Introduction to TQFTs

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Topological **Q**uantum **F**ield **T**heory

- scary name, but actually quite simple;
- graphical calculus: pictures → linear algebra.

Motivations

- Physics: toy models for quantum gravity; Chern–Simons theory.
- Mathematics: invariants for manifolds, knots (Donaldson theory for 4-manifolds).

References

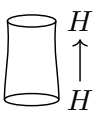
- 1988: Witten's original paper "TQFT".
- 1989: Atiyah's axioms: "TQFT".
- Baez, "This week's finds" on Frobenius algebras, 268.
- Kock, "Frobenius algebras and 2d TQFTs".
- (Lauda–Pfeiffer, "Open-closed strings: 2d extended TQFTs and Frobenius algebras".)
- Turaev: "YB equation and invariants of links" (1988).
- Fock–Tatitscheff–Thomas: "TQFTs from Hecke algebras".

I Intro and definitions

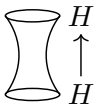
I.1 Why the name?

quantum: Hilbert / vector spaces;
independent systems $\widehat{=}$ tensor product.

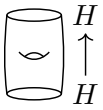
field theory: spread out in all of space.
$\rightsquigarrow$ QFT: Hilbert space to space regions.
Here: to all of space.

Evolution (space $\subset$ spacetime) $\widehat{=}$ linear map (unitary) 

topological: evolution only depends on the topology of spacetime, not on its geometry (like a metric).

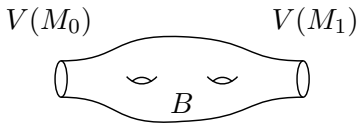
→ very restrictive. For example 

has to be the identity!

To get non-trivial evolution: 

Basic idea

- space $\widehat{=}$ manifold M $\widehat{=}$ vector space $V(M)$ (space of states / wave functions);
- time evolution $\widehat{=}$ cobordism B between "incoming" (at $t = t_0$) and "outcoming" (at $t = t_1$) manifold:



$\widehat{=}$ linear map $V(M_0) \longrightarrow V(M_1)$.

To distinguish incoming and outcoming: everything is oriented (M_0, B, M_1):

$$\partial B = M_0 \sqcup M_1, \qquad \partial^- B = M_0, \quad \partial^+ B = M_1$$

$\rightarrow$ the orientation of B induces an orientation on ∂B, either compatible (M_1) or reverse (M_0).

Example.

$M_{\text{in}} = M_0$  $M_1 = M_{\text{out}}$

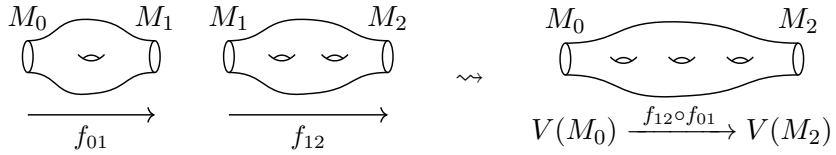
Another possibility:



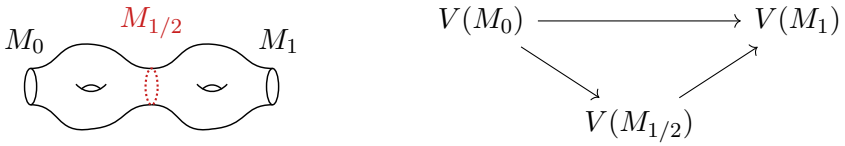
$M_{\text{in}} = M_0 \sqcup M_1$
$M_{\text{out}} = \emptyset$

$V(M_0 \sqcup M_1) \cong V(M_0) \otimes V(M_1)$
$V(\emptyset) = \mathbb{K}$ ground field!
$(V(M \sqcup \emptyset) = V(M) \otimes V(\emptyset) = V(M)$, so $V(\emptyset) = \mathbb{K})$

- **Gluing:**



But this also allows us to decompose into simpler pieces:



"TQFT = representation theory for manifolds with cobordisms": the linear map associated to a cobordism M is obtained via "reading" M from left to right (or bottom to top).

I.2 Formal definition (Atiyah axioms)

Definition. A ***d*-dimensional TQFT** is an assignment (functor) giving

- a $\mathbb{K}$-vector space $V(M)$ to each closed oriented manifold M of dimension $d-1$;
- a vector $v(B) \in V(\partial B)$ to each oriented d-dimensional manifold B (with boundary)
  $\rightsquigarrow$ $v(B)\colon V(\partial^- B) \to V(\partial^+ B)$, see Remark below

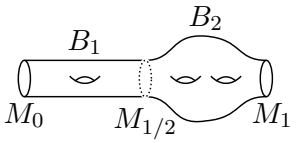
such that

(1) $V(M)$ and $v(B)$ only depend on the topology of M and B;

(2) $V(M^*) = V(M)^*$, where M^* is M with the opposite orientation;

(3) $V(M_1 \sqcup M_2) = V(M_1) \otimes V(M_2)$;

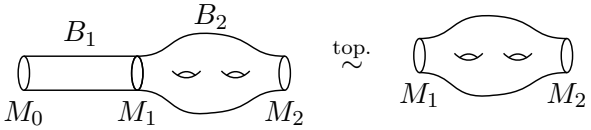
(4) Gluing: $v(B_1 \#_{M_{1/2}} B_2) = v(B_2) \circ v(B_1)$.



Remark. $\mathrm{Hom}(V, W) \cong V^* \otimes W$, so a vector in $V^* \otimes W \mathrel{\widehat{=}}$ a linear map $V \to W$. So $v(B)$ is a linear map $V(\partial^- B) \to V(\partial^+ B)$.

I.3 Easy consequences

- If the TQFT is non-trivial, then $v(M \times [0,1]) = \mathrm{id}_{V(M)}$. Since



  we get $v(B_2) \circ v(M \times [0,1]) = v(B_2)$ for all B_2.

- $V(\emptyset) \cong \mathbb{K}$ if the TQFT is non-trivial.

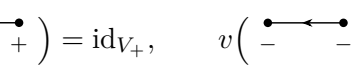
- If B is closed, then $v(B) \in \mathbb{K}$ $\rightsquigarrow$ topological invariant of B

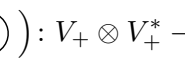


$V(\emptyset) \cong \mathbb{K} \longrightarrow V(\emptyset) \cong \mathbb{K}$
$\mathrm{Hom}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}$ acting by multiplication.

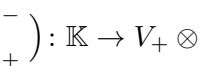
I.4 Easiest example: 1d TQFT

0-dimensional manifolds = finite sets of points, orientation: $+$ and $-$.
Cobordisms = oriented edges .
TQFT: $V(+) = V_+$, $V(-) = V_- = V_+^*$ by Axiom (2).





: $V_+ \otimes V_+^* \to \mathbb{K}$ evaluation map $(v, \varphi) \mapsto \varphi(v)$



: $\mathbb{K} \to V_+ \otimes V_+^*$ "coevaluation" $1 \mapsto \sum_i e_i \otimes e^i$ (independent of the chosen basis)

Remark.  is simply a swap ($\mathrel{\widehat{=}} v \otimes w \mapsto w \otimes v$).

There is a bijection {1d TQFTs} $\longleftrightarrow$ {vector spaces}.

Example. $v($$): \mathbb{K} \to \mathbb{K}$, $1 \mapsto \sum_i e_i \otimes e^i \mapsto \sum_i e^i(e_i) = \dim(V)$.

Remark. Slight generalisation: instead of $\mathbb{K}$-vector spaces, we can use A-modules where A is a (unital commutative) ring.

II 2d TQFT and Frobenius algebras

Closed 1d manifolds = circles:

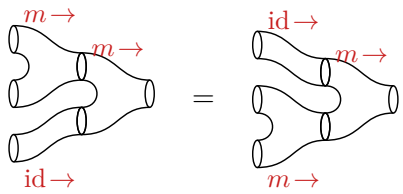
$$V\left(\circlearrowleft\right) = A, \qquad V\left(\circlearrowright\right) = A^*.$$

Cobordisms: oriented surfaces → classified by the number of boundary components and the genus.

II.1 Elementary cobordisms

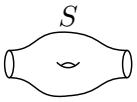
-  $m\colon A \otimes A \to A$ multiplication map.

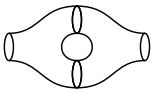
  Multiplication is associative: $m(m(a,b),c) = m(a,m(b,c))$:



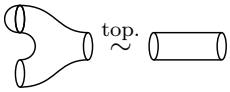
$m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m)$

-  $\Delta\colon A \to A \otimes A$ "comultiplication", "coassociative".

Remark. On a 

$=$  we get $v(S) = m \circ \Delta$.

- $\mathbb{K} \xrightarrow{\varepsilon} A$ unit (image of 1).

  Indeed 

so $\mathrm{id}_A = m \circ (\varepsilon \otimes \mathrm{id}_A)$.

-  $A \xrightarrow{\eta} \mathbb{K}$ "counit" (typically: trace).

Q: Is multiplication commutative?

A: Yes, since 

(our pictures are not embedded in $\mathbb{R}^3$!).

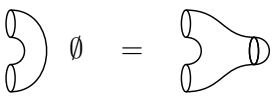
II.2 Frobenius algebras

Algebraic content of 2d TQFT = Frobenius algebra.

Definition. A **Frobenius algebra** A is a vector space with a (unital, associative) multiplication and a pairing (non-degenerate bilinear form) $\sigma\colon A \times A \to \mathbb{K}$ such that

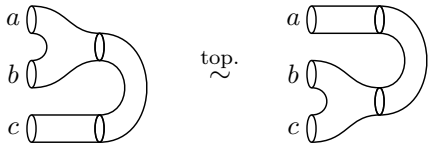
$$\sigma(ab,c) = \sigma(a,bc) \qquad \textbf{"Frobenius identity"}.$$

How to construct σ from the TQFT?

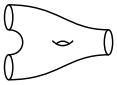
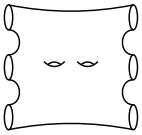


so $\sigma = \eta \circ m$.

Let's check the Frobenius identity graphically:

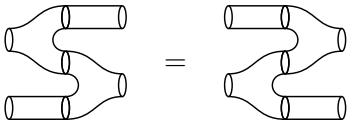


Exercise.

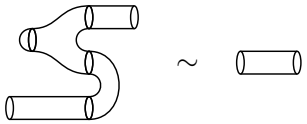
(1) Decompose  and  into elementary cobordisms.

(2) Using the graphical calculus, prove the following identities in a 2d TQFT $(A, m, \Delta, \varepsilon, \eta)$:

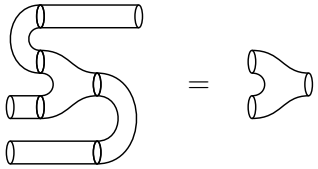
(a) $(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta)$



(b) $(\mathrm{id}_A \otimes \sigma) \circ (\Delta \otimes \mathrm{id}_A) \circ (\varepsilon \otimes \mathrm{id}_A) = \mathrm{id}_A$



(c) (needs β, see below) $(\mathrm{id}_A \otimes \sigma) \circ (\mathrm{id}_A \otimes m \otimes \mathrm{id}_A) \circ (\beta \otimes \mathrm{id}_A \otimes \mathrm{id}_A) = m$

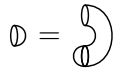


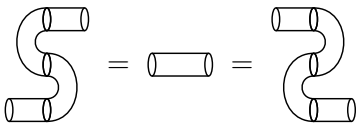
Theorem. There is a bijection (equivalence of categories)

$$\boxed{\text{2d TQFTs} \longleftrightarrow \text{commutative Frobenius algebras.}}$$

II.3 How to reconstruct the TQFT from a Frobenius algebra?

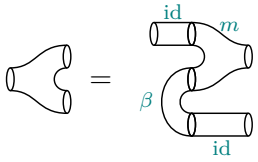
→ We have to construct the 4 basic cobordisms.

- m comes from the multiplication in A.
- η is defined by $\eta(a) = \sigma(a, 1)$.  ⇝ this satisfies $\sigma = \eta \circ m$.
- Non-degeneracy of the pairing σ gives the existence of  $\mathbb{K} \xrightarrow{\beta} A \otimes A$ such that

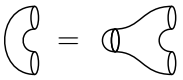
 "snake relation"

i.e. $(\mathrm{id}_A \otimes \sigma) \circ (\beta \otimes \mathrm{id}_A) = \mathrm{id}_A$ (see Remark below).

- Comultiplication is then defined via

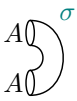


- Finally, the unit ε is defined by ($\rightsquigarrow \varepsilon$ is already given)



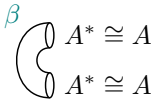
$\beta = \Delta \circ \varepsilon.$

Remark.



$= A \; A^*$ non-degeneracy asserts that this is an iso: $A \cong A^*$.

Then:



explains the existence of β.

Summary: 2d TQFT

2d cobordism	*alg. operation*	*Feynman diagram*	*principle*
	multiplication	$A \otimes A \to A$	merging
	unit	$\mathbb{K} \to A$	creation
	comultiplication	$A \to A \otimes A$	splitting
	counit	$A \to \mathbb{K}$	annihilation

II.4 Examples of Frobenius algebras

- $A = M_n(\mathbb{K})$ matrix ring, $\sigma(a,b) = \mathrm{tr}(ab)$.
  $\hookrightarrow$ non-commutative.
- $A/\mathbb{K}$ field extension of finite degree, $\sigma(a,b) = \varepsilon(ab)$ with any non-zero $\mathbb{K}$-linear $\varepsilon\colon A \to \mathbb{K}$.
  $\hookrightarrow$ commutative.
- $A = \mathbb{H}$ quaternions, $\sigma(a,b) = \mathrm{Re}(ab) \to$ non-commutative.
- $A = \mathbb{K}[G]$ group algebra $\to a \in A$ is a finite formal combination $a = \sum_{g\in G} a_g\, g$,

$$\sigma(a,b) = \varepsilon(ab) \quad \text{where } \varepsilon\Big(\sum_{g\in G} a_g\, g\Big) = a_{\mathrm{id}}.$$

  $\hookrightarrow$ commutative iff G is (but $\varepsilon(ab) = \varepsilon(ba)$).
- $A = H^*(X, \mathbb{K})$ cohomology of a compact oriented manifold X, multiplication by cup-product,

$$\sigma(a,b) = \int_X a \wedge b.$$

  $\hookrightarrow$ not commutative, but graded-commutative: $b \wedge a = (-1)^{d^\circ(a) d^\circ(b)}\, a \wedge b$.
  $\hookrightarrow$ more simple: $A = \Lambda(V)$ with wedge product, $\sigma(a,b) = \langle \mathrm{vol}, a \wedge b \rangle$.

III Generalisation 1: application to knots and braids

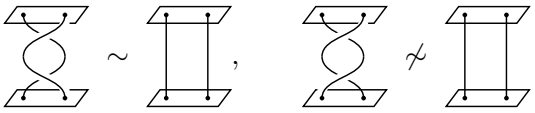
Let's go back to 1d. Can the swap  be upgraded to something like ? which would be $\neq$ from .

III.1 Embedded setting

0-dimensional manifolds $\widehat{=}$ points (with $\pm$ signs);

1-dimensional cobordisms which are embedded into $\mathbb{R}^2 \times [0,1]$.

Instead of diffeomorphisms, we use **isotopies**, i.e. diffeomorphisms of $\mathbb{R}^2 \times [0,1]$ which can be continuously deformed to id (by staying a diffeomorphism at each moment).



Interpretation as movement of particles (space-time diagram with 2d space).

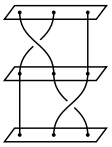
Definition. A braid with n strands is an embedding $\beta\colon \bigsqcup_{i=1}^{n}[0,1] \to \mathbb{R}^2 \times [0,1]$ such that

$$\beta(0_i) = (i,0,0) \ \ \forall i \qquad \text{and} \qquad \{\beta(1_i)\} = \{(i,0,1)\},$$

and which is foliation-preserving. We consider braids up to isotopies of $\mathbb{R}^2 \times [0,1]$ fixing $\{(i,0,0),(i,0,1)\}$.

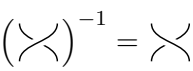
Remark. From the TQFT perspective,  is possible $\to$ notion of tangles; using duality we restrict to braids (better behaved).

Observation. Braids form a group!

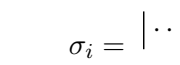
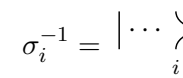
Multiplication = concatenation 

identity element $|\,|\,|\cdots|$

clearly associative

inverse: , indeed: 

Every braid can be decomposed into

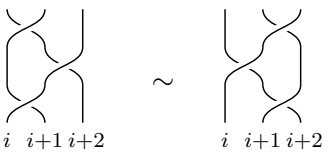
$\sigma_i =$  or $\sigma_i^{-1} =$ 

Theorem (Artin)**.** The braid group B_n is generated by the σ_i (and σ_i^{-1}) and the relations are

$$\sigma_i\sigma_i^{-1} = 1,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2 \qquad |\ \ \ |\ \ \ |\ \ \ |,$$

$$\text{and} \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{"braid relation"}$$



Remark. $P_n = \pi_1(\mathcal{C}_n)$ where $\mathcal{C}_n$ is the configuration space of n (ordered) points,

$$\mathcal{C}_n = \{x_1, \dots, x_n \in \mathbb{R}^2 \mid x_i \neq x_j \ \forall i \neq j\} = \{z \in \mathbb{C}^n \mid z_i \neq z_j \ \forall i \neq j\},$$

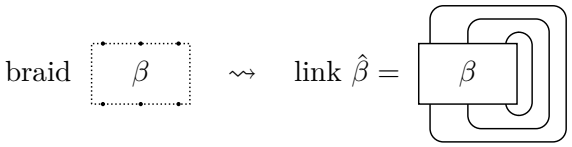
and $B_n = \pi_1(\mathcal{C}_n/\mathfrak{S}_n)$, where $\mathcal{C}_n/\mathfrak{S}_n$ is the configuration space of n unordered points.

Link to the symmetric group (permutations): any braid gives a permutation at $t = 1$ of the n particles:

$$\begin{array}{l} B_n \longrightarrow \mathfrak{S}_n \\ \beta \longmapsto (i \mapsto \text{1st coordinate of } \beta(1_i)) \end{array} \qquad \text{is a group homomorphism.}$$

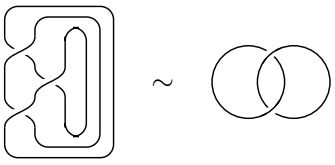
Elements of the kernel are called "pure braids", group P_n.

III.2 Link to knots and links: closing procedure of a braid



Example.

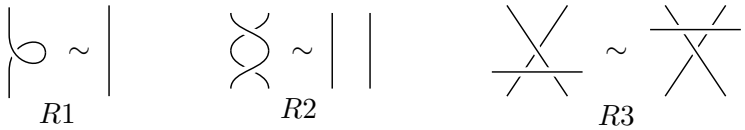
- trivial braid in $B_n \longmapsto n$ unknots;
- $\times \longmapsto$ unknot $\sim \bigcirc$;
- $\sigma_1\sigma_2\sigma_1 \longmapsto$ Hopf link



Definition. A link is an embedding $\bigsqcup_{i=1}^n S^1 \to \mathbb{R}^3$ (or S^3), considered up to isotopy (of $\mathbb{R}^3$). If $n = 1$, we speak about a knot.

Remark. Knots or links do not form a group! That's why braids are so important.

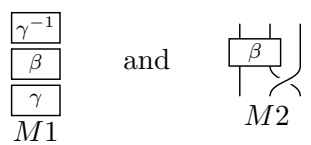
Theorem (Reidemeister)**.** Two link diagrams represent the same link iff there is a finite sequence of the following moves between them:



Remark. R2 $\widehat{=}\ \sigma_i\sigma_i^{-1} = 1$; R3 $\widehat{=}$ braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

Theorem (Alexander)**.** $\beta \mapsto \hat{\beta}$ is surjective onto the set of all links.

Exercise: Check that the closure does not change under



"Markov moves"

⇝ actually all relations.

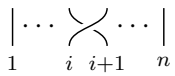
III.3 Braid invariants from "TQFT"

- $\widehat{=}$ vector space V

  braid $\widehat{=}$ linear map $V^{\otimes n} \to V^{\otimes n}$

It can be decomposed into  $R\colon V \otimes V \to V \otimes V$.

Remark.  $\widehat{=} R^{-1}$ (R is invertible), and



$$\widehat{=}\ \mathrm{id}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}^{\otimes(n-i-1)} =: R_{i,i+1}.$$

The matrix R has to satisfy one condition:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \qquad \text{Yang–Baxter equation.}$$

$\rightsquigarrow$ A solution is called an R-matrix.
$\rightsquigarrow$ $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ is automatic, and $R_{i,i+1}R_{i+1,i+2}R_{i,i+1} \widehat{=} R_{12}R_{23}R_{12}$.

Example.

- R is simply the swap $R(v \otimes w) = w \otimes v$.
  $\hookrightarrow$ "true" TQFT: the linear map associated to β is simply a permutation of $V^{\otimes n}$.
- $V \cong \mathbb{C}^2$, basis (v_1, v_2):

$$(*) \qquad R(v_i \otimes v_j) = v_j \otimes v_i + \varepsilon_{ij}\, v_i \otimes v_j \qquad \text{with} \qquad \varepsilon_{ij} = \begin{cases} q - q^{-1} & \text{if } i < j, \\ q - 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

$$R = \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \end{array}$$

$$R_{12} = R \otimes \mathrm{id} = \begin{pmatrix} R & 0_4 \\ 0_4 & R \end{pmatrix}, \qquad R_{23} = \mathrm{id} \otimes R = \begin{pmatrix} q\mathbb{1}_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & (q - q^{-1})\mathbb{1}_2 & \mathbb{1}_2 & 0_2 \\ 0_2 & \mathbb{1}_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & q\mathbb{1}_2 \end{pmatrix}.$$

The YB equation is satisfied! (`matrixcalc.org`)

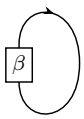
Remark. The main source of R-matrices are quantum groups (deformations of Lie algebras). We have $V \otimes W \ncong W \otimes V$ as representations of quantum groups.
See also this introductory video to quantum groups (in French) on the YouTube channel Thomaths.

Remark. For R-matrices from quantum groups: $R^2 = (q - q^{-1})R + \mathbb{1}$ "Hecke relation".

Remark. More generally: $V \cong \mathbb{C}^n$, basis $(v_1, \ldots, v_n)$; the same formula $(*)$ gives an R-matrix.

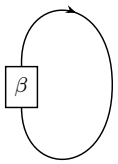
Using an R-matrix, we get $B_n \longrightarrow \mathrm{GL}(V^{\otimes n})$, a representation of the braid group!

Proposition. Closure $\widehat{=}$ trace, i.e.



$$\widehat{=}\ \begin{array}{l} \mathbb{K} \longrightarrow \mathbb{K} \\ x \longmapsto \mathrm{tr}(\beta)\, x \end{array} \qquad (\beta \text{ considered as an element of } \mathrm{GL}(V^{\otimes n})).$$

Proof. Let $W = V^{\otimes n}$.



$$\begin{array}{ll} & \mathbb{K} \\ & \uparrow_{\text{evaluation}} \\ W \otimes W^* & \\ & \uparrow_{\beta\otimes \mathrm{id}_{W^*}} \\ W \otimes W^* & \\ & \uparrow_{\text{coevaluation}} \\ & \mathbb{K} \end{array} \qquad \begin{aligned} 1 &\longmapsto \textstyle\sum_i e_i \otimes e^i \\ &\longmapsto \textstyle\sum_i \beta(e_i) \otimes e^i \\ &\longmapsto \textstyle\sum_i e^i(\beta(e_i)) = \sum_{i,j} e^i(\beta_i^j e_j) \\ &\quad = \mathrm{tr}(\beta). \end{aligned}$$

□

In summary: embedded 1d TQFT $\longleftrightarrow$ R-matrices $\longrightarrow$ representations of braid groups.

Exercise (From braid to link invariants)**.** To get a link invariant by closing the braid, we have to ensure that our construction is invariant under the Markov moves M1 and M2.

a) Show that $L \mapsto v(\hat{\beta})$ is invariant under M1. $\quad \to \mathrm{tr}(\gamma^{-1}\beta\gamma) = \mathrm{tr}\,\beta.$

M2 is not satisfied. For R given by $(*)$, define $\mu = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$. Consider

$$L \overset{I}{\longmapsto} q^{-2w(\beta)}\, v(\hat{\beta}\,\mu^{\otimes n}), \qquad \beta \in B_n \text{ with } \hat{\beta} = L,$$

where $w(\beta) = \#\,\text{positive crossings} - \#\,\text{negative crossings}$ (called the "writhe"); positive crossings are the σ_i $\nearrow\!\!\searrow$, negative ones the σ_i^{-1} $\searrow\!\!\nearrow$.

b) Compute $I(\bigcirc)$ and $I(\textcircled{\scriptsize 1})$.

For $\bigcirc$: $w(\beta) = 0$, $I = \mathrm{tr}(\mu) = q + q^{-1}$.

For $\textcircled{\scriptsize 1}$: $w(\beta) = 1$,

$$I = q^{-2}\,\mathrm{tr}\,(R \circ \mu^{\otimes 2}) = q^{-2}\,\mathrm{tr}\left(\begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix}\begin{pmatrix} q^{-2} & & & \\ & 1 & & \\ & & 1 & \\ & & & q^2 \end{pmatrix}\right) = q + q^{-1}.$$

d)* Show that I is invariant under M1 and M2.

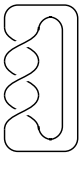
$\hookrightarrow$ For M1: $w(\gamma^{-1}\beta\gamma) = w(\beta)$ and $\mu \otimes \mu$ commutes with R. Hence

$$q^{-2w(\gamma^{-1}\beta\gamma)}\,\mathrm{tr}\,(\gamma^{-1}\beta\gamma\,\mu^{\otimes n}) = q^{-2w(\beta)}\,\mathrm{tr}\,(\gamma^{-1}\beta\,\mu^{\otimes n}\gamma) = I(\hat{\beta}).$$

For M2: 

$$\begin{aligned} & q^{-2(w(\beta)+1)}\,\mathrm{tr}\,\big((\beta\otimes\mathrm{id})(\mathrm{id}\otimes R)\,\mu^{\otimes(n+1)}\big) \\ &\quad = q^{-2w(\beta)-2}\,\mathrm{tr}\Big(\beta\,(\mu^{\otimes(n-1)}\otimes \mathrm{id}_V)\big(\mathrm{id}_{V^{\otimes(n-1)}} \otimes \underbrace{\mathrm{tr}_2\,((\mu\otimes\mu)R)}_{q^2\mu}\big)\Big) \\ &\quad = q^{-2w(\beta)}\,\mathrm{tr}\,(\beta\,\mu^{\otimes n}) = I(\hat{\beta}). \end{aligned}$$

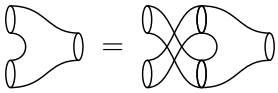
c) Compute the invariant for the trefoil knot ($\hat{\beta}$ with $\beta = \sigma_1^3 \in B_2$).



$$\begin{aligned} I(\hat{\beta}) &= q^{-6}\,\mathrm{tr}\,((\mu\otimes\mu)R^3) \\ &= q^{-6}(q + q^3 - q + q^{-1} - q^{-3} + q - q^{-1} + q^5) \qquad \text{(matrix calc)} \\ &= q^{-1} + q^{-3} + q^{-5} - q^{-9}. \end{aligned}$$

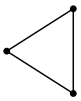
IV Generalisation 2: state sums

Let's come back to 2d TQFTs. As for 1d TQFTs, it is boring that



Another building block could be: a triangle!

Idea here: build up surfaces via triangulations ("cellular approach").

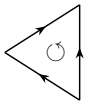


edge $\widehat{=} V$

triangle $\widehat{=}$ linear map

Here: we ignore the vertices (somehow removed from the surface, which becomes open).

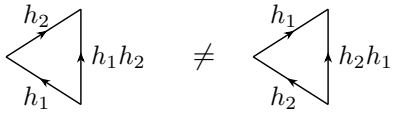
A triangle gives a multiplication (when oriented):



$\widehat{=}\ V \otimes V \to V$ multiplication on V.

Changing the orientation of an edge $\widehat{=} V \longleftrightarrow V^*$.

Very similar to the previous 2d TQFT, but here: not commutative!



Remark. The algebraic structure we will get here is a special class of (non-commutative) Frobenius algebras.

Here: associativity $\widehat{=}$ flip.

IV.1 Ciliated surfaces

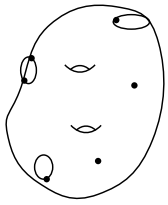
What kind of geometric objects do you get by gluing triangles together?

→ polygons 

→ closed surfaces with marked points; for instance

 torus with 1 marked point, or  sphere with 3 marked points.

Definition. A **ciliated surface** S is a surface from which we remove a finite number of discs, together with ≥ 1 marked points (in the interior or on the boundary, at least 1 marked point for each boundary component).



The boundary of S is then a disjoint union of segments.

Fact. A ciliated surface can always be decomposed into triangles and "1-gons". You need the 1-gon only for , , $S_{0,1}$, $S_{0,2}$ and .

Simplest examples:

- 1-gon , triangle ;
- 2-gon = triangle glued to a 1-gon;
- cone = = triangle + 2-gon = $S_{0,2}\setminus$ 1-gon;
- $S_{0,1}$ = gluing of 2 , $S_{0,2}$ = gluing of 2 ;
- = (2 triangles);
- annulus =

IV.2 TQFT for ciliated surfaces

Segment $\widehat{=}$ V

Triangle $\widehat{=}$ multiplication (not commutative)

1-gon $\widehat{=}$ trace function $V \xrightarrow{\mathrm{tr}} \mathbb{K}$ and unit $\mathbb{K} \to V$.

> We impose that $V \to V$ is the identity.
> $\hookrightarrow$ Then is the unit.
> (We suppose that $\mathrm{tr}\colon V \to V^*$, $a \mapsto \mathrm{tr}(a\,\cdot)$ is an iso, $\widehat{=}$.)

Remark. Defining $\sigma(a,b) = \mathrm{tr}(ab)$ gives a Frobenius algebra.

Proposition. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$.

Proof (from the 2-gon). By rotation of 180°:

a b $\sim$ b a and a ab b = 2-gon = b ba a

□

Remark. $\widehat{=}\ \mathrm{tr}(xyz)$; cyclic invariance shows well-definedness.

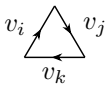
Gluing:

$$(V^* \otimes V^* \otimes \underline{V^*}) \otimes (\underline{V} \otimes V \otimes V) \longrightarrow V^* \otimes V^* \otimes V \otimes V$$

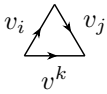
(contraction, or natural pairing)

Remark. Like in GR, we deal with tensors here.

Concretely: V with basis (v_i); the dual basis of $V^* \mathrel{\widehat{=}}$ a basis (v^i) of V via the trace $\rightsquigarrow$ $\operatorname{tr}(v^i v_j) = \delta^i_j$.



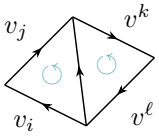
$\mathrel{\widehat{=}}\ \operatorname{tr}(v_i v_j v_k)$,



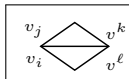
$\mathrel{\widehat{=}}\ \operatorname{tr}(v_i v_j v^k)$ structure constants

$$\rightsquigarrow \quad v_i v_j = \sum_k c_{ij}^k v_k, \quad \text{and } c_{ij}^k = \operatorname{tr}(v_i v_j v^k), \quad \text{since} \quad \operatorname{tr}(v_i v_j v^k) = \sum_\ell c_{ij}^\ell \underbrace{\operatorname{tr}(v_\ell v^k)}_{\delta_\ell^k} = c_{ij}^k.$$

Gluing of 2 triangles:



$$\begin{aligned}\langle v_i v_j, v^k v^\ell \rangle &= \operatorname{tr}(v_i v_j v^k v^\ell) \\ &= \sum_{m,n} c_{ij}^m \, c_n^{k\ell} \, \operatorname{tr}(v_m v^n) \\ &= \sum_m c_{ij}^m \, c_m^{k\ell}\end{aligned}$$



$\mapsto \operatorname{tr}(v^\ell v_i v_j v^k) = \sum_m c_{im}^\ell \, c_j^{km}$

$(v_i v_j)v^k = v_i(v_j v^k)$ gives equality

Remark. Invariance under flip $\mathrel{\widehat{=}}$ associativity of the product!

In general, the TQFT is given by a **state sum**:

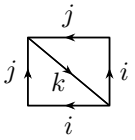
> **state:** assignment of a basis vector to each internal edge;
>
> **weight of a state:** product of all structure constants (one such constant per face).

To get the value of the TQFT for given exterior values (by linearity chosen to be basis vectors), we sum all possible states together:

$$\sum_{\substack{\text{internal edges} \\ \to \{1,\dots,n\}}} \prod_{f \in \text{faces}} c_f \qquad (c_f \text{ structure constant})$$

$\rightsquigarrow$ independent of the triangulation (all related by flips).

Example. $V = \mathbb{C}[1, s]$ with $s^2 = (q - q^{-1})s + 1$, $\operatorname{tr}(a + bs) = a$ ("constant term").
$\rightsquigarrow \operatorname{tr}\big((a + bs)(c + ds)\big) = ac + bd = \operatorname{tr}\big((c + ds)(a + bs)\big)$.
$\rightsquigarrow$ Here: dual basis = basis.
Value of $S_{0,3}$:



$$\longmapsto \sum_{i,j,k} c_{ijk} \, c^{ijk} = \sum_{i,j,k} c_{ijk}^2 = 1 + 1 + 1 + 1 + (q - q^{-1})^2 = q^2 + 2 + q^{-2}.$$

(Same for $S_{1,1}$, since the Hecke group is too small.)

In summary: TQFT for triangulated surfaces $\longleftrightarrow$ algebras with symmetric non-degenerate traces.

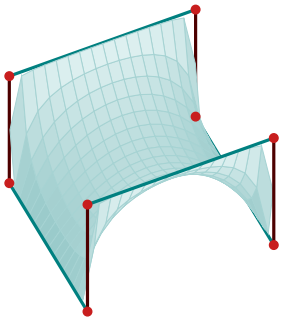
Example.

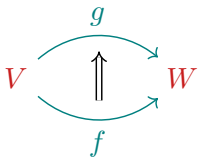
- G group, $V = \mathbb{C}[G]$ = finite formal sums $\sum a_g \, g$ ($a_g \in \mathbb{C}$). Multiplication using the group law, $\operatorname{tr}\big(\sum a_g \, g\big) = a_{\mathrm{id}} \to$ this is symmetric.
- Hecke algebra: $V = (\mathbb{C}[q, q^{-1}])[\mathfrak{S}_n]$, $\sigma_i \sigma_j = \sigma_j \sigma_i$, braid relation, $\operatorname{tr}\big(\sum a_g \, g\big) = a_{\mathrm{id}}$ and $\sigma_i^2 = (q - q^{-1})\sigma_i + 1$.

V Opening

Many other generalisations and related topics:

- using categorical language (replace vector spaces by a symmetric monoidal category);
- extended TQFTs (for manifolds with corners)
  ↪ in 2d: vector spaces to points (objects in a category)
  linear maps to 1d objects (morphisms in the category)
  intertwiners to 2d objects (2-morphisms)





- open-closed TQFTs → mixing ciliated surfaces and normal ones;
- CFTs, CohFTs, ...