Plan & Idea

- Centered around a math object – Multiple zeta values

Anyone Familiar? → makes you aware of the existence of this research domain idea of questions directions

- Idea of this mini-course:
  - give a bit of historical intro & motivation

Today [• math context & some conj / results
overview
some exercises scattered / corrections later + some tomorrow
some reading in case you're interested.

Tomorrow • physics by mathematician
dive in concrete stuff F integrals / graphs

- Personal motivation
  ↳ relation of the lie alg of MZVs with OT / KV thesis.

---

0. Intro

1) The Basel Pb

In 1650, P. Mengoli formulates the so called Basel pb:

What is the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \; ? \qquad = \frac{\pi^2}{6}$$

Solved by Euler in 1735 ∴ Accepted proof in 1741, even that needed completion in the next century

→ lots of very nice proofs, see ex/reading sheet

All of this happens ~120 years before we start the Rieman zeta function Fun, but we'll call it $\zeta(2)$ nonetheless.

- Question : What about $\zeta(n)$ $n \in \mathbb{N}$??

- Thm (Euler, 1735)

$$\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}$$

↑ Bernoulli numbers

def by the power series identity

$$\frac{t}{e^t - 1} = \sum_{k \geq 0} B_k \frac{t^k}{k!}$$

→ We have a nice exact expression for every even zeta value.

- ~1760, Lambert π irrational
- ~1882, Lindenmann π transcendental

= non algebraic

i.e not the root of a non-zero pol. with int coeffs.

Transcendental ⇒ irrational.

$\zeta(2k)$ all transcendental!

- $\zeta(2k+1)$?

Thm Apéry 1975

$\zeta(3)$ is irrational

Rk: His proof doesn't generalize at all. Fun others are by Beukers

Thm Rivoal (00)

Infinitely many $\zeta(2k+1)$ are irrational

Thm Zudilin (01)

Atleast one of the 4 numbers
$\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational

Aaand this is pretty much all we know about odd zeta values.

Conj (Transcendence)

The numbers $\pi, \zeta(3), \ldots, \zeta(2k+1) \ldots$ are algebraically indep: $\forall k \geq 0$ and non-zero polynomial $P \in \mathbb{Z}[x_0 \ldots, x_Q]$

$$P(\pi, \zeta(3), \ldots, \zeta(2k+1)) \neq 0$$

→ new family of math constants ?
all transcendental.

Is this math. enough?
Well those things pop up in ≠ places...

2. Anomalous moment of the electron

. disdaimer

Magnetic moment of the electron = intrisic phys. prop of e, quantifying its fundamental magnetic behaviour. indep

In 1928, Dirac predicts the emm value as prop. to the particle spin. $\mu_e = g \frac{e}{2m} s$

m mass
s spin
e = elem charge
g = g-factor

But experiments measure a slight deviation called anomalous magnetic moment due to interactions with other particles + fields

UNIVERSITÉ DE GENÈVE ETH zürich swissmaprs.ch Swiss National Science Foundation SwissMAP The Mathematics of Physics National Centre of Competence in Research

QED ~~theory~~ enabling the computation of this deviation

1948 Schwinger computes the one-loop (1st order) contribution of this a_e.
(result engraved on his tombstone!)

From then more & more precisions in both experiments & computations
→ makes it the most accurately tested prediction in the history of physics (10^{-9}) and QED one of the most stringently tested theories.

So... From the correction at
- order 2 we get $\zeta(2)$, $\zeta(3)$ to appear
- order 3 $\zeta(5)$
  $\zeta(2)\cdot\zeta(3)$

→ Who are those numbers?
What's their structure?
How to compute them?

Pretty recent field : ~40 years
- Gancharov, Deligne, Brown, Broadhurst
  Zagier, Hoffman...
- important for computations of F. integrals.

SRS
SwissMAP Research
Station in Les Diablerets

1) MZVs: Def, shuffle product, classic. conjectures.

1. Back to Euler : product of MZV

$$\zeta(2)\cdot\zeta(2) = \sum_{n\geq 1}\frac{1}{n^2}\cdot\sum_{m\geq 1}\frac{1}{m^2}$$

$$= \sum_{n>m\geq 1} + \sum_{m>n\geq 1} + \sum_{m=n\geq 1}$$

$$= \text{“}\zeta(2,2)\text{”} + \text{“}\zeta(2,2)\text{”} \qquad \zeta(4)$$

double zeta values CV objects.

$$\cdot\ \zeta(n)\cdot\zeta(m) = \zeta(n+m) + \zeta(n,m) + \zeta(m,n) \qquad n,m\geq 2$$

Linear combs of zeta and double zeta values

Product of more factors? ⇝ Def of MZVs

DEF • Let $s=(s_1,\dots,s_e)\in\mathbb{N}^e$ — multi-index
if $s_1\geq 2$ — admissible

• The MZV assoc. to the index s is

$$\zeta(s)=\zeta(s_1,\dots,s_e)=\sum_{n_1>n_2>\dots>n_e\geq 1}\frac{1}{n_1^{s_1}\cdots n_e^{s_e}}$$

- weight $w(s)=s_1+\cdots+s_e$
- length/depth $e(s)=e$.
- $\zeta(\emptyset)=1$.

⚠ ≠ conventions coexist $n_1>n_2$ $1\leq n_1<n_2$. I will mess it up.

Lemma: CV $\Leftrightarrow$ $s_1\geq 2$

Pf: $\zeta(s)\leq\zeta(2,1,\dots,1)$ $\forall$ s

Use $\sum_{k=1}^{n}\frac{1}{k}\leq 1+\log n$ (by comp with $\int 1/x$

Let $\mathcal{Z}$ = $\mathbb{Q}$-vector space gen. by all MZVs

$= \langle 1, \zeta(2), \dots, \zeta(2,1), \dots \rangle_{\mathbb{Q}}$

- $\mathcal{Z}_k = \langle \zeta(s) \mid w(s) = k \rangle_{\mathbb{Q}}$

  $\mathcal{Z}_0 = \mathbb{Q}$ $\quad$ $\mathcal{Z}_2 = \langle \zeta(2) \rangle$

  $\mathcal{Z}_1 = \{0\}$ $\quad$ $\mathcal{Z}_3 = \langle \zeta(3), \zeta(2,1) \rangle$

- $F_\ell \mathcal{Z} = \langle \zeta(s) \mid \ell(s) \le \ell \rangle_{\mathbb{Q}}$

  $\subset F_\ell \mathcal{Z} \subset F_{\ell+1}\mathcal{Z} \subset \dots$

- $F_\ell \mathcal{Z}_k = \langle \zeta(s) \mid w(s) = k \ \& \ \ell(s) \le \ell \rangle_{\mathbb{Q}}$

ex $F_1 \mathcal{Z} = \langle \zeta(n) \mid n \ge 2 \rangle$ single zeta values

$F_3 \mathcal{Z}_4 = \langle \zeta(s) \mid w(s) = 4, \ \ell(s) \le 3 \rangle$

$= \langle \zeta(4), \zeta(3,1), \zeta(2,2), \zeta(2,1,1) \rangle$

Thm Stuffle (or series shuffle) product

Mult. of real numbers induces an algebra structure on $\mathcal{Z}$ compatible with the weight & length filtration in the sense that

$$F_{\ell_1}\mathcal{Z}_{k_1} \cdot F_{\ell_2}\mathcal{Z}_{k_2} \subseteq F_{\ell_1+\ell_2}\mathcal{Z}_{k_1+k_2}$$

$\forall \ell_1, \ell_2, k_1, k_2 \ge 0$

Every product of MZV can be written as a $\mathbb{Q}$-lin comb. of MZV.

Rk : formula for stuffle product is recursive and that that fun / useful.

- Dimensions of vs Z_k, etc?

For each weight k, how many admissible indices do we have?

2^{k-2}

For given weight k & length ℓ

($\ell+1$ elements fixed are already) $\rightarrow \binom{k-2}{k-\ell-1} = \binom{k-2}{\ell-1}$

$\sum_{\ell=1}^{k-1} \leftarrow = 2^{k-2}$

- But Euler (again) already had noticed stg weird:

Thm Euler's sum formula

If $s \geq 3$, $\zeta(s) = \sum_{j=1}^{s-2} \zeta(s-j, j)$

- $\zeta(3) = \zeta(2,1)$
- $\zeta(4) = \zeta(3,1) + \zeta(2,2)$
- $\zeta(5) = \zeta(4,1) + \zeta(3,2) + \zeta(2,3)$

: Linear relations between MZVs exist $\rightarrow$ dimensions are going to drop.

- ex: $\zeta(2,1,1) = \zeta(4)$

$\zeta(3,1) = \frac{1}{4}\zeta(4)$

$\zeta(2,2) = \frac{3}{4}\zeta(4)$

many ways to produce such relations.

Always appear between MZVs of same weight...

Conj Zagier 94

1. The weight k defines a grading on the $\mathbb{Q}$-alg $\mathcal{Z}$:

$$Z = \bigoplus_{k \geq 0} Z_k$$

$\Rightarrow$ There are no relations between MZVs of $\neq$ weights

no trivial $\mathbb{Q}$-lin.

2. $\forall k$, $\dim_{\mathbb{Q}} Z_k = d_k$

where $(d_k)_{k}$ is def following sequence recursion:

$$\begin{cases} d_0 = 1, \ d_1 = 0, \ d_2 = 1 \\ d_k = d_{k-2} + d_{k-3} \end{cases}$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
d_k	1	0	1	1	1	2	3	5	7	10	15	22	32	46
2^{k-2}	-	-	1	2	4	8	16	32	64	128	256	512	1024	2048

R_k

- conj. by numerical experiments (PSLQ algo)
- eg $\dim_{\mathbb{Q}}(\mathcal{Z}_k) = d_k$ only known for $k \leq 4$

i.e we don't even know if there's a single k for which $\dim_{\mathbb{Q}}(\mathcal{Z}_k) > 1$

Thm | Gencharov (01), Terasoma 02

$\forall k$, $\dim_{\mathbb{Q}}(\mathcal{Z}_k) \leq d_k$

Proofs comes from algebraic geom and doesn't consist in producing enough $\mathbb{Q}$-lin relations between MZVs, ignores existing families...

Conj Hoffman 97

$\forall k$, the MZVs $\zeta(s_1, \ldots, s_e)$ with $s_i \in \{2, 3\}$ form a basis of $\mathcal{Z}_k$

Thm Brown 12 (Annals of Math paper)

— generate $\mathcal{Z}$.

| Again hardcore alg. geom techniques...

SRS
SwissMAP Research
Station in Les Diablerets

Rk: Hoffman conj ⇒ Zagier conj ⇒ transcendentality of all MZVs

pf: Assume $\zeta(s)$ algebraic for s of weight >0
Then it satisfies a pdy. eq with rational coeffs

$$\sum_{k=0}^{\ell} a_k \zeta(s)^k = 0$$

But then $a_\ell \zeta(s)^\ell \in \mathbb{Z}\omega_\ell \cap \bigoplus_{\ell'<\ell} \mathbb{Z}\omega_{\ell'}$

$\Rightarrow a_\ell = 0$ ∎

Everything seems delightfully out of reach.

BREAK?

2. Integral representation & shuffle product. polylogs

$$\frac{1}{1-t} = \sum_{k\geq 1} t^{k-1}$$

$$\bullet \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \sum_{k\geq 1}^{\infty} t_2^{k-1}\, dt_2$$

$$= \int_0^1 \frac{dt_1}{t_1} \sum_{k=1}^{\infty} \frac{t_1^k}{k}$$

$$= \sum_{k\geq 1} \frac{1}{k} \int_0^1 t_1^{k-1}\, dt_1 = \sum_{k\geq 1} \frac{1}{k^2} = \zeta(2)$$

$$\bullet \int_{1\geq t_1\geq t_2\geq t_3\geq 0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{1-t_3} = \zeta(2,1)$$

SRS
SwissMAP Research
Station in Les Diablerets

• General set up

Let: $\Delta^n = \{(t_1,\dots,t_n) \mid 1 \geq t_1 \geq t_2 \geq \dots \geq t_n \geq 0\}$ n-simplex

and the two diff one-forms on $]0,1[$ (containing = order of ver)

- $\omega_0(t) = \frac{dt}{t}$
- $\omega_1(t) = \frac{dt}{1-t}$

Let $s = (s_1,\dots,s_e)$ and $r_i = \sum_{k=1}^{i} s_k$

We def a diff form of degree $r_e = wt(s)$ by

$$\omega_s = \omega_0(t_1) \wedge \dots \wedge \omega_0(t_{r_1-1}) \wedge \omega_1(t_{r_1}) \wedge \omega_0(t_{r_1+1}) \wedge \dots \wedge \omega_0(t_{r_2+1}) \wedge \omega_1(t_{r_2}) \wedge \dots \wedge \omega_0(t_{r_{e-1}+1}) \wedge \dots \wedge \omega_0(t_{r_e-1}) \wedge \omega_1(t_{r_e})$$

back to ex $\int_0^1 \frac{dt_1}{t} \int_0^{t_1} \frac{dt_2}{1-t_2}$ $\omega_{(2)} = \omega_0(t_1)\wedge\omega_1(t_2) = \frac{dt_1}{t_1} \wedge \frac{dt_2}{1-t_2}$

$$\omega_{(2,1)} = \frac{dt_1}{t_1} \wedge \frac{dt_2}{1-t_2} \wedge \frac{dt_3}{1-t_3} = \omega_0(t_1)\wedge\omega_1(t_2)\wedge\omega_1(t_3)$$

Thm (Kontsevich)

Let $s = (s_1,\dots,s_e)$ an admissible multi index.
Then the MZV $\zeta(s)$ can be obtained by a cv improper integral

$$\zeta(s) = \int_{\Delta^{wt(s)}} \omega_s$$

Reasons why that's cool

- MZVs are periods = cplx numbers whose real and im parts can be written as ass' cv integrals of rational fcts on $\mathbb{Q}$ on algebraic domains of $\mathbb{R}^n$.