = 4\mathcal{K}(\kappa)/\omega$$

$$\underbrace{2\pi^2}_{V_{S^3}} \int_0^{\mathcal{T}} \left(\frac{d\rho}{dt} \right)^2 dt = 2\pi n \quad \longrightarrow \quad \boxed{\lambda n = \frac{8\pi}{3(1-2\kappa)^{3/2}} \left[(2\kappa - 1)\mathcal{E}(\kappa) + (1-\kappa)\mathcal{K}(\kappa) \right]}$$

$\mathcal{K}(\kappa), \mathcal{E}(\kappa)$ elliptic integrals of 1st and 2nd kind

Plug the solution into the action:

$$\Delta_{-1} = R E_{cl} = V_{S^3} T_{00} = \frac{2\pi^2 \kappa (1-\kappa)}{\lambda n (1 - 2\kappa)}$$

Organizes as a
double scaling limit : $\lambda \to 0 \qquad n \to \infty \qquad \lambda n = fixed$

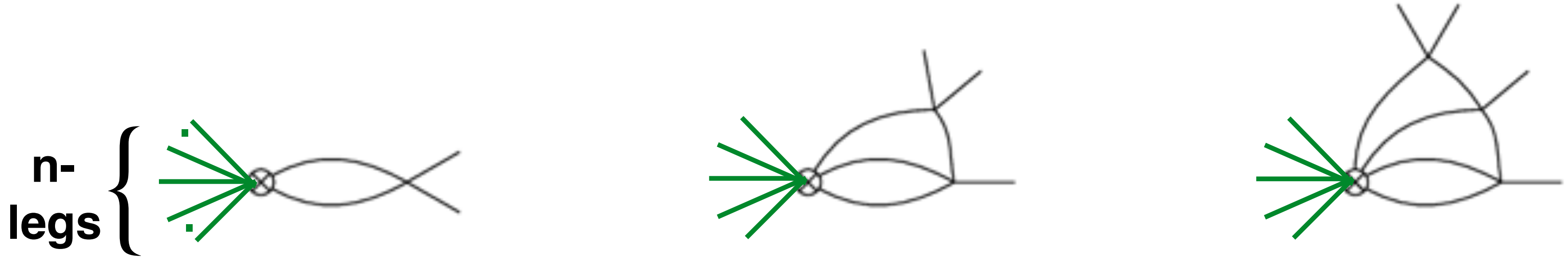
Weak coupling limit

$\lambda n \ll 1$

$$\Delta_{-1} = \frac{2\pi^2 \kappa \left(1-\kappa\right)}{\lambda n \left(1-2\kappa\right)^2}$$

$$\Delta_{-1} = n \left[1 + \frac{3\lambda n}{16\pi^2} - \frac{17\lambda^2 n^2}{256\pi^4} + \mathcal{O}\left(\lambda^3 n^3\right)\right]$$

Classical computation resums infinite number of leading Feynman diagrams



Strong coupling limit

$\lambda n \gg 1$

$$\Delta_{-1} = \frac{2\pi^2 \kappa \left(1-\kappa\right)}{\lambda n \left(1-2\kappa\right)^2}$$

$$\Delta_{-1} = \left(\frac{3\Gamma\left(\frac{3}{4}\right)}{2^{5/4}\Gamma\left(\frac{1}{4}\right)}\right)^{4/3} \lambda^{1/3} n^{4/3} + \mathcal{O}\left(n^{2/3}\lambda^{-1/3}\right)$$

Same behaviour as the large charge expansion:

$$\Delta \sim (Q,n)^{\frac{d}{d-1}}$$

Leading quantum correction:

$$\rho(t) = \sqrt{n}\, x_0\, \mathrm{cn}(\omega t|\kappa) \qquad \phi = \rho(t) + \eta(\vec{x}, t)$$

$$\mathcal{L}_2 = \sum_{a=2}^{N} \frac{1}{2}\tilde{\phi}_a \mathcal{O}_1 \tilde{\phi}_a + \frac{1}{2}\eta \mathcal{O}_2 \eta$$

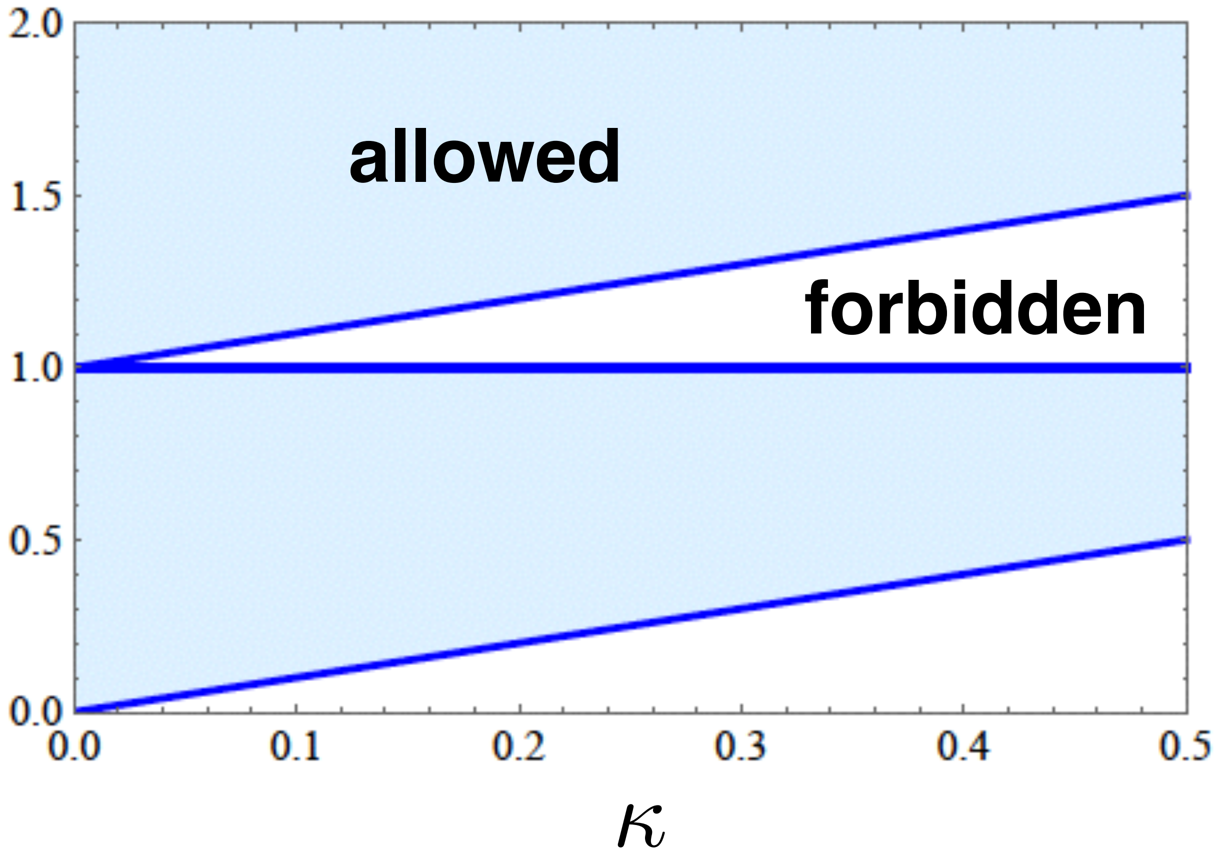
$$\mathcal{O}_j = -\partial_t^2 + \Delta_S - m^2 - \frac{j(j+1)}{2}\lambda\, \rho^2(t)\,, \qquad j = 1, 2$$

$$\mathcal{O}_j = \frac{m^2}{1-2\kappa} L_j\,, \qquad j = 1, 2$$

Lame operator: $L_j = -\partial_z^2 + j(j+1)\ \kappa\ sn(z|\kappa)^2 - \Lambda_j(\ell)$

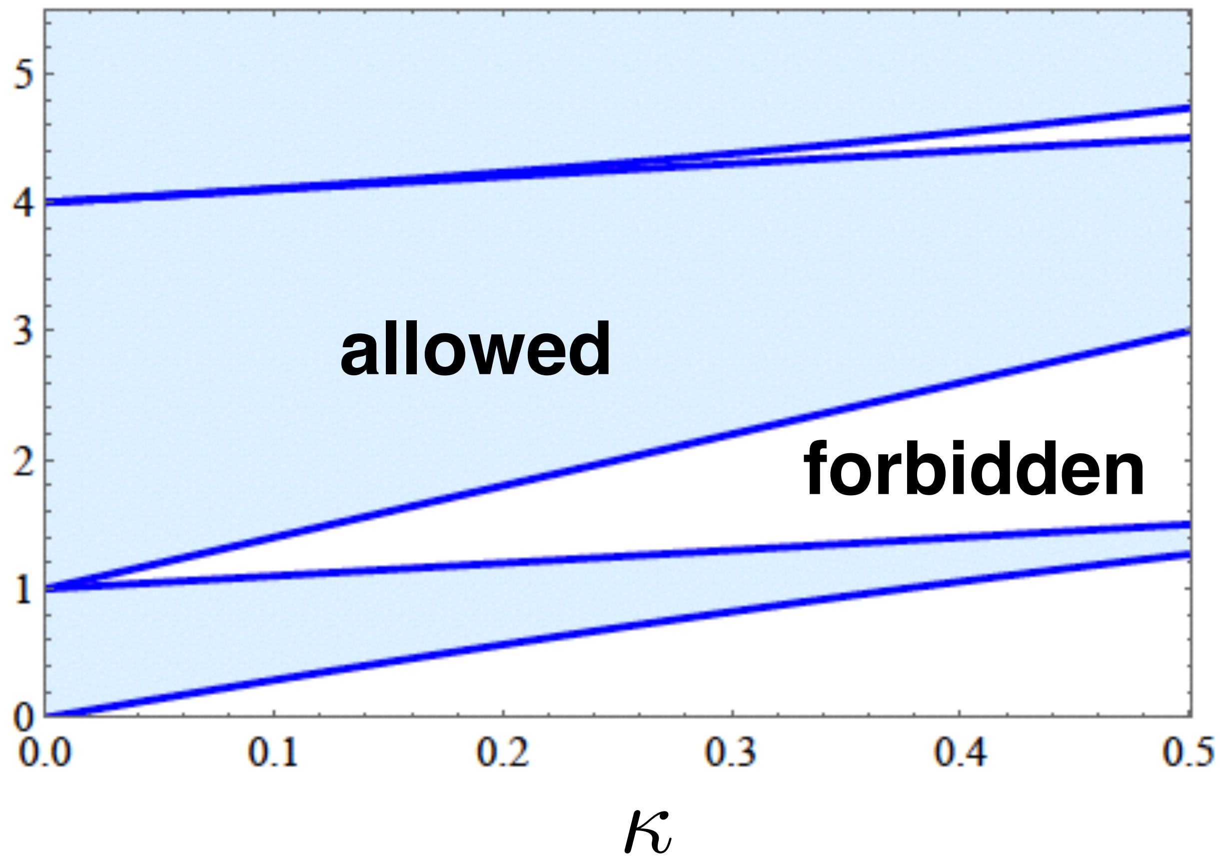
Lame operator: $L_j \xi_{\ell,\pm}(z) = 0$

j=1



(N-1) transverse modes

j=2



Longitudinal mode

Bloch solutions and stability angles of the Lame potential

$$\xi_{\ell,\pm}(z) = \prod_{I=1}^{j} \frac{H(z \pm \alpha_i \,|\, \kappa)}{\Theta(z \,|\, \kappa)} e^{\mp z Z(\alpha_i \,|\, \kappa)}$$

$$\nu_{j,\ell} = \pm 4iK(\kappa) \sum_{i=1}^{j} Z(\alpha_i \,|\, \kappa)$$

parameters α_i solve a set of j transcendental equations

$$E_{n,q_\ell} = \delta E_1 + \frac{1}{2\mathcal{T}} \sum_{\ell=0}^{\infty} (n_\ell \left[(N-1)\nu_{1,\ell} + \nu_{2,\ell}\right] + 2q_{1,\ell}\nu_{1,\ell} + 2q_{2,\ell}\nu_{2,\ell})$$

Neutral operators **perturbative LO+NLO** result (N=1)

$$\Delta_{n,\{q_\ell\}} = n + \sum_{\ell=1}^{\infty} q_\ell \ell + \frac{1}{6}\left[n^2 - 2\left(2+\sum_{\ell=1}^{\infty}\frac{(\ell-1)q_\ell}{\ell+1}\right)n + O(n^0)\right]\epsilon$$

$$-\frac{1}{324}\left[17n^3 - \left(67+3\sum_{\ell=1}^{\infty}\frac{(\ell-1)\left(17\ell^4+78\ell^3+135\ell^2+98\ell+12\right)q_\ell}{\ell(\ell+1)^3(\ell+2)}\right)n^2 + O(n,n^0)\right]\epsilon^2 + O(\epsilon^3)$$

ℓ=1 are descendants

Exciting mode-ℓ adds ℓ derivatives to the operator in the ($\ell/2$, $\ell/2$) reps of SO(3,1)

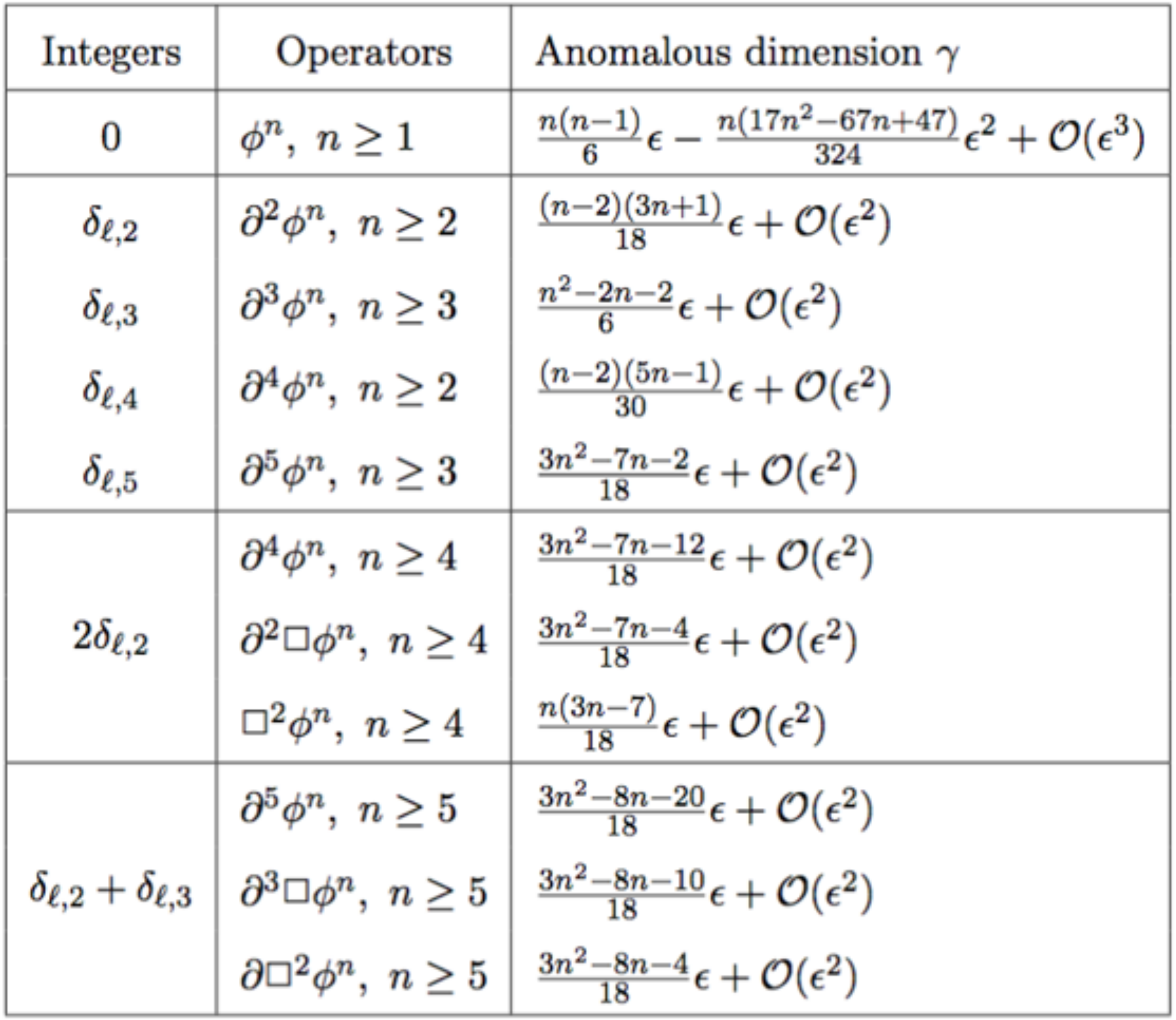
Exciting ℓ=2 mode twice

$4 \oplus 2 \oplus 0$

Exciting ℓ=2 and ℓ=3 mode once

$5 \oplus 3 \oplus 1$

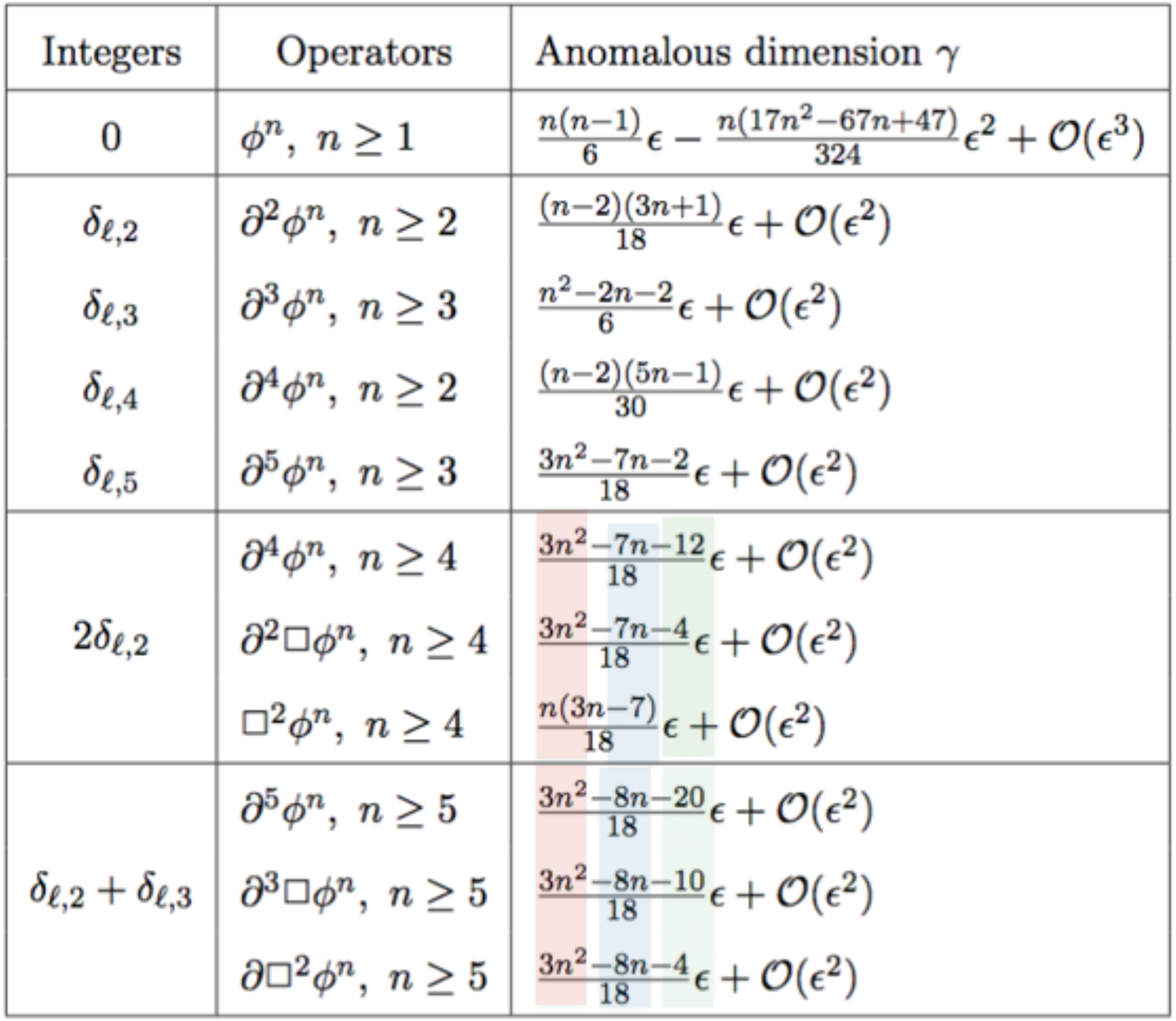
q_l



Integers	Operators	Anomalous dimension γ
0	$\phi^n,\ n \geq 1$	$\frac{n(n-1)}{6}\epsilon - \frac{n(17n^2-67n+47)}{324}\epsilon^2 + \mathcal{O}(\epsilon^3)$
$\delta_{\ell,2}$	$\partial^2\phi^n,\ n \geq 2$	$\frac{(n-2)(3n+1)}{18}\epsilon + \mathcal{O}(\epsilon^2)$
$\delta_{\ell,3}$	$\partial^3\phi^n,\ n \geq 3$	$\frac{n^2-2n-2}{6}\epsilon + \mathcal{O}(\epsilon^2)$
$\delta_{\ell,4}$	$\partial^4\phi^n,\ n \geq 2$	$\frac{(n-2)(5n-1)}{30}\epsilon + \mathcal{O}(\epsilon^2)$
$\delta_{\ell,5}$	$\partial^5\phi^n,\ n \geq 3$	$\frac{3n^2-7n-2}{18}\epsilon + \mathcal{O}(\epsilon^2)$
$2\delta_{\ell,2}$	$\partial^4\phi^n,\ n \geq 4$	$\frac{3n^2-7n-12}{18}\epsilon + \mathcal{O}(\epsilon^2)$
	$\partial^2\Box\phi^n,\ n \geq 4$	$\frac{3n^2-7n-4}{18}\epsilon + \mathcal{O}(\epsilon^2)$
	$\Box^2\phi^n,\ n \geq 4$	$\frac{n(3n-7)}{18}\epsilon + \mathcal{O}(\epsilon^2)$
$\delta_{\ell,2}+\delta_{\ell,3}$	$\partial^5\phi^n,\ n \geq 5$	$\frac{3n^2-8n-20}{18}\epsilon + \mathcal{O}(\epsilon^2)$
	$\partial^3\Box\phi^n,\ n \geq 5$	$\frac{3n^2-8n-10}{18}\epsilon + \mathcal{O}(\epsilon^2)$
	$\partial\Box^2\phi^n,\ n \geq 5$	$\frac{3n^2-8n-4}{18}\epsilon + \mathcal{O}(\epsilon^2)$

one-loop perturbative results

Spectroscopy of composite operators



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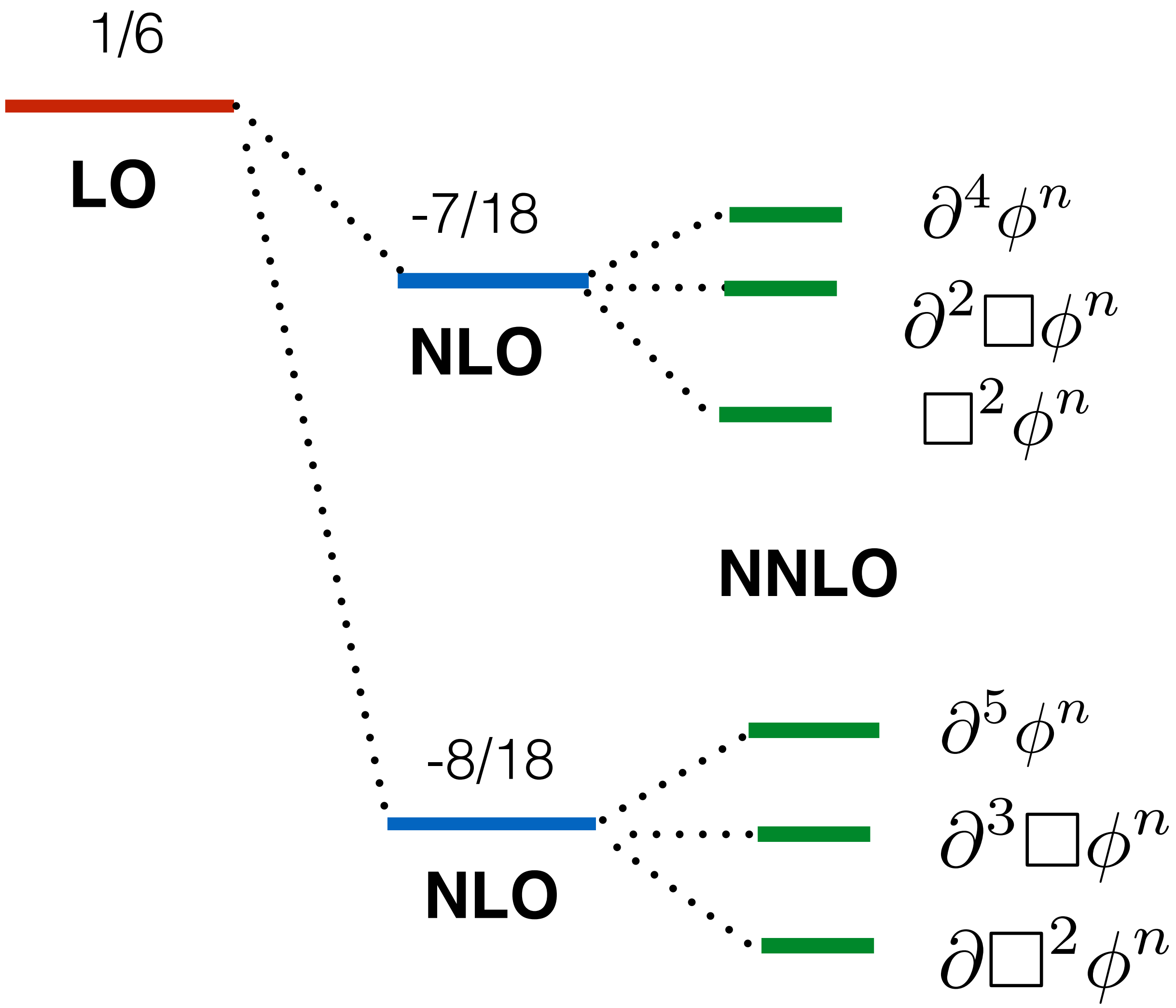
NNLO in semiclassics and so beyond our computation



Neutral operators to NLO

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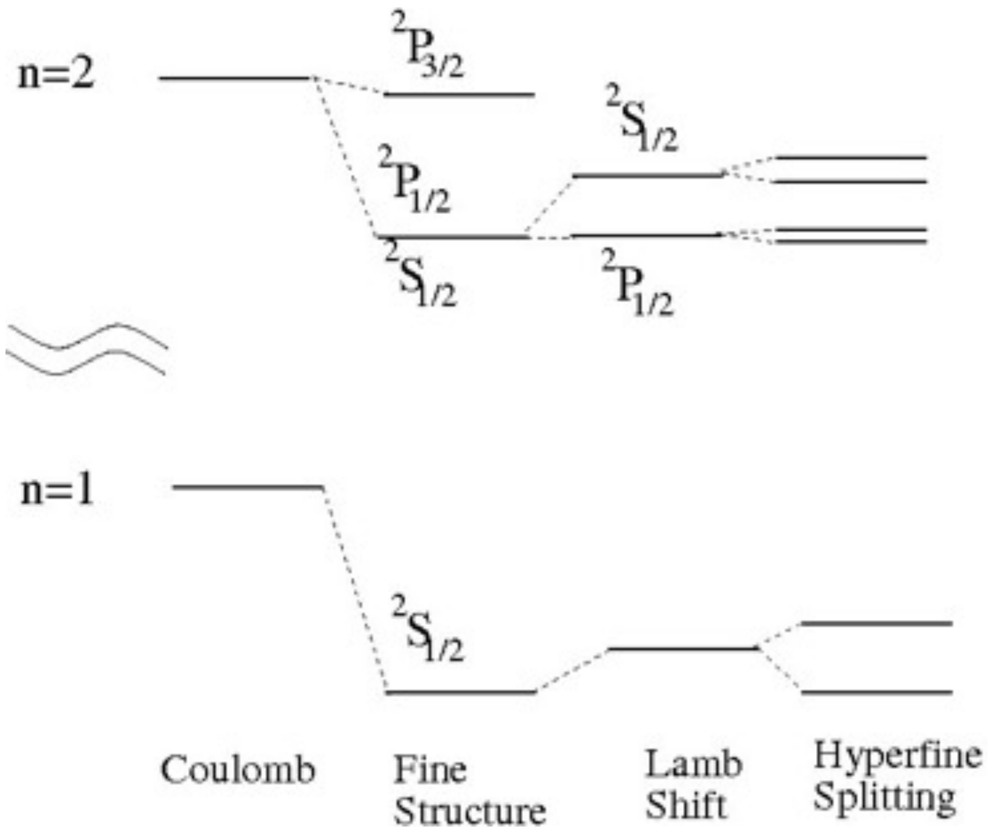
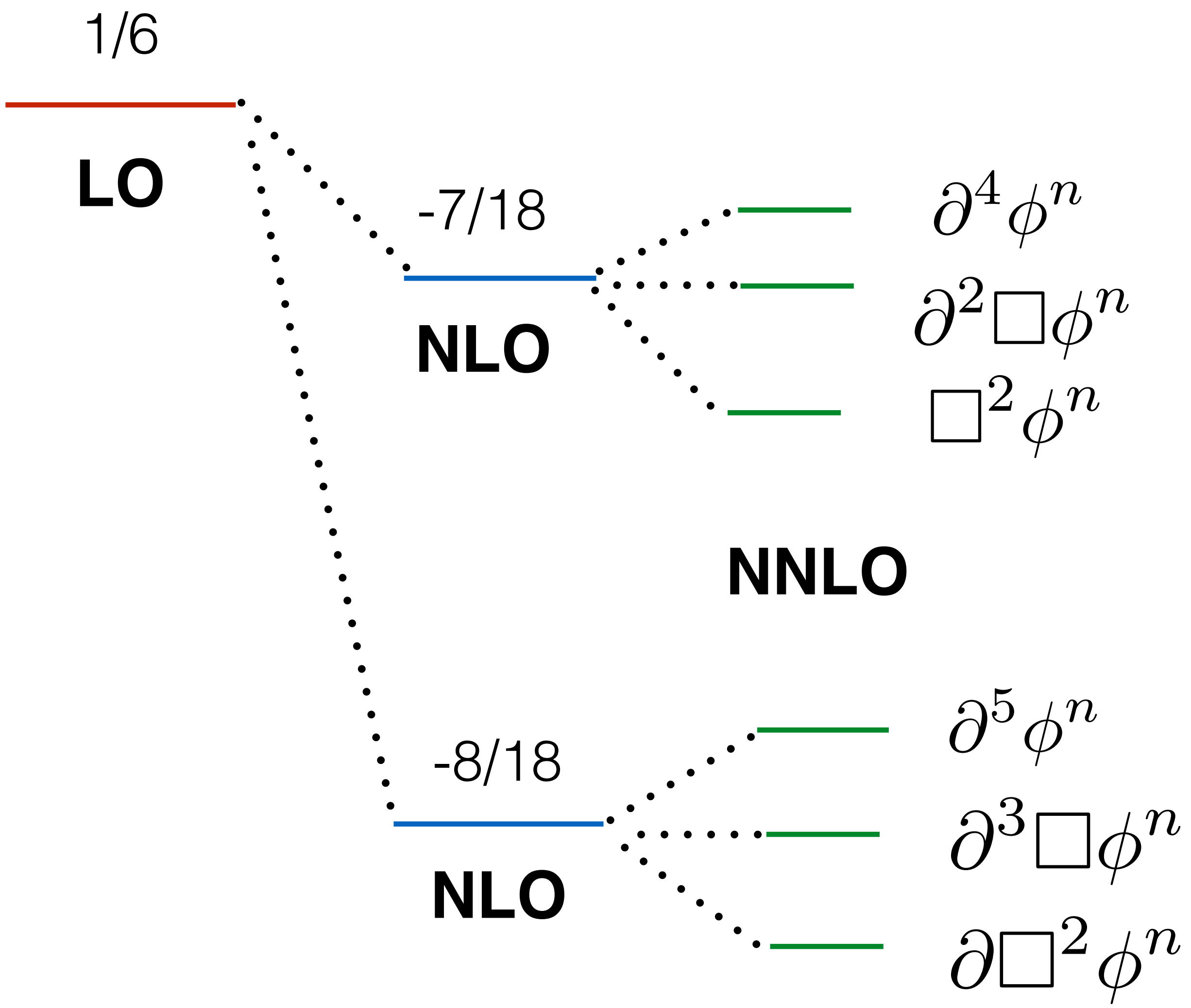
$$-\frac{1}{324}\left[17n^3 - \left(67 + 3\sum_{\ell=1}^{\infty} \frac{(\ell-1)\left(17\ell^4+78\ell^3+135\ell^2+98\ell+12\right)q_\ell}{\ell(\ell+1)^3(\ell+2)}\right)n^2 + O(n,n^0)\right]\epsilon^2 + O(\epsilon^3)$$



hydrogen atom



1/6

LO

-7/18

NLO

$\partial^4\phi^n$

$\partial^2\Box\phi^n$

$\Box^2\phi^n$

NNLO

-8/18

NLO

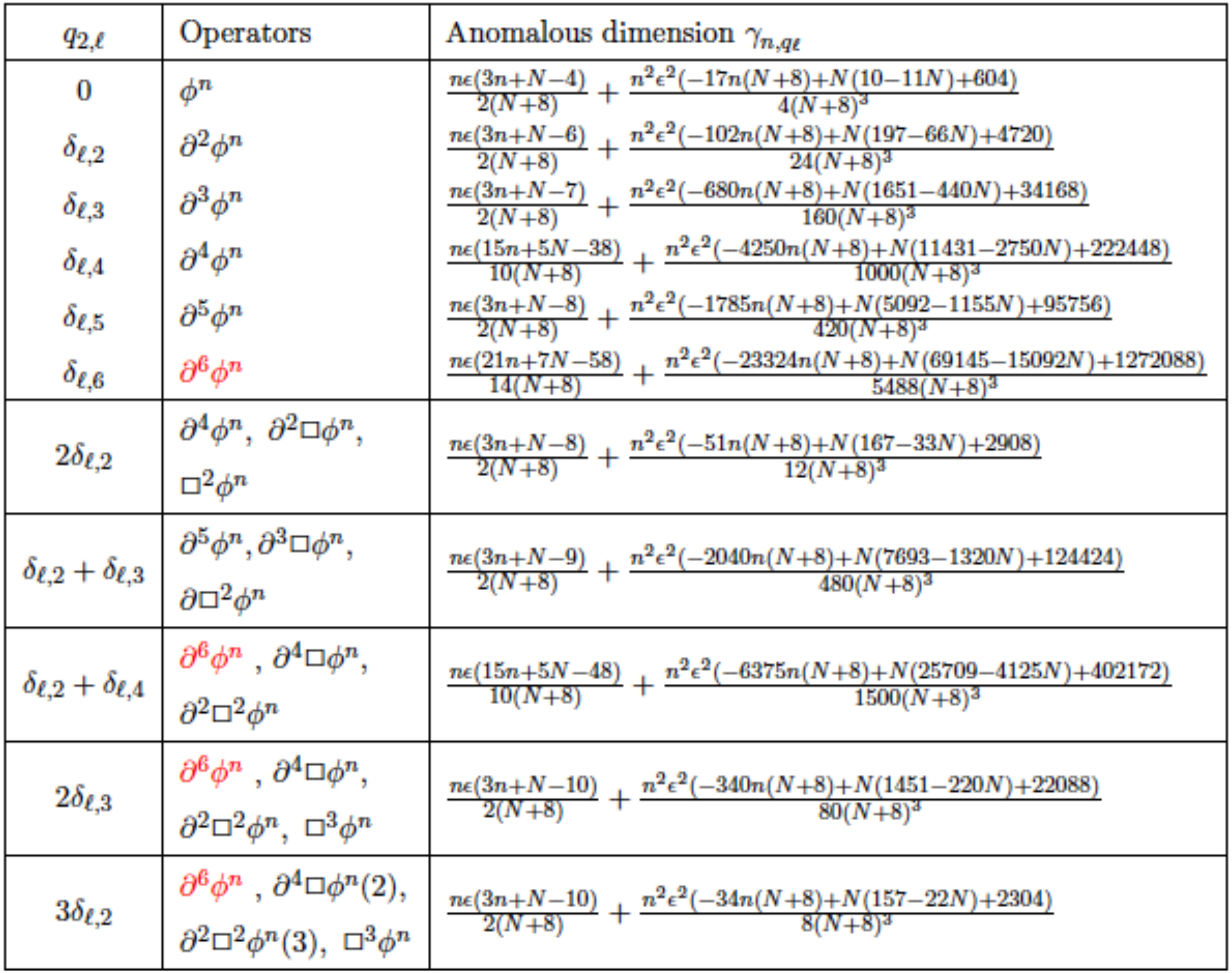
$\partial^5\phi^n$

$\partial^3\Box\phi^n$

$\partial\Box^2\phi^n$

Mixing of the composites

At spin-6 level and to 1-loop, for example, there are **four operators**. Three of them mix via degree-3 polynomial and one remains rational. To compare with us, find the eigenvalues of 3x3 matrix and expand them at large n. The LO and NLO terms will agree with semiclassics.



$q_{2,\ell}$	Operators	Anomalous dimension $\gamma_{n,q\ell}$
0	ϕ^n	$\frac{n\epsilon(3n+N-4)}{2(N+8)}+\frac{n^2\epsilon^2(-17n(N+8)+N(10-11N)+604)}{4(N+8)^3}$
$\delta_{\ell,2}$	$\partial^2\phi^n$	$\frac{n\epsilon(3n+N-6)}{2(N+8)}+\frac{n^2\epsilon^2(-102n(N+8)+N(197-66N)+4720)}{24(N+8)^3}$
$\delta_{\ell,3}$	$\partial^3\phi^n$	$\frac{n\epsilon(3n+N-7)}{2(N+8)}+\frac{n^2\epsilon^2(-680n(N+8)+N(1651-440N)+34168)}{160(N+8)^3}$
$\delta_{\ell,4}$	$\partial^4\phi^n$	$\frac{n\epsilon(15n+5N-38)}{10(N+8)}+\frac{n^2\epsilon^2(-4250n(N+8)+N(11431-2750N)+222448)}{1000(N+8)^3}$
$\delta_{\ell,5}$	$\partial^5\phi^n$	$\frac{n\epsilon(3n+N-8)}{2(N+8)}+\frac{n^2\epsilon^2(-1785n(N+8)+N(5092-1155N)+95756)}{420(N+8)^3}$
$\delta_{\ell,6}$	$\partial^6\phi^n$	$\frac{n\epsilon(21n+7N-58)}{14(N+8)}+\frac{n^2\epsilon^2(-23324n(N+8)+N(69145-15092N)+1272088)}{5488(N+8)^3}$
$2\delta_{\ell,2}$	$\partial^4\phi^n,\ \partial^2\Box\phi^n,$ $\Box^2\phi^n$	$\frac{n\epsilon(3n+N-8)}{2(N+8)}+\frac{n^2\epsilon^2(-51n(N+8)+N(167-33N)+2908)}{12(N+8)^3}$
$\delta_{\ell,2}+\delta_{\ell,3}$	$\partial^5\phi^n,\partial^3\Box\phi^n,$ $\partial\Box^2\phi^n$	$\frac{n\epsilon(3n+N-9)}{2(N+8)}+\frac{n^2\epsilon^2(-2040n(N+8)+N(7693-1320N)+124424)}{480(N+8)^3}$
$\delta_{\ell,2}+\delta_{\ell,4}$	$\partial^6\phi^n$, $\partial^4\Box\phi^n,$ $\partial^2\Box^2\phi^n$	$\frac{n\epsilon(15n+5N-48)}{10(N+8)}+\frac{n^2\epsilon^2(-6375n(N+8)+N(25709-4125N)+402172)}{1500(N+8)^3}$
$2\delta_{\ell,3}$	$\partial^6\phi^n$, $\partial^4\Box\phi^n,$ $\partial^2\Box^2\phi^n,\ \Box^3\phi^n$	$\frac{n\epsilon(3n+N-10)}{2(N+8)}+\frac{n^2\epsilon^2(-340n(N+8)+N(1451-220N)+22088)}{80(N+8)^3}$
$3\delta_{\ell,2}$	$\partial^6\phi^n$, $\partial^4\Box\phi^n(2),$ $\partial^2\Box^2\phi^n(3),\ \Box^3\phi^n$	$\frac{n\epsilon(3n+N-10)}{2(N+8)}+\frac{n^2\epsilon^2(-34n(N+8)+N(157-22N)+2304)}{8(N+8)^3}$

Instabilities at strong coupling

$$\Delta_n = \frac{1}{\epsilon}\left[\left(\frac{\sqrt{\frac{3\pi}{2}}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}n\epsilon\right)^{\frac{4-\epsilon}{3-\epsilon}}\left(\underset{\substack{\Big\downarrow\\ \textbf{LO}}}{1}+\underset{\substack{\Big\downarrow\\ \textbf{NLO}}}{\alpha\epsilon}+\mathcal{O}\epsilon^2\right)+\mathcal{O}n^{\frac{2-\epsilon}{3-\epsilon}}\right]$$

$$\alpha = \frac{1}{18}\left(6\gamma_E - 9 + 3\log\frac{4}{3} + 32\int_0^\infty dk\left(\tilde{\Sigma}(k) - \frac{3}{16k(1+k^2)}\right)\right)$$

The integrand has a branch cut leading to complex energies and signaling instability

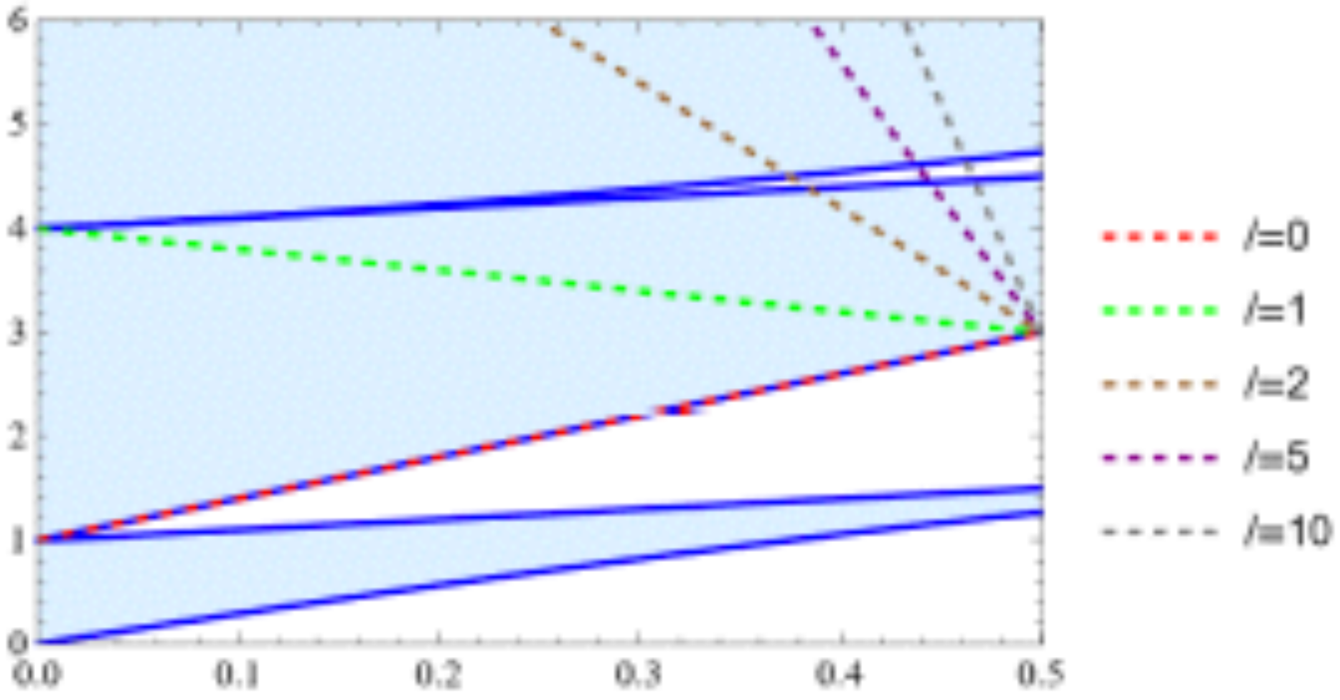
The stability angles are complex for any integer ℓ in the range:

$$\sqrt{\frac{4-5\kappa}{1-2\kappa}} \leq \ell + 1 \leq \sqrt{2}\sqrt{1+\frac{\sqrt{1-\kappa+\kappa^2}}{1-2\kappa}}$$

$$L_j = -\partial_z^2 + j(j+1)\ \kappa\ sn(z|\kappa)^2 - \Lambda_j(\ell)$$

$$\Lambda_2(\ell) = 6\kappa + (1-2\kappa)\left(1+\frac{2\ell}{d-2}\right)^2$$

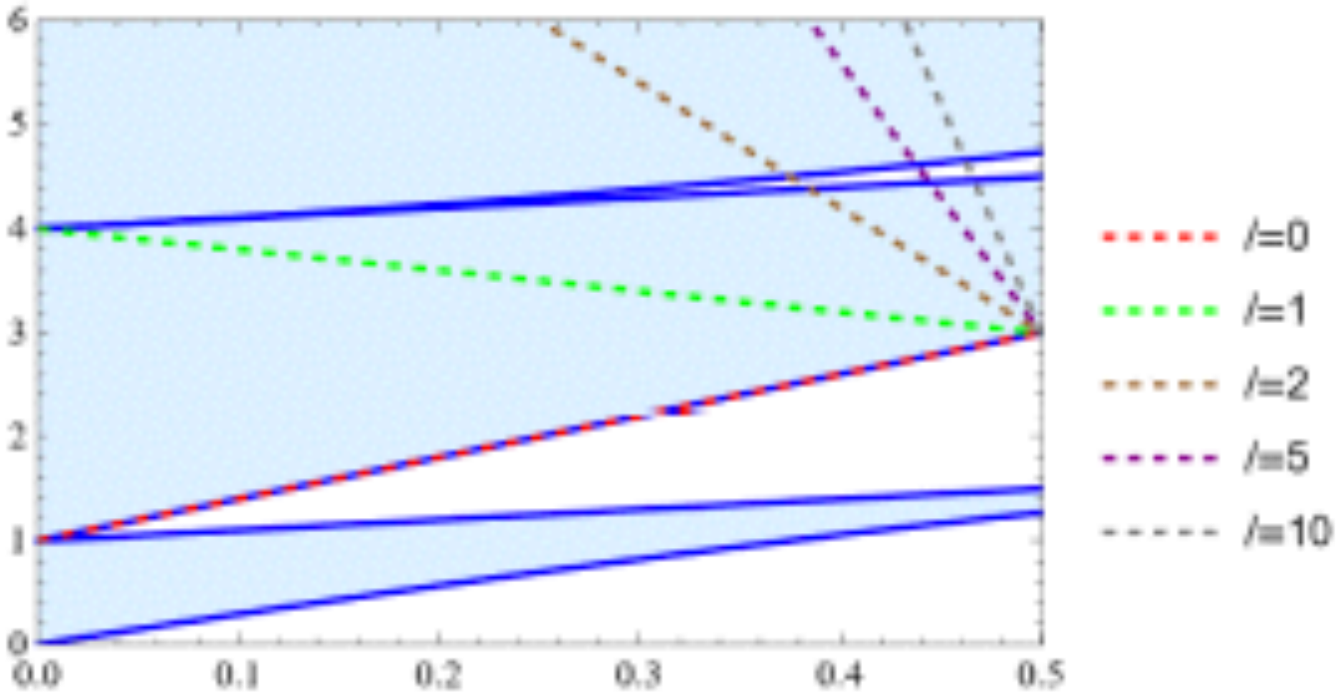
Here the spectral parameter $\Lambda_2(\ell)$ takes values inside the forbidden zones (gap) of the Lame operator L_2 and the saddle point becomes unstable



$$\sqrt{\frac{4-5\kappa}{1-2\kappa}} \leq \ell+1 \leq \sqrt{2}\sqrt{1+\frac{\sqrt{1-\kappa+\kappa^2}}{1-2\kappa}}$$

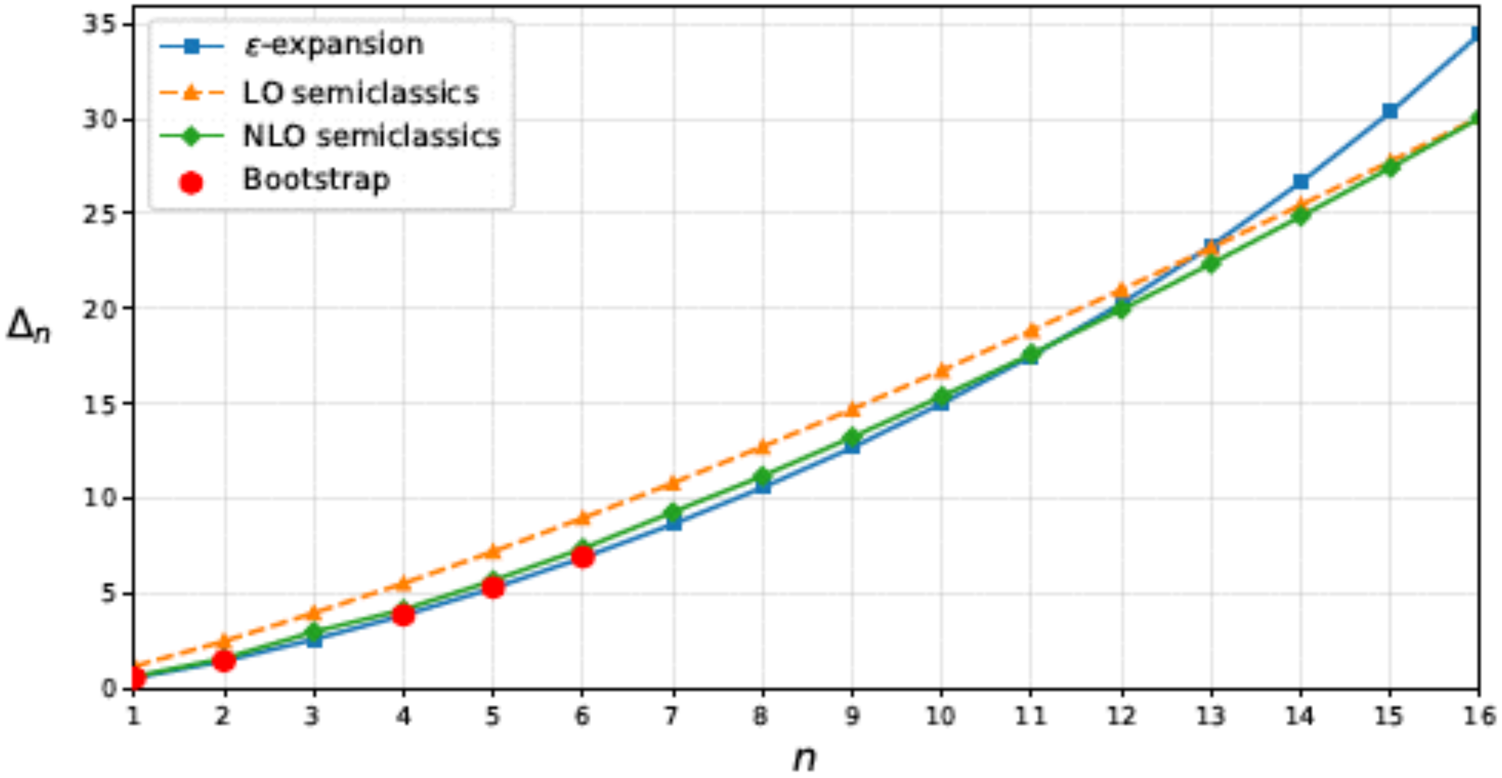
The values of $\kappa=\kappa(\lambda n)$ for which there is at least one complex stability angles display the following pattern:

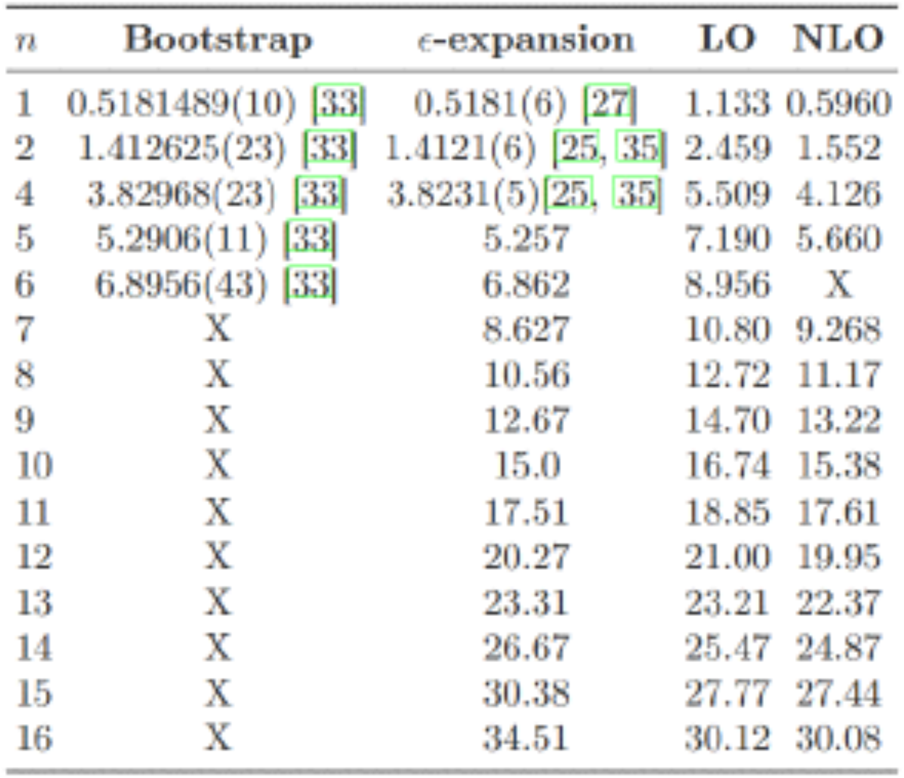
For $\lambda n \rightarrow \infty$, the stability angles form a continuum and the complex ones occur in a certain interval



3d Ising CFT

We take ε→1 and compare the results for Φ^n to bootstrap and the 5-loop ε-expansion:





n	**Bootstrap**	**ϵ-expansion**	**LO**	**NLO**
1	0.5181489(10) [33]	0.5181(6) [27]	1.133	0.5960
2	1.412625(23) [33]	1.4121(6) [25, 35]	2.459	1.552
4	3.82968(23) [33]	3.8231(5)[25, 35]	5.509	4.126
5	5.2906(11) [33]	5.257	7.190	5.660
6	6.8956(43) [33]	6.862	8.956	X
7	X	8.627	10.80	9.268
8	X	10.56	12.72	11.17
9	X	12.67	14.70	13.22
10	X	15.0	16.74	15.38
11	X	17.51	18.85	17.61
12	X	20.27	21.00	19.95
13	X	23.31	23.21	22.37
14	X	26.67	25.47	24.87
15	X	30.38	27.77	27.44
16	X	34.51	30.12	30.08

We observe the expected convergence as n increases.
Moreover, for n>11, the semiclassical LO is closer to the ε-expansion than the NLO.
For n~10 the NLO semiclassical expansion supersede the 5-loop ε-expansion becoming *de facto* the state-of-the-art.

QFT applications

EFT: Higgs SMEFT

$$L_H = \partial_\mu H^\dagger \partial^\mu H - \lambda_H (HH^\dagger)^2 + \sum_i c_i O_i$$

$$\mu \frac{d}{d\mu} c_i \sim \gamma_{c_i}$$

Requires knowledge of anomalous dimensions for the operators.
Our results apply outside of CFT

uncharged operators

$$\phi^n(x) = [\phi(x)\bar{\phi}(x)]^{n/2}$$

for n=2: Higgs mass $HH^\dagger$

for n=4: Higgs quartic $(HH^\dagger)^2$

for n=6: Higgs sextic $(HH^\dagger)^3$

……

Example 2: O(N) sextic model in d=3-ε

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi_a\right)^2 - \frac{\lambda^2}{6}(\phi_a\phi_a)^3\,, \qquad a = 1,\ldots,N$$

Fixed point: $$\frac{\lambda_*^2}{4\pi^2} = \frac{\epsilon}{3N+22} + \mathcal{O}\left(\epsilon^2\right)$$

Spatially homogeneous classical solution v(t) of:

$$\frac{d^2v}{dt^2} + \mu^2 v + \lambda^2 v^5 = 0$$

$$v(t) = \frac{a\,\mathrm{cn}\left(\omega t|\, m\right)}{\sqrt{1 + b\,\mathrm{cn}\left(\omega t|\, m\right)^2}}\,, \qquad \omega = \mu\left(1 + m(m-1)\right)^{-1/4}$$

- *Bohr-Sommerfeld quantisation condition* $I=2\pi n$*:*

$$u = cn\left(\omega t \right| m)$$

$$\begin{aligned} I &= 16\pi r^2 \omega a^2 \int \frac{\sqrt{1-u^2}\sqrt{m\left(u^2-1\right)+1}}{\left(bu^2+1\right)^3}\, du \\ &= -\frac{16\pi r^2 \omega a^2}{8b^2(b+1)\sqrt{1-m}(b(m-1)+m)}\Bigg[\big((3b+4)b^3 \\ &\quad -6(b+1)^2 b^2 m + (b+1)^3(3b-1)m^2\big)\,\Pi\left(-b\left|\frac{m}{m-1}\right.\right) \\ &\quad -(b(3b+2)(m-1)-m)(b(m-1)+m)\mathbb{K}\left(\frac{m}{m-1}\right) \\ &\quad +b(m-1)(b(3b(m-1)+4m-2)+m)\mathbb{E}\left(\frac{m}{m-1}\right)\Bigg] \end{aligned}$$

➡

$$\Delta_0 = \Omega_{d-1} r^d T_{00}|_{d=3} = \frac{\sqrt{2}\pi(r\omega)^2}{3gn}\sqrt{1-m+m^2-2(m-2)(m+1)(2m-1)(r\omega)^2}$$

Leading quantum correction

$$\mathcal{L}_2 = \frac{1}{2}\eta\mathcal{O}_{||}\eta + \frac{1}{2}\tilde{\phi}_a\mathcal{O}_{\perp}\tilde{\phi}_a$$

$$\mathcal{O}_{||,\ell} = -\partial_t^2 - J_\ell^2 - \mu^2 - 5\lambda^2 v^4(t)$$

$$\mathcal{O}_{\perp,\ell} = -\partial_t^2 - J_\ell^2 - \mu^2 - \lambda^2 v^4(t)$$

Unaware of the exact analytic expressions for the stability angles

However can be computed perturbatively in the weakly coupled regime and the results have been confirmed by Feynman diagrams computations

Outlook

- In a generic scalar QFT/CFT, I showed how to semiclassically compute anomalous dimensions for neutral composite operators

- Large order behaviour of the series (resurgence)
- Condensed matter applications (3d Ising)
- Test dualities between different CFTs in their neutral sectors
- OPE coefficients
- Instabilities and relation to level-crossing and chaos
- ….

Thank you!

Overlap of two expansions

	1-loop	**2-loop**	**3-loop**	
Δ_{-1}	$Q^2\lambda_0$	$Q^3\lambda_0^2$	$Q^4\lambda_0^3$	$\ldots.$
Δ_0	$Q\lambda_0$	$Q^2\lambda_0^2$	$Q^3\lambda_0^3$	$\ldots.$
Δ_1		$Q\lambda_0^2$	$Q^2\lambda_0^3$	$\ldots.$
Δ_2			$Q\lambda_0^3$	$\ldots.$
$\vdots$				

Pheno application: Higgsplosion

Multi-boson production

$$h^* \to nh$$

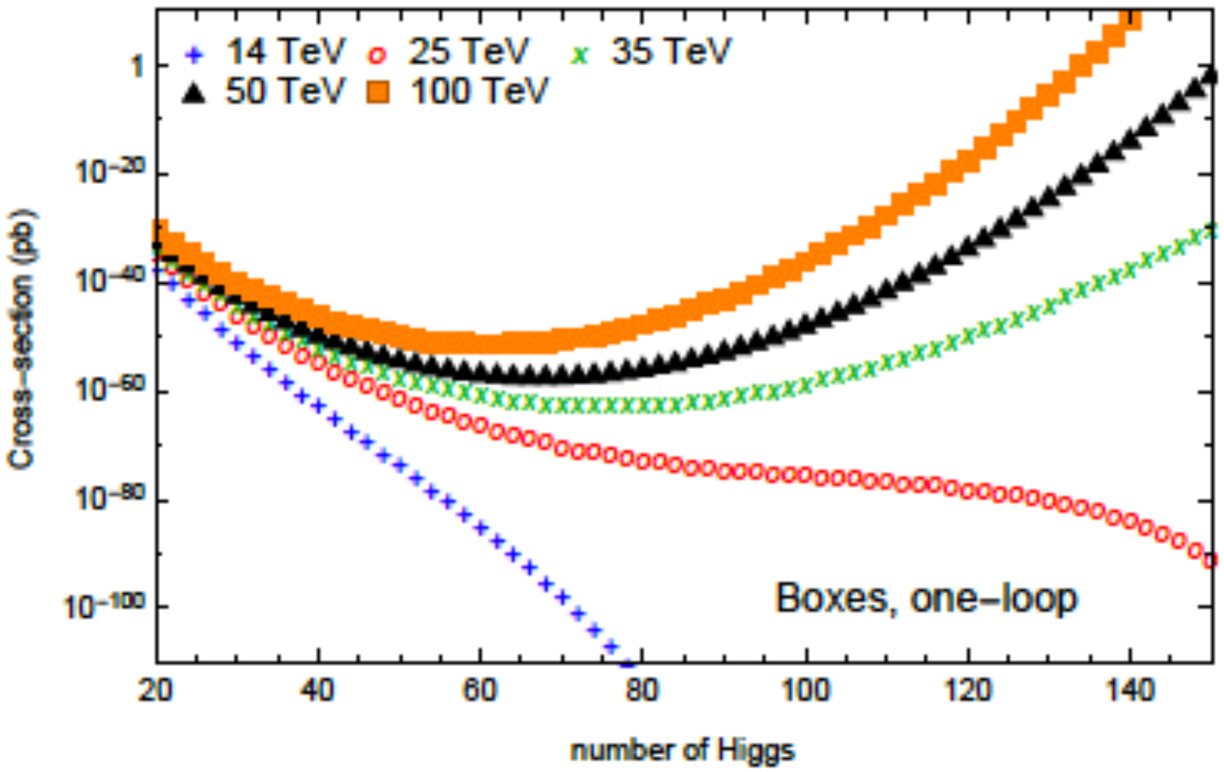
$$\boxed{\lambda\phi^4}$$

Consider the $1 \to n$ amplitude

$$A^{tree} = n!\lambda^{\frac{n-1}{2}} e^{-\frac{5}{6}En}$$

$$A = A^{tree} e^{B\lambda n}$$

$$\sigma(1 \to n) = e^{F(\lambda n, E)}$$



[Degrande, Khoze, Mattelaer, 2016]

$$\boxed{n \approx \sqrt{s}/m}$$