Polylogs

Def Let $s = (s_1, \dots, s_e)$ positive index

$t \in \mathbb{C}$ $|t| < 1$

$$Li_s(t) = \sum_{n_1 > n_2 > \dots > n_e \geq 1} \frac{t^{n_1}}{n_1^{s_1} \dots n_e^{s_e}}$$

(multiple) polylog funct° of 1 variable)

Similarly, $$Li_s(t_1, \dots, t_e) = \sum_{n_1 > n_2 > \dots > n_e \geq 1} \frac{t_1^{n_1} t_2^{n_2} \dots t_e^{n_e}}{n_1^{s_1} \dots n_e^{s_e}}$$

with $|t_1| < 1$ $\quad |t_i| \leq 1 \quad \forall i \geq 2$.

Prop: cplx analysis : if $s_1 \geq 2$, $Li_s(t)$ is holomorphic on the open unit disc. It and can be extended continuously to the closed disc $|t| \leq 1$ and satisfies

$$Li_s(1) = \zeta(s)$$

$$Li_2(1) = \sum_{n_1 \geq 1} \frac{1}{n^2}$$

$\int_0^t$ int along straight path

Sweet relations $$Li_1(t) = \sum_{n_1 \geq 1} \frac{t^{n_1}}{n_1} = -\log(1-t) = \int_0^t \frac{dt_1}{1-t_1}$$

$$Li_2(t) = \sum_{n_1 \geq 1} \frac{t^{n_1}}{n_1^2} = -\int_0^t \log(1-t_1) \frac{dt_1}{t_1}$$

primitive of $\frac{Li_1(t)}{t}$ vanishing at $t=0$

Prop functional equations

- $$\int_0^t Li_{s_1, \dots, s_e}(t_1) \frac{dt_1}{t_1} = Li_{s_1+1, \dots, s_e}(t)$$
- $$\int_0^t Li_{s_1, \dots, s_e}(t_1) \frac{dt_1}{1-t_1} = Li_{1, s_1, \dots, s_e}(t)$$

proof

$$\int_0^t \sum_{n_1 > n_2 \cdots > n_e \geq 1} \frac{t_1^{n_1 - 1}}{n_1^{s_1} \cdots n_e^{s_e}} \, dt_1$$

$$= \sum_{n_1 > \cdots > n_e \geq 1} \frac{t_1^{n_1}}{n_1^{s_1 + 1} \cdots n_e^{s_e}}$$

$$\int_0^t \sum_{n_1 > \cdots > n_e \geq 1} \frac{t_1^{n_1}}{n_1^{s_1} \cdots n_e^{s_e}} \frac{dt_1}{1 - t_1}$$

$$\frac{t_1^{n_1 + k - 1}}{n_1^{s_1} \cdots n_e^{s_e}}$$

$$\rightsquigarrow \quad \frac{t_1^{n_1 + k} \, n_0}{(n_1 + k) n_1^{s_1} \cdots}$$

n_0

Thm If $0 < t < 1$ real number

$$Li_s(t) = \int_{\Delta^{wt(s)}(t)} \omega_s$$

proof by induction on $wt(s)$ using functional eq.

$$\Delta^n(t) = \{ t \leq t_n \leq \cdots \leq t_1 \leq 0 \}$$

Appear in Feynman ~~integral~~ amplitudes computation

(not all of them but huge class of them)

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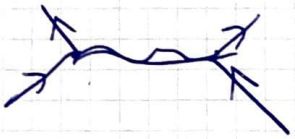
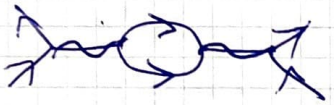
Feynman amplitudes

"scattering amplitudes"

We represent interactions between particles using graphs



two particles scattering (ie interacting) here exchanging a photon then scattering apart.



same, but photon spontaneously becomes a positron/e pair that later annihilates & forms a photon again.

- one has to consider all possible interactions (∞, yay!) (more and more loops/orders)
- have to associate $G \longmapsto I_G$ s.th we can work with

I_G amplitude given by an integral

function of masses & momenta of particles.

= Q answer to "how likely is this scatter"?

Predictions are obtained by summing over all graphs.

$\sum_G I_G$ diverges?

We'll work in scalar Euclidean space-time $\mathbb{R}^D$, $D>0$ even

norm $x^2 = \sum x_i^2$

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Feynmann graphs

following Brown

- $G = (V_G, E_G^{int}, E_G^{ext})$
  $N_G = \# E_G^{int}$
  G connected for us today
  together with the data
- masses $m_e \in \mathbb{R}_{\geq 0}$ $\forall e \in E_G^{int}$
- momenta $q_i \in \mathbb{R}^d$ $\forall i \in E_G^{ext}$

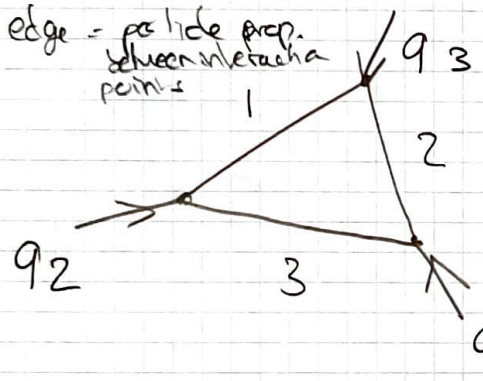
convention: all arrows incoming
$q_i < 0$ means outgoing particle

- internal edges "classic edges"
  $E_G^{int} \to Sym^2 V_G$
- external edges particles linked to a single vertex.

subject to momentum conservation

$$\sum_{i \in E_G^{ext}} q_i = 0$$

edge = particle prop. between interaction points



To each internal edge we assign a momentum variable $k_e \in \mathbb{R}^D$
+ orientation
→ integration variable
since we don't know the internal momenta def by dig struct + cons. laws
we integrate over all poss

$$p_e = \begin{cases} k_e & \text{if } e \in E^{int} \\ q_e & \text{if } e \in E^{ext} \end{cases}$$

δ restricts the domain of int to where mom is conserved
→ Dirac

$$I_G(q,m) = \int_{\mathbb{R}^{D \times N_G}} \prod_{e \in E_G^{int}} \frac{d^D k_e}{k_e^2 + m_e^2} \cdot \prod_{v \in V_G} \delta\Big(\underbrace{\sum_{\substack{e \text{ edges} \\ \text{which meet} \\ \text{at } v}} p_e}_{\text{mom cons at every vertex}}\Big)$$

"Momentum space FI"

- may diverge
- looks terrifying.

⇒ use parametric representation to benefit from iterated integral setting

Rq: No coupling constants — simplified setting
Panzer, Brown

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How to make it palatable ?

① Schwinger's trick

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha x}\, dx \qquad x > 0$$

Idea: we introduce a new variable (= Schwinger parameter) for every internal edge and replace $\frac{1}{k_e^2 + m_e^2}$ with $\int_0^\infty e^{-(k_e^2+m_e^2)\alpha_e} d\alpha_e$

$$\rightsquigarrow \int_0^\infty \int_{\mathbb{R}^{D \cdot N_G}} \exp\left(-\sum_e (k_e^2+m_e^2)\alpha_e\right) d \cdot \prod \delta(\Sigma_{pe}) \prod_{e \in E^{int}_G} d\alpha_e \; (d\alpha_1 \cdots d\alpha_{N_G})$$

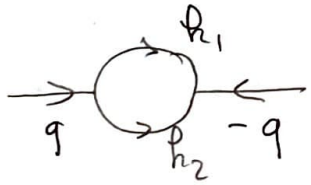
② Gauss $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$ $\quad x, b \in \mathbb{R}^D$

$$\int_{\mathbb{R}^D} e^{Q(x)} dx = \frac{\pi^{D/2}}{\sqrt{\det A}} e^{b^T A^{-1} b}$$

$Q(x) = -x^T A x + 2 b^T x$, A sym + def pos $D \times D$

$\rightsquigarrow$ D-dimensional equivalent

Example



(dirac fund.) $k_1 + k_2 = q$ (or int domain)

$$-\sum_e (k_e^2+m_e^2)\alpha_e$$
$$= \alpha_1(k_1^2+m_1^2) + \alpha_2(k_2^2+m_2^2)$$
$$= \alpha_1 k_1^2 + \alpha_2 (q-k_1)^2 + \alpha_1 m_1^2 + \alpha_2 m_2^2$$

complete the square

$$= (\alpha_1+\alpha_2)\left(k_1 - q\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^2 + q^2 \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} + \alpha_1 m_1^2 + \alpha_2 m_2^2$$

variable shift $k_1 = k_1 - q\frac{\alpha_2}{\alpha_1+\alpha_2}$

$$= (\alpha_1+\alpha_2)k^2 + \text{——————}$$

Integral is translation inv.

Deduce that $I_G(q,m) = \pi^{D/2} \int_0^\infty \frac{d\alpha_1 d\alpha_2}{(\alpha_1+\alpha_2)^{D/2}} \exp\left(\frac{-\alpha_1\alpha_2}{\alpha_1+\alpha_2} q^2 - m\chi^2 - m_3^2 \alpha_3\right)$

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Works in g a) (complete the square, etc)
one gets the determinant of a certain matrix that we then reexpress in terms of graph theoretical data.

$$I_G(q,m) = \text{trivial factors} \times \int_0^\infty \frac{\prod_{e\in E^{int}} d\alpha_e}{\Psi_G^{D/2}} \exp\left(-\frac{\phi_G(q)}{\Psi_G} - \sum_{e\in E^{int}} m_e^2 \alpha_e\right)$$

$$\frac{\pi^{Dh_G/2}}{(2\pi)^{Dh_G}}$$

where Ψ_G and $\phi_G(q)$ are called $\boxed{\text{Symanzik polyn}}$

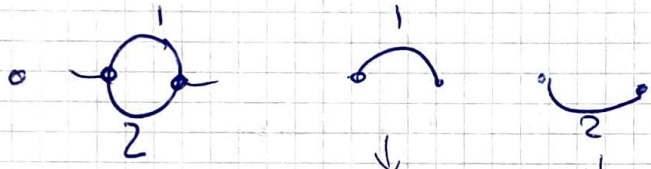
$\Psi_G \in \mathbb{Z}[\alpha_e, e \in E_G^{int}]$

. doesn't depend on external momenta nor masses

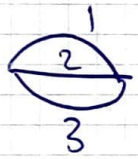
$$\Psi_G = \sum_{\substack{T\subseteq G \\ \text{spanning tree}}} \prod_{e\notin T} \alpha_e$$

T spanning tree of G
↳ $V_T = V_G$
T tree, subset of E_G^{int}

eg :



$$\Psi_G = \alpha_2 + \alpha_1$$



$$\Psi_G = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3)$$

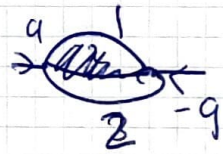
$\phi_G(q) \in \mathbb{Z}[\alpha_e, e \in E_G^{int}, q_i \cdot q_j]$ ← Euclidean · product

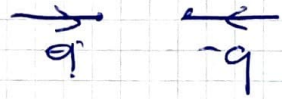
scalar prod of ext. momenta

$$\phi_G(q) = -\sum_{\substack{T=T_1\cup T_2 \\ \text{spanning 2-tres}}} q^{T_1} \cdot q^{T_2} \prod_{e\notin T} \alpha_e$$

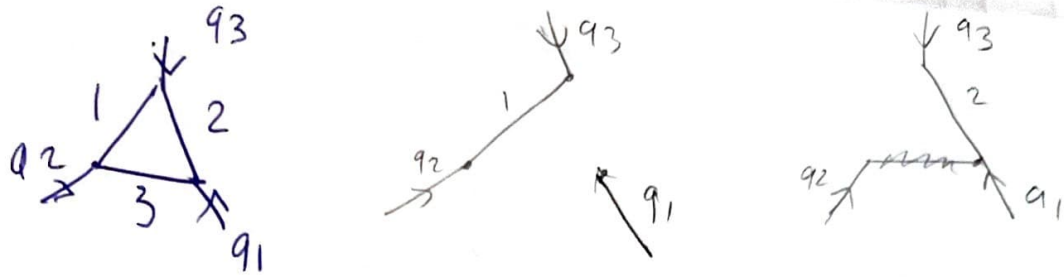
↳ forest with 2 components, simply connected, spans

q^{T_i} = total momentum entering T_i





$$-(q\cdot -q)\,\alpha_1\alpha_2$$
$$= +q^2\alpha_1\alpha_2$$



$$-q_1(q_2+q_3)\alpha_2\alpha_3 - \cdots -$$
$$= q_1^2\alpha_2\alpha_3 + q_2^2\alpha_2\alpha_3 + q_3^2\alpha_1\alpha_2$$

Rq: Ψ_G is homogeneous of degree h_G = # loops of G

$\phi_G(q)$ hom. in α_e of degree h_G+1

Now we just want to get rid of the exp factor

- ~~Dim regularization~~ ~~since~~ this homogeneity means we can rescale $\alpha_e \to \lambda\alpha_e$.

change of var $\alpha_e = \lambda\beta_e$ $\quad \sum \beta_e = 1 \quad \lambda \in \mathbb{R}^{>0}$

$$\prod d\alpha_e = \lambda^{N_G-1} d\lambda \wedge \Omega_G$$

Up to now: • started from product of propagators $\frac{1}{k_e^2+m_e^2}$ integrated over internal momenta k_e

- Schwinger parameters, get exponentials
- Gaussian integrals $\rightsquigarrow$ collapses int. into det.
- det are related to trees

$\to$ parametric form depending only on graph. conn + ext data.

- To get rid of the exp factor, one more step / integration to get a projective integral

$$\Omega_G = \sum_i (-1)^i \beta_i \, d\beta_1 \wedge \cdots \wedge \widehat{d\beta_i} \wedge \cdots d\beta.$$

$$= \int_{\substack{\beta_e>0 \\ \sum\beta_e=1}} \frac{\Omega_G}{\Psi_G^{D/2}} \underbrace{\int_0^\infty e^{-\lambda} \left(\text{usual stuff}\right) \lambda^{N_G - D/2\, h_G} \frac{d\lambda}{\lambda}}_{\text{computable}}$$

$$= \prod_{\substack{\Gamma \text{ value} \\ \text{sd}}} (-S_d(G)) \left(\frac{\Psi_G}{\phi_G(q)(\sum m_e\alpha_e)\Psi_G}\right)^{-S_d G}$$

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Where $sdG = -N_G + D/2\, h_G$ "superficial degree of div of the graph"

>0 div
≤ 0 cv

logarithmic div when $= 0$
(as special case in a reg)

$$\Rightarrow I_G = \underbrace{\Gamma(N_G - D/2\, h_G)}_{\text{hello poss. poles}} \int \frac{\Omega_G}{\Psi_G^{D/2}} \cdot \left(\frac{\Psi_G}{\phi_G(a) + (\sum m_e^2 x_e)\Psi} \right.$$

$\sigma \subseteq \mathbb{P}^{N_G - 1}(\mathbb{R})$ — sd(

$\sigma = \{(B_1 : \dots : B_{N_G}) \in \mathbb{P}^{N_G-1}(\mathbb{R}) \mid B_i \geq 0\}$

'nam of deg 0

proj. m

If pole $\rightarrow$ Dimensional reg

replace D with $D - \varepsilon$, perform Taylor exp in ε

compute coeffs of ε^i

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Panorama

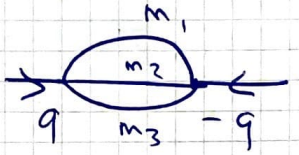
- The general 1-loop diagram can be expressed using

$$\left.\begin{array}{l} Li_2(x) = \sum_{n\geq 1}^{\infty} \frac{x^n}{n^2} \quad |x|\leq 1 \\ \log(x) \end{array}\right\} = \int_{0\leq t_1\leq t_2\leq x} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$$

with x some function of masses & momenta

- General 2-loop : not known.

ex Sunrise



solved for all m, q.
not !

iterated elliptic integrals

- A huge class of amplitudes are known & expressible using multiple polylogs

$$Li_{n_1,\dots,n_r}(x_1,\dots,x_r) = \sum_{1\leq k_1\leq\dots\leq k_r} \frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}$$

$|x_i|<1$, extends to a multivalued function on

$$\left(\mathbb{C}\setminus\{0,1\}\right)^r \setminus \bigcup_{1\leq i\leq j\leq r} \{x_i x_{i+1}\cdots x_j = 1\}$$

$$\cong \mathcal{M}_{0,n+3}(\mathbb{C})$$

Single scale processes : our nice stuff

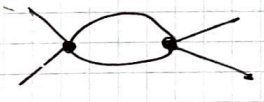
$I_G(q,m)$ is a function of a single variable

(e.g. all masses are 0 or only 1 momentum)

$\rightsquigarrow$ I_G boils down to a number.

Residues in massless ϕ^4 $\quad D=4$ all $m_e=0$

G in ϕ^4 $\leftrightarrow$ all vertices of G are of degree 4



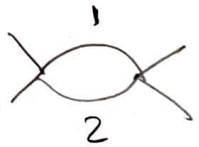
$N_G = 2h_G \Rightarrow \Gamma\left(N_G - \frac{D}{2}h_G\right)$ has a pole

Residue

$$I_G = \int_\sigma \frac{\Omega_G}{\Psi_G^2}$$

This thing will cv iff G is "primitive"

i.e. all of its proper subgraphs verify $N_\gamma > 2h_\gamma$

Ex : ∘ 

$$I_G = \int_\sigma \frac{\alpha_1 d\alpha_2 - \alpha_2 d\alpha_1}{(\alpha_1+\alpha_2)^2}$$

$$= \int_{\alpha_2=0}^{\infty} \frac{d\alpha_2}{(1+\alpha_2)^2} = 1$$

∘ Next interesting graphs

$I_G = 6\zeta(3)$

WS3 detailed fun

$I_G = 20\zeta(5)$ $I_G = 36\zeta(3)^2$

next with 6 loops → lin combs of $\zeta(5,3)$, $\zeta(5)\cdot\zeta(3)$, $\zeta(8)$ | ham of weight 8

Encl

Cool stuff we actually know

- infinite class of such graphs s.t. residue is a MZV
- for a while there was hope for all of them, but nope (8-loop 9-loop planar)
- not every MZV seems to occur, only a specific subspace → to investigate
- All odd zeta values do appear!

Thm (Brown-Schnetz 12) conj by Broadhurst Kreimer

Zig zag graphs

Z_5

$$I_{Z_n} = \frac{4(2n-2)!}{n!(n-1)!}\left(1-\frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3)$$

(here I is a residue again)

But no other family of graphs for which amplitudes are known for all #s of loops in this theory ... no conjectures

Conj (B-Schnetz)

Let G be a prim Γ in ϕ^4 ($N_G = 2h_G$) with h_G loops. Then $0 < I_G \leq I_{Z_{h_G}}$

~~work in progress~~

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