Introduction to TQFTs - exercises

Exercise 1. *Decomposing surfaces*
Let $S_{g,n}$ denote the (connected oriented) surface of genus g with n boundary components. Decompose $S_{g,n}$ into elementary cobordisms. You can start with $S_{1,3}$ and $S_{2,5}$.

Exercise 2. *Graphical calculus*
Let $(A, m, \Delta, \varepsilon, \eta)$ be a 2d TQFT. We define $\sigma = \eta \circ m$ and $\beta = \Delta \circ \varepsilon$. Using the graphical calculus, proof the following identities:

a) $(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta)$,

b) $(\mathrm{id}_A \otimes \sigma) \circ (\Delta \otimes \mathrm{id}_A) \circ (\varepsilon \otimes \mathrm{id}_A) = \mathrm{id}_A$,

c) $(\mathrm{id}_A \otimes \sigma) \circ (\mathrm{id}_A \otimes m \otimes \mathrm{id}_A) \circ (\beta \otimes \mathrm{id}_A \otimes \mathrm{id}_A) = m$.

Exercise 3. *Markov moves*
To any braid $\beta \in B_n$, we associate a link $\widehat{\beta}$ giving by the closure of β. Show that the isotopy class of the associated link does not change under the following moves on β, called *Markov moves*:

(M1) $\gamma\beta\gamma^{-1}$, for any $\gamma \in B_n$,

(M2) $\iota(\beta)\sigma_n$, where $\iota : B_n \to B_{n+1}$ is the natural inclusion of braids on n strings into braids on $n+1$ strings (keeping σ_i for $i = 1, ..., n-1$).

Remark. *A theorem of Markov asserts that two braids have isotopic closures if and only if they are related by a finite number of Markov moves.*

Exercise 4. *From braid to link invariants*
In this problem, we define an invariant on links, i.e. an algebraic object (here a Laurent polynomial) which does only depend on the isotopy class of the link.

Consider a link L, which can be written as the closure of a braid $L = \widehat{\beta}$, with $\beta \in B_n$. Denote by $v : B_n \to \mathrm{GL}(V^{\otimes n})$ the braid representation coming from some R-matrix.

To get a link invariant from the braid invariant, we have to assure that the construction is invariant under the Markov moves $M1$ and $M2$.

a) Show that $L \mapsto \operatorname{tr} v(\beta)$ is invariant under $M1$.

Unfortunately this construction is not invariant under $M2$. To remedy this, consider the special case of $V \cong \mathbb{C}^2$ with basis (v_1, v_2). We further consider $\mu = \operatorname{diag}(q^{-1}, q)$ and the R-matrix of type A_1, i.e. $R(v_i \otimes v_j) = v_j \otimes v_i + \varepsilon_{ij}\, v_i \otimes v_j$, with $\varepsilon_{ij} = q - q^{-1}$ if $i < j$, equals $q - 1$ if $i = j$ and else vanishes. We then define

$$I(L) = q^{-2w(\beta)} \operatorname{tr}(v(\beta)\mu^{\otimes n}) \in \mathbb{C}[q, q^{-1}]\,,$$

where $w(\beta)$ is the number of positive crossings (σ_i's) minus the number of negative crossings (σ_i^{-1}'s) in β, called the *writhe*.

b) Compute I for the closure of id and σ_1 in B_2.

c) Compute I for the trefoil knot, the closure of $\sigma_1^3 \in B_2$.

d) Show that R commutes with $\mu \otimes \mu$, and that $\operatorname{tr}_2(R(\mu \otimes \mu)) = q^2\mu$, where tr_2 denotes the partial trace.

e)* Show that I is a link invariant, i.e. that it is invariant under the Markov moves $M1$ and $M2$.

References:

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