Introduction to rigorous Statistical Mechanics through the Ising model

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Preamble

These notes are written for a short course given at the Swissmap Summer School 2026. The first chapter is an introduction to the general principles of classical statistical physics (local specification, thermodynamic functions and limits, phase transitions) in the setup of the (ferromagnetic) Ising model. The second contains a brief informal overview of the scaling limits of the model, focusing on dimension 2. The appendices contain "crash courses" on probability and distribution theory for those interested (they are not needed to read the notes/follow the course). Appendix C contains the solutions to some of the Exercises.

As time is strongly constraint, several results are only mention without proofs, or some proofs are only sketched. References to further reading are given for the interested reader. The notes are likely to contain imprecisions/typos, you can send them to me at sebastien.ott@epfl.ch if you find some!

Some standard references for further reading:

Pedagogical introduction: [10]

Advanced reading: [12] for more on Gibbs measures, [11] for more on CFT from lattice models, [18, 20] for more on lattice statistical physics (also treat quantum models).

If you find typos/errors or have suggestions/constructive comments, you can send them to me at sebastien.ott@epfl.ch.

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Chapter 1

The Ising model

1.1 Definition of the model

1.1.1 Parameters, spins, energy

The Ising model attempts to model the ferro/para-magnetic phase transition. The model depends on two parameters: a *magnetic field* $h \in \mathbb{R}$, and the *inverse temperature* $\beta = \frac{1}{T} \geq 0$. The Ising model is a random attribution of *spins* (microscopic magnetic moments) to atoms of a solid (sites of a lattice). The simplifying assumption is that spin can take only two values: up $(+1)$ or down (-1). The *spin space* is therefore $\Omega_0 = \{-1, +1\} \equiv \{+, -\}$. The *configuration space* is then Ω:

$$\Omega = \Omega_0^{\mathbb{Z}^d}, \quad \Omega_\Lambda^\xi = \{\omega \in \Omega :\ \omega_i = \xi_i \ \forall i \in \Lambda^c\}, \quad \Lambda \Subset \mathbb{Z}^d,\ \xi \in \Omega,$$

where Ω_Λ^ξ are the finite volume configuration spaces; in words: they prescribe a given configuration, ξ, outside of a finite volume Λ.

Notation. $\Lambda \Subset \mathbb{Z}^d$ means $\Lambda \subset \mathbb{Z}^d$ and $|\Lambda| < \infty$.

Notation. For $\Lambda \subset \mathbb{Z}^d$, $\omega \in \Omega$, $\omega_\Lambda \in \Omega_0^\Lambda$ is the restriction of ω to Λ. Also, denote $\omega_\Lambda \omega'_{\Lambda'} \in \Omega_0^{\Lambda \cup \Lambda'}$ for Λ, Λ' disjoint the configuration on $\Lambda \cup \Lambda'$ that restrict to ω on Λ and to ω' on Λ'.

We can then define the *Hamiltonian* (interaction energy function) that defines the model:

$$H(\omega) = -\sum_{\{i,j\} \subset \mathbb{Z}^d :\, |i-j|=1} \omega_i \omega_j, \quad H_\Lambda(\omega) = -\sum_{\{i,j\} \subset \mathbb{Z}^d :\, |i-j|=1,\, \{i,j\} \cap \Lambda \neq \varnothing} \omega_i \omega_j.$$

Notation. $|\cdot|$ denotes the Euclidean norm.

Notation. $i \sim j$ if $|i - j| = 1$.

$$\partial\Lambda = \{j \in \Lambda^c :\ \exists i \in \Lambda,\ i \sim j\}, \quad \bar{\Lambda} = \Lambda \cup \partial\Lambda.$$

Also use the sum shorthand

$$\sum_{\{i,j\} \subset \mathbb{Z}^d :\, |i-j|=1,\, \{i,j\} \cap \Lambda \neq \varnothing} \equiv \sum_{i \sim j \in \Lambda}.$$

The quantity of central interest will be the *magnetization*:

$$M_\Lambda(\omega) = \sum_{i\in\Lambda} \omega_i, \quad m_\Lambda(\omega) = \frac{1}{|\Lambda|} M_\Lambda(\omega). \tag{1.1}$$

We are now in position to define the Ising model in finite volume.

1.1.2 Finite volume measures

Definition 1.1.1. The *Ising model* in finite volume is the collection of probability measures on Ω, $\mu^\xi_{\Lambda;\beta,h}$, $\Lambda \Subset \mathbb{Z}^d$, $\xi \in \Omega$ given by

$$\mu^\xi_{\Lambda;\beta,h}(A) = \frac{1}{Z^\xi_{\Lambda;\beta,h}} \sum_{\omega\in\Omega} \mathbb{1}_{\Omega^\xi_\Lambda}(\omega)\mathbb{1}_A(\omega) e^{-\beta H_\Lambda(\omega) + hM_\Lambda(\omega)}.$$

$Z^\xi_{\Lambda;\beta,h} = \sum_{\omega\in\Omega^\xi_\Lambda} e^{-\beta H_\Lambda(\omega)+hM_\Lambda(\omega)}$ is a normalization constant called the *partition function*. We will denote

$$\mu^\xi_{\Lambda;\beta,h}(f) = \frac{1}{Z^\xi_{\Lambda;\beta,h}} \sum_{\omega\in\Omega} \mathbb{1}_{\Omega^\xi_\Lambda}(\omega) f(\omega) e^{-\beta H_\Lambda(\omega)+hM_\Lambda(\omega)}$$

the *expected value of* f under $\mu^\xi_{\Lambda;\beta,h}$.

Notation. We will denote $+$ for $\xi \equiv +1$ the constant $+1$ configuration, and $-$ for $\xi \equiv -1$ the constant -1 configuration.

A very useful observation is the following *compatibility condition*: for different volumes $\Lambda \subset \Lambda'$,

$$\mu^\xi_{\Lambda';\beta,h}(A) = \sum_{\omega\in\Omega} \mu^\xi_{\Lambda';\beta,h}(\{\omega\})\mu^\omega_{\Lambda;\beta,h}(A) \equiv \mu^\xi_{\Lambda';\beta,h}\big(\mu^{\cdot}_{\Lambda;\beta,h}(A)\big) \tag{1.2}$$

In words: if one first sample a configuration in Λ' according to $\mu^\xi_{\Lambda';\beta}$, and then *resample* the configuration in Λ conditionally on what was obtained in Λ^c using $\mu^{\cdot}_{\Lambda;\beta}$, the obtained configuration is still sampled according to $\mu^\xi_{\Lambda';\beta}$. The compatibility condition says that the family of finite volume measures forms a *local specification*: a compatible collection of probability kernels (Markov kernels) defined by

$$(A, \xi) \mapsto \mu^\xi_{\Lambda;\beta,h}(A), \quad \Lambda \Subset \mathbb{Z}^d.$$

Exercise 1.1. Check identity (1.2).

1.2 Infinite volume measures

Ω being uncountable, defining a probability measure on it is not straightforward. The first point is to find what is the suitable σ-algebra. From the physical perspective, one can "only" do measurements on a finite number of spins a a time, so the natural thing to take is the smallest σ-algebra that allows such measurements.

Definition 1.2.1. For $(\Omega_0, \mathcal{F}_0)(= (\{-1,+1\}, \mathcal{P}(\{-1,1\})))$ a measurable space, define the sets of *cylinders*

$$\mathrm{Cyl}_\Lambda = \big\{ \times_{i\in\mathbb{Z}^d} A_i : \ A_i \in \mathcal{F}_0 \, \forall i \in \Lambda, \ A_i = \Omega_0 \, \forall i \in \Lambda^c \big\}, \quad \Lambda \Subset \mathbb{Z}^d.$$

Define then the *product σ-algebra* on $\Omega = \Omega_0^{\mathbb{Z}^d}$ as

$$\mathcal{F} = \mathcal{F}_0^{\otimes \mathbb{Z}^d} = \sigma\big(\cup_{\Lambda \Subset \mathbb{Z}^d} \mathrm{Cyl}_\Lambda \big) = \bigcap_{\mathcal{F}' \supset \cup_{\Lambda\Subset\mathbb{Z}^d} \mathrm{Cyl}_\Lambda \ \sigma-\text{algebra}} \mathcal{F}'.$$

In words, this is the smallest σ-algebra that allows to observe events depending on finitely many sites.

Infinite volume measures are then defined by lifting the compatibility condition 1.2 to infinite volume measures. This idea was put forth by Dobrushin, Lanford, and Ruelle (independently). The resulting formalism is the called *DLR formalism* and (1.3) is called the *DLR equation*.

Definition 1.2.2. An *infinite volume Gibbs measure* for the Ising model at parameters (β, h) is a probability measure μ on Ω (equipped with the product σ-algebra $\mathcal{F}$) such that for any $\Lambda \Subset \mathbb{Z}^d$,

$$\mu(A) = \mu\big(\mu^{\cdot}_{\Lambda;\beta,h}(A)\big), \quad \forall A \in \mathcal{F}. \tag{1.3}$$

Denote $\mathcal{G}(\beta, h)$ the set of infinite volume Gibbs measures.

Fact 1. To verify that two measures μ, ν on $(\Omega, \mathcal{F})$ are equal, it is sufficient to check that they agree on cylinders or that their expectation values agree on a set generating local functions (functions that depend on a finite number of sites).

Two very useful classes of local functions for the Ising model are

$$\sigma_V(\omega) = \prod_{i\in V} \omega_i, \quad n_V(\omega) = \prod_{i \in V} \mathbb{1}_{\omega_i = +1}, \quad V \Subset \mathbb{Z}^d. \tag{1.4}$$

Exercise 1.2. Show that the collections given in (1.4) both generate local functions. *Hint: work with functions supported on a fixed $V \Subset \mathbb{Z}^d$. Show that they can be expressed in terms of one-another, and that the σ_V's form an orthogonal basis for a suitable scalar product.*

Exercise 1.3. Remembering that product on finite spaces are compact, show that $|\mathcal{G}(\beta, h)|$ is non-empty. *Hint: fix some boundary condition ξ, and look at the sequence of measures $\mu^\xi_{\Lambda_L;\beta,h}$, $L \geq 1$, $\Lambda_L = \{-L, \dots, L\}^d$.*

The set $\mathcal{G}(\beta, h)$ has a very nice structure.

Fact 2 (Non-trivial)**.** $\mathcal{G}(\beta, h)$ is a Choquet simplex.

Exercise 1.4. Show that if $\mu, \mu' \in \mathcal{G}(\beta, h)$, then $t\mu + (1-t)\mu' \in \mathcal{G}(\beta, h)$ for all $t \in [0, 1]$.

1.3 Ensembles

In probabilistic terms, ensembles are conditional versions of the measures, where one fixes certain quantities to be intensive (be independent of the system size).

Definition 1.3.1 (Canonical ensemble)**.** The canonical ensemble allows to fix the magnetization density. Let $M \in \mathbb{Z}$ be a fixed number (the *total magnetization*).

Configuration space: $\Omega_\Lambda^\xi(M) = \{\omega \in \Omega : \omega_{\Lambda^c} = \xi_{\Lambda^c}, \sum_{i\in\Lambda} \omega_i = M\}$.

Partition functions: $Z_{\Lambda;\beta,M}^{\xi,\text{can}} = \sum_{\omega\in\Omega_\Lambda^\xi(M)} e^{\beta \sum_{i\sim j\in\Lambda} \omega_i\omega_j}$.

Finite volume measure: $\mu_{\Lambda;\beta,N}^{\xi,\text{can}}(A) = \frac{1}{Z_{\Lambda;\beta,N}^{\xi,\text{can}}} \sum_{n\in\Omega_\Lambda^\xi(N)} e^{\beta\sum_{i\sim j\in\Lambda} n_i n_j}$.

Definition 1.3.2 (Grand-canonical ensemble)**.** The grand-canonical ensemble fixes noting, every quantity is regulated using energy potentials.

Configuration space: $\Omega_\Lambda^\xi = \{n \in \Omega : n_{\Lambda^c} = \xi\}$.

Partition functions: $Z_{\Lambda;\beta,h}^{\xi,\text{GC}} = \sum_{\omega\in\Omega_\Lambda^\xi} e^{\beta\sum_{i\sim j\in\Lambda}\omega_i\omega_j + h\sum_{i\in\Lambda}\omega_i}$.

Finite volume measure: $\mu_{\Lambda;\beta,h}(A) = \frac{1}{Z_{\Lambda;\beta,h}^{\xi,\text{GC}}} \sum_{\omega\in\Omega_\Lambda^\xi} \mathbb{1}_A(\omega) e^{\beta\sum_{i\sim j\in\Lambda}\omega_i\omega_j + h\sum_{i\in\Lambda}\omega_i}$.

Remark 1.3.1. An important (trivial) identity is

$$Z_{\Lambda;\beta,\mu}^{\xi,\text{GC}} = \sum_{M=-|\Lambda|}^{|\Lambda|} Z_{\Lambda;\beta,M}^{\xi,\text{can}} e^{hM}.$$

In particular, the total magnetization in a box under $\mu_{\Lambda;\beta,h}^\xi$ is a discrete random variable with values in $\{-|\Lambda|, \ldots, |\Lambda|\}$ and probability mass function $p(M) = \frac{Z_{\Lambda;\beta,M}^{\xi,\text{can}} e^{hM}}{Z_{\Lambda;\beta,\mu}^{\xi,\text{GC}}}$.

Exercise 1.5. Show that

$$\frac{\partial}{\partial h} \ln\Big(Z_{\Lambda;\beta,h}^{\xi,\text{GC}}\Big) = \sum_{i\in\Lambda} \mu_{\Lambda;\beta,h}^\xi(\sigma_i), \quad \frac{\partial}{\partial \beta} \ln\Big(Z_{\Lambda;\beta,h}^{\xi,\text{GC}}\Big) = \sum_{i\sim j\in\Lambda} \mu_{\Lambda;\beta,h}^\xi(\sigma_i\sigma_j).$$

1.4 Thermodynamic functions

The name of the following two thermodynamic functions is natural from the *lattice gas* perspective: the lattice gas is obtained from the Ising model by putting a particle at places where the spin is $+1$, and no particles when the spin is -1.

Definition 1.4.1 (Pressure)**.** Let $\Lambda_L = \{-L, \ldots, L\}^d$. The limit

$$\Psi(\beta, h) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \ln\Big(Z_{\Lambda_L;\beta,h}^{\xi,\text{GC}}\Big), \tag{1.5}$$

is called the *pressure*. It exists and is independent of ξ, see Theorem 1.4.3.

Definition 1.4.2 (Free energy)**.** Let $\Lambda_L = \{-L, \dots, L\}^d$. For $m \in [-1, 1]$, the limit

$$\mathrm{f}(\beta, m) = -\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \ln\Big(Z^{\xi,\mathrm{can}}_{\Lambda_L;\beta,\lfloor m|\Lambda_L|\rfloor}\Big), \tag{1.6}$$

is called the *free energy*. It exists and is independent of ξ, see Theorem 1.4.1.

Theorem 1.4.1. *For any $\beta \geq 0, m \in [-1, 1]$, the limit (1.6) exists in $\mathbb{R}$ and is independent of ξ. Moreover, it is a convex, hence continuous, function of m.*

The existence will rely on (the generalization of) a very well known results about sequences: *Fekete's Lemma.*

Lemma 1.4.2 (Multi-dimensional Fekete)**.** *Let $a = (a_{L_1,\dots,L_d})_{L_1,\dots,L_d\geq 1}$ be a multi-index sequence. Suppose that a is* quasi-superadditive*: for any $i = 1, \dots, d$,*

$$a_{L_1,\dots,L_i+L_i',\dots,L_d} \geq a_{L_1,\dots,L_i,\dots,L_d} + a_{L_1,\dots,L_i',\dots,L_d} + g(\min(L_i, L_i'))$$

with $g(r) = o(r)$ as $r \to \infty$. Then,

$$\lim_{L_1,\dots,L_d\nearrow\infty} \frac{a_{L_1,\dots,L_d}}{\prod_{i=1}^d L_i}$$

exists in $(-\infty, +\infty]$.

Proof of Theorem 1.4.1. The idea both for convexity and existence uses the same observation. For Λ, Λ' disjoints, and M, M' one has

$$Z^{\xi,\mathrm{can}}_{\Lambda\cup\Lambda';\beta,M+M'} \geq Z^{\xi,\mathrm{can}}_{\Lambda;\beta,M} Z^{\xi}_{\Lambda';\beta,M'} e^{O(|\partial\Lambda|+|\partial\Lambda'|)}.$$

This gives that the sequence

$$a_{L_1,\dots,L_d} = -\ln\Big(Z^{\xi,\mathrm{can}}_{\Lambda_{L_1,\dots,L_d};\beta,\lfloor m|\Lambda_{L_1,\dots,L_d}|\rfloor}\Big)$$

with $\Lambda_{L_1,\dots,L_d} = \times_{k=1}^d \{-L_k, \dots, L_k\}$, is quasi-superadditive. One can then apply Lemma 1.4.2. Details are left as exercise for motivated readers. □

Exercise 1.6. Prove Lemma 1.4.2. Start with $d = 1$. *Hint: start with $g = 0$, write large values of $L_1, \dots, L_d$ using Euclidean division by any fixed number, and look at what sub-additivity gives you.*

Theorem 1.4.3. *For any $\beta \geq 0, h \in \mathbb{R}$, the limit (1.5) exists and is independent of ξ. Moreover, Ψ is convex, and Ψ, f are related by* Legendre-Fenchel duality*:*

$$\mathrm{f}(\beta, m) = \sup_{h\in\mathbb{R}} \big(hm - \Psi(\beta, h)\big), \quad \Psi(\beta, h) = \sup_{m\in[-1,1]} \big(mh - \mathrm{f}(\beta, m)\big). \tag{1.7}$$

Proof. One can proceed via a sub-additivity argument as in the proof of Theorem 1.4.1 to get existence, and proving (1.7) after. We will take a more direct path towards (1.7), which shows both existence and (1.7) at the same time.

Remembering 1.3.1, one has the bounds

$$\max_{M\in\{-|\Lambda|,\dots,|\Lambda|\}} Z^{\xi,\mathrm{can}}_{\Lambda;\beta,M} e^{hM} \le Z^{\xi,\mathrm{GC}}_{\Lambda;\beta,h} \le (|\Lambda|+1) \max_{M\in\{-|\Lambda|,\dots,|\Lambda|\}} Z^{\xi,\mathrm{can}}_{\Lambda;\beta,M} e^{hM}.$$

In particular,

$$\begin{aligned}\limsup_{L\to\infty} \frac{1}{|\Lambda_L|} \ln\Big(Z^{\xi,\mathrm{GC}}_{\Lambda_L;\beta,h}\Big) &\le \limsup_{L\to\infty} \max_{-|\Lambda_L|\le M\le|\Lambda_L|} \Big(\frac{Mh}{|\Lambda_L|} + \frac{1}{|\Lambda_L|}\ln\Big(Z^{\xi,\mathrm{can}}_{\Lambda_L;\beta,M}\Big)\Big)\\ &= \sup_{m\in[-1,1]} \big(mh - \mathrm{f}(\beta,m)\big),\end{aligned}$$

and

$$\begin{aligned}\liminf_{L\to\infty} \frac{1}{|\Lambda_L|} \ln\Big(Z^{\xi,\mathrm{GC}}_{\Lambda_L;\beta,h}\Big) &\ge \liminf_{L\to\infty} \max_{-|\Lambda_L|\le M\le|\Lambda_L|} \Big(\frac{Mh}{|\Lambda_L|} + \frac{1}{|\Lambda_L|}\ln\Big(Z^{\xi,\mathrm{can}}_{\Lambda_L;\beta,M}\Big)\Big)\\ &= \sup_{m\in[-1,1]} \big(mh - \mathrm{f}(\beta,m)\big).\end{aligned}$$

In particular, $\Psi(\beta,h)$ exists and is equal to $\sup_{m\in[-1,1]}\big(mh-\mathrm{f}(\beta,m)\big)$.

The convexity is left as an exercise. Convexity and continuity implies that the Legendre-Fenchel dual of the dual is the function itself, see Exercises. □

Exercise 1.7. Show that for $\Lambda \Subset \mathbb{Z}^d$, $\xi\in\Omega$, $\Psi_\Lambda(\beta,h) = \frac{1}{|\Lambda|}\ln\Big(Z^{\xi,\mathrm{GC}}_{\Lambda;\beta,h}\Big)$ is convex. Deduce that Ψ is convex. *Hint: remember Hölder's inequality.*

Exercise 1.8. Suppose $g:\mathbb{R}\to\mathbb{R}$ is convex and continuous. Define $g^*:\mathbb{R}\to\mathbb{R}$ by

$$g^*(x) = \sup_{y\in\mathbb{R}}(xy - g(y)).$$

Show that $(g^*)^* = g$.
For motivated people: extend to $g:(-\infty,+\infty]\to(-\infty,+\infty]$ convex and lower-semi-continuous.

Remark 1.4.1. One can take much more permissive limits: by approximating any sequence of volume that has boundary negligible compare to its size via a union of hyper-cubes, one can show that the same limit is obtained by taking the limit of $\frac{1}{|\Lambda|}\ln\Big(Z^{\xi,\mathrm{GC}}_{\Lambda;\beta,\mu}\Big)$ along any Λ growing to Z^d with a boundary negligible compare to its volume, and the same of the canonical ensemble. in the same fashion, in the free energy definition, one can take any sequences of magnetisation densities that converges to m. See [18, 20, 10] for more details.

1.5 Correlation inequalities and consequences

1.5.1 A family of probability measures

Consider $G=(\Lambda,J)$ a (un-directed) weighted graph: Λ is a finite set, and $J:\Lambda^2\to[0,+\infty)$ satisfies

$$J_{ij} = J_{ji}, \quad J_{ii} = 0.$$

For $\underline{h} \in \mathbb{R}^\Lambda$, introduce the probability measures $\mu_{J,\underline{h}}$ on $\{-1,1\}^\Lambda$ via

$$\mu_{J,\underline{h}}(\{\omega\}) \propto \exp\Big(\tfrac{1}{2}\sum_{i,j\in\Lambda} J_{ij}\omega_i\omega_j + \sum_{i\in\Lambda} h_i\omega_i\Big). \tag{1.8}$$

Exercise 1.9. Show that for any $\Lambda \Subset \mathbb{Z}^d$, $\xi \in \Omega$, $\beta \geq 0, h \in \mathbb{R}$, $\mu^\xi_{\Lambda;\beta,h}$ (its restriction to Ω_Λ) is of the form (1.8).

1.5.2 GKS inequality

Recall the correlation functions

$$\sigma_V(\omega) = \prod_{i\in V}\omega_i.$$

Theorem 1.5.1 (Griffiths-Kelly-Sherman inequalities)**.** *In the setup of Section 1.5.1, if $h_i \geq 0$ for all $i \in \Lambda$,*

1. *$\mu_{J,\underline{h}}(\sigma_V) \geq 0$ for all $V \subset \Lambda$;*
2. *$\mu_{J,\underline{h}}(\sigma_V\sigma_W) \geq \mu_{J,\underline{h}}(\sigma_V)\mu_{J,\underline{h}}(\sigma_W)$ for all $V, W \subset \Lambda$.*

Proof. Introduce

$$Z_{J,\underline{h}}(f) := \sum_{\omega\in\{-1,1\}^\Lambda} f(\omega)e^{\sum_{\{i,j\}\subset\Lambda} J_{ij}\omega_i\omega_j + \sum_{i\in\Lambda} h_i\omega_i}.$$

Then, $\mu_{J,\underline{h}}(\sigma_V) = \frac{Z_{J,\underline{h}}(\sigma_V)}{Z_{J,\underline{h}}(1)}$. As $Z_{J,\underline{h}}(1) > 0$, the first inequality is equivalent to show that $Z_{J,\underline{h}}(\sigma_V) \geq 0$. Using a Taylor expansion of the exponential, one gets

$$\begin{aligned}
Z_{J,\underline{h}}(\sigma_V) &= \sum_{\omega\in\{-1,1\}^\Lambda}\prod_{i\in V}\omega_i \prod_{\{i,j\}\subset\Lambda}\Big(\sum_{n\geq 0}\frac{J_{ij}^n}{n!}\omega_i^n\omega_j^n\Big)\prod_{i\in\Lambda}\Big(\sum_{n\geq 0}\frac{h_i^n}{n!}\omega_i^n\Big) \\
&= \sum_{m\in\mathbb{N}^\Lambda}\sum_{n\in\mathbb{N}^{E_\Lambda}}\prod_{\{i,j\}\in E_\Lambda}\frac{J_{ij}^{n_{ij}}}{n_{ij}!}\prod_{i\in\Lambda}\frac{h_i^{m_i}}{m_i!}\prod_{i\in\Lambda}\Big(\sum_{\omega_i\in\{-1,1\}}\omega_i^{\sum_{j\neq i} n_{ij}+m_i+\mathbb{1}_V(i)}\Big)
\end{aligned}$$

where $E_\Lambda = \{\{i,j\} \subset \Lambda\}$. Every term in the double sum is non-negative as $1^a + (-1)^a \geq 0$ for all integers a.

The second inequality is equivalent to $Z_{J,\underline{h}}(\sigma_V\sigma_W)Z_{J,\underline{h}}(1) \geq Z_{J,\underline{h}}(\sigma_V)Z_{J,\underline{h}}(\sigma_W)$. Now,

$$\begin{aligned}
&Z_{J,\underline{h}}(\sigma_V\sigma_W)Z_{J,\underline{h}}(1) - Z_{J,\underline{h}}(\sigma_V)Z_{J,\underline{h}}(\sigma_W) \\
&= \sum_{\omega,\omega'} e^{\sum_{\{i,j\}} J_{ij}(\omega_i\omega_j+\omega'_i\omega'_j)+\sum_i h_i(\omega_i+\omega'_i)}\prod_{i\in V}\omega_i\Big(\prod_{j\in W}\omega_j - \prod_{j\in W}\omega_j\Big) \\
&= \sum_{\tau\in\{-1,1\}^\Lambda}\sum_{\omega\in\{-1,1\}^\Lambda} e^{\sum_{\{i,j\}} J_{ij}\omega_i\omega_j(1+\tau_i\tau_j)+\sum_i h_i\omega_i(1+\tau_i)}\prod_{i\in V}\omega_i\prod_{j\in W}\omega_j\Big(1-\prod_{j\in W}\tau_j\Big) \\
&= \sum_{\tau\in\{-1,1\}^\Lambda}\Big(1-\prod_{j\in W}\tau_j\Big)Z_{J^{(\tau)},\underline{h}^{(\tau)}}(\sigma_V\sigma_W)
\end{aligned}$$

where we changed variables to $(\omega_i, \tau_i)_{i\in\Lambda} = (\omega_i, \omega_i\omega'_i)_{i\in\Lambda}$, and

$$J_{ij}^{(\tau)} = J_{ij}(1+\tau_i\tau_j) \geq 0, \quad h_i^{(\tau)} = h_i(1+\tau_i) \geq 0.$$

As $1 - \prod_{j\in W} \tau_j \geq 0$ for all $\tau \in \{-1,1\}^\Lambda$, we can apply the first point to conclude. □

Exercise 1.10. Show that for $h \geq 0$, $0 \in \Lambda \Subset \mathbb{Z}^d$, $(\beta, h) \mapsto \mu_{\Lambda;\beta,h}(\sigma_0)$ is non-decreasing in both β and h.

1.5.3 FKG inequality

The space $\{-1,+1\}^\Lambda$ is naturally equipped with a partial order:

$$\omega \leq \omega' \text{ if } \omega_i \leq \omega'_i \text{ for all } i \in \Lambda.$$

A function $f : \{-1,1\}^\Lambda \to \mathbb{R}$ is called *non-decreasing* if $\omega \leq \omega'$ implies $f(\omega) \leq f(\omega')$.

Let μ, μ' be two probability measures on $\{-1,+1\}$. Say that μ *is stochastically dominated by* μ', denoted $\mu \preccurlyeq \mu'$ if for any non-decreasing function f,

$$\mu(f) \leq \mu'(f).$$

Theorem 1.5.2 (Fortuin-Kastelyn-Ginibre inequality)**.** *In the setup of Section 1.5.1, for any $f, g : \{-1,1\}^\Lambda \to \mathbb{R}$ non-decreasing,*

$$\mu_{J,\underline{h}}(fg) \geq \mu_{J,\underline{h}}(f)\mu_{J,\underline{h}}(g).$$

Corollary 1.5.3. *For any $\underline{h} \leq \underline{h}'$, $\mu_{J,\underline{h}} \preccurlyeq \mu_{J,\underline{h}'}$. In particular, for any $\xi \leq \xi' \in \Omega$, any $\Lambda' \subset \Lambda \Subset \mathbb{Z}^d$, any $h \leq h' \in \mathbb{R}$, and any $\beta \geq 0$,*

$$\mu^-_{\Lambda';\beta,h} \preccurlyeq \mu^\xi_{\Lambda;\beta,h} \preccurlyeq \mu^{\xi'}_{\Lambda;\beta,h'} \preccurlyeq \mu^+_{\Lambda';\beta,h'}.$$

Exercise 1.11. Deduce Corollary 1.5.3 from Theorem 1.5.2.

All definitions extend to infinite volume measures in the obvious way. The condition for stochastic ordering is only on local functions (functions that depend on finitely many coordinates).

1.5.4 Construction of FKG-extremal measures

We will now use the FKG inequality, more precisely Corollary 1.5.3, to construct two infinite volume measures (which might be equal). They play an extremely important role in the study of the Ising model, and are one of the "miracles" that makes working with the Ising model so pleasant.

Construction of FKG-extremal states

The easy part of the construction is the construction of infinite volume *states*.

Definition 1.5.1 (Infinite volume Gibbs state)**.** Let Loc be the set of local functions:

$$\mathrm{Loc} := \{f : \Omega \to \mathbb{R} : \ \exists \Lambda \Subset \mathbb{Z}^d,\ f(\omega) = f(\omega')\, \forall \omega, \omega' \in \Omega\, \omega_{\Lambda^c} = \omega'_{\Lambda^c}\}.$$

A *Gibbs state* at parameters β, h is a linear functional $\langle \cdot \rangle : \mathrm{Loc} \to \mathbb{R}$ that satisfies

- $\langle 1 \rangle = 1$, $\langle 0 \rangle = 0$;
- $\langle f \rangle \geq 0$ for all non-negative function $f \in \mathrm{Loc}$;
- $\langle \mu^{\cdot}_{\Lambda;\beta,h}(f) \rangle = \langle f \rangle$ for all $f \in \mathrm{Loc}$, and all $\Lambda \Subset \mathbb{Z}^d$.

In words, a state is a version of expectation that only applies to local functions. From a state, a classical result in functional analysis, the *Riesz-Markov-Kakutani Theorem* ensures that there is a unique probability measure μ on Ω equipped with the product σ-algebra such that $\mu(f) = \langle f \rangle$ for all $f \in \mathrm{Loc}$. Moreover, the third constraint on the Gibbs state $\langle \cdot \rangle$ ensures that $\mu \in \mathcal{G}(\beta, h)$.

Let us construct two Gibbs states. By Corollary 1.5.3, for any $V \Subset \mathbb{Z}^d$,

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{+}_{\Lambda;\beta,h}(n_V)$$

exists in $[0, 1]$ (as limits of non-increasing sequences), and does not depend on the sequence of volumes. In particular, one can use this to construct an infinite volume state $\langle \cdot \rangle^{+}_{\beta,h}$ by setting

$$\langle n_V \rangle^{+}_{\beta,h} = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mu^{+}_{\Lambda;\beta,h}(n_V), \quad V \Subset \mathbb{Z}^d.$$

Then, as any function $f \in \mathrm{Loc}$ can be written as $f = \sum_{k=1}^{m} \alpha_k n_{V_k}$ for some $m \geq 1$, some $V_1, \ldots, V_m \Subset \mathbb{Z}^d$, and some $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$, one can set

$$\langle f \rangle^{+}_{\beta,h} = \sum_{k=1}^{m} \alpha_k \langle n_{V_k} \rangle^{+}_{\beta,h}.$$

It is then direct to check that this provides a linear function on Loc. The Gibbs state property follows from the fact that it is a limit of $\mu^{+}_{\Lambda;\beta,h}$ that satisfy the compatibility condition.

In the same fashion, one can construct $\langle \cdot \rangle^{-}_{\beta,h}$ using $\mu^{-}_{\Lambda;\beta,h}$. Let $\mu^{+}_{\beta,h}, \mu^{-}_{\beta,h}$ be the Gibbs measures associated to these two states.

Exercise 1.12. Check that $\langle \cdot \rangle^{+}_{\beta,h}, \langle \cdot \rangle^{-}_{\beta,h}$ are Gibbs states. When $h = 0$, show that if X has law $\mu^{+}_{\beta,0}$, then $-X$ has law $\mu_{\beta,0^-}$.

Exercise 1.13. Check that $\mu^{+}_{\beta,h}, \mu^{-}_{\beta,h}$ are FKG: for any $f, g \in \mathrm{Loc}$ non-decreasing,

$$\mu^{+}_{\beta,h}(fg) \geq \mu^{+}_{\beta,h}(f)\mu^{+}_{\beta,h}(g).$$

Check that for any $\mu \in \mathcal{G}(\beta, h)$, $\mu^{-}_{\beta,h} \preccurlyeq \mu \preccurlyeq \mu^{+}_{\beta,h}$. *Hint: use the DLR equation and the fact that μ^+, μ^- are defined in terms of limits of finite volume measures.*

1.5.5 Properties of μ^+, μ^-

Lemma 1.5.4. *$\mu^+_{\beta,h}, \mu^-_{\beta,h}$ are translation invariant:*

$$\mu^+_{\beta,h}(\sigma_V) = \mu^+_{\beta,h}(\sigma_{x+V}) \quad \forall x \in \mathbb{Z}^d,$$

and the same for $-$, and have decaying covariances:

$$\lim_{|x|\to\infty} \mu^+_{\beta,h}(\sigma_V; \sigma_{x+W}) = \mu^+_{\beta,h}(\sigma_V \sigma_{x+W}) - \mu^+_{\beta,h}(\sigma_V)\mu^+_{\beta,h}(\sigma_W) = 0, \quad \forall V, W \subset \mathbb{Z}^d,$$

and the same for $-$.

Note that as the σ_V's are generating local functions, the same decoupling holds for any local functions f, g replacing σ_V, σ_W. Note also that we could use the n_V's instead of the σ_V's and obtain an equivalent statement.

Proof. As $\mu^+_{\beta,h} = \lim_{\Lambda\to\mathbb{Z}^d} \mu^+_{\Lambda;\beta,h}$ does not depend on the sequence of volume, we have that for any $V \Subset \mathbb{Z}^d$, and any $x \in \mathbb{Z}^d$,

$$\mu^+_{\beta,h}(\sigma_{V+x}) = \lim_{L\to\infty} \mu^+_{\Lambda_L+x;\beta,h}(\sigma_{V+x}) = \lim_{L\to\infty} \mu^+_{\Lambda_L;\beta,h}(\sigma_V) = \mu^+_{\beta,h}(\sigma_V),$$

where we used that the family of finite volume measures are translation covariant: $\mu^\xi_{\Lambda;\beta,h}(A) = \mu^{\theta_x\xi}_{\Lambda+x;\beta,h}(\theta_x A)$ with θ_x the translation by x, which follows from the translation invariance of the Hamiltonian.

We show the decay of covariances for the n_V's as other local functions are finite linear combination of them. For $V, W \Subset \mathbb{Z}^d$, and $x \in \mathbb{Z}^d$ we have that

$$\begin{aligned}\mu^+_{\beta,h}(n_V n_{W+x}) = \lim_{L\to\infty} \mu_{\Lambda_L;\beta,h}(n_V n_{W+x}) &\geq \lim_{L\to\infty} \mu^+_{\Lambda_L;\beta,h}(n_V)\mu^+_{\Lambda_L;\beta,h}(n_{W+x}) \\ &= \mu^+_{\Lambda_L;\beta,h}(n_V)\mu^+_{\beta,h}(n_{W+x}) = \mu^+_{\Lambda_L;\beta,h}(n_V)\mu^+_{\beta,h}(n_W),\end{aligned}$$

by the FKG inequality. Now, for $K_x = \lfloor |x|/3 \rfloor$, we have that for any x with $|x|$ large enough,

$$V \subset \Lambda_{K_x}, \quad W + x \subset x + \Lambda_{K_x}, \quad \mathrm{dist}\big(\Lambda_{K_x}, (x + \Lambda_{K_x}\big) > 100.$$

In particular, using again monotonicity (FKG),

$$\begin{aligned}\lim_{|x|\to\infty} \mu^+_{\beta,h}(n_V n_{W+x}) &\leq \lim_{|x|\to\infty} \mu^+_{\Lambda_{K_x}\cup(x+\Lambda_{K_x});\beta,h}(n_V n_{W+x}) \\ &= \lim_{|x|\to\infty} \mu^+_{\Lambda_{K_x};\beta,h}(n_V)\mu^+_{x+\Lambda_{K_x};\beta,h}(n_{W+x}) \\ &= \lim_{|x|\to\infty} \mu^+_{\Lambda_{K_x};\beta,h}(n_V)\mu^+_{\Lambda_{K_x};\beta,h}(n_W) = \mu^+_{\beta,h}(n_V)\mu^+_{\beta,h}(n_W).\end{aligned}$$

Combining the two bounds gives the wanted decay of covariances. □

1.5.6 Relation between μ^+, μ^- and Ψ

From the previous Section, we get that if $\mu^+ = \mu^-$, then $|\mathcal{G}(\beta, h)| = 1$ (no first order phase transition). The next results links this property of Gibbs measures to differentiability properties of Ψ.

Lemma 1.5.5. *For any $\beta \geq 0$, the function $h \mapsto \Psi(\beta, h)$ is non-differentiable at at most countable many points. Letting $\mathcal{D}_\beta$ be the set of h's where it is differentiable, one has*

$$\mu^-_{\beta,h}(\sigma_0) = \frac{\partial}{\partial h}\Psi(\beta, h) = \mu^+_{\beta,h}(\sigma_0).$$

Proof. Let $\Psi_\beta(h) \equiv \Psi(\beta, h)$. As $h \mapsto \Psi_\beta(h)$ is a convex function, it is differentiable at all but at most countable many values of h. Denote $\mathcal{D}_\beta$ the set of values of h where Ψ_β is differentiable. For those values,

$$\begin{aligned}
\Psi'_\beta(h) &= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\frac{\partial}{\partial h}\ln\big(Z^+_{\Lambda_L;\beta,h}\big) \\
&= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_L} \mu^+_{\Lambda_L;\beta,h}(\sigma_i) \geq \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_L} \mu^+_{\beta,h}(\sigma_i) = \mu^+_{\beta,h}(\sigma_0)
\end{aligned}$$

where we used that the derivative of a differentiable convex function that is the limit of differentiable convex functions is the limit of the derivatives, and that $\mu^+_{\beta,h}$ is translation invariant. Now, for any $K \geq 1$,

$$\begin{aligned}
\lim_{L\to\infty} \frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_L} \mu^+_{\Lambda_L;\beta,h}(\sigma_i) &\leq \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\Big(|\Lambda_L \setminus \Lambda_{L-K}| + \sum_{i\in\Lambda_{L-K}} \mu^+_{i+\Lambda_K;\beta,h}(\sigma_i)\Big) \\
&= \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_{L-K}} \mu^+_{\Lambda_K;\beta,h}(\sigma_0) = \mu^+_{\Lambda_K;\beta,h}(\sigma_0).
\end{aligned}$$

One can then take the infimum over K and use that, by Corollary 1.5.3, $\inf_{K\geq 1} \mu^+_{\Lambda_K;\beta,h}(\sigma_0) = \lim_{K\to\infty} \mu^+_{\Lambda_K;\beta,h}(\sigma_0) = \mu^+_{\beta,h}(\sigma_0)$ to obtain $\Psi'_\beta(h) \leq \mu^+_{\beta,h}(\sigma_0)$, and thus $\Psi'_\beta(h) = \mu^+_{\beta,h}(\sigma_0)$.

Proceeding in the same fashion using μ^- instead of μ^+, one obtains $\Psi'_\beta(h) = \mu^-_{\beta,h}(\sigma_0)$.

□

With a little more work (still using monotonicity and convexity), one can show the following more refined result.

Lemma 1.5.6. *For any $\beta \geq 0$, $h \in \mathbb{R}$, one has*

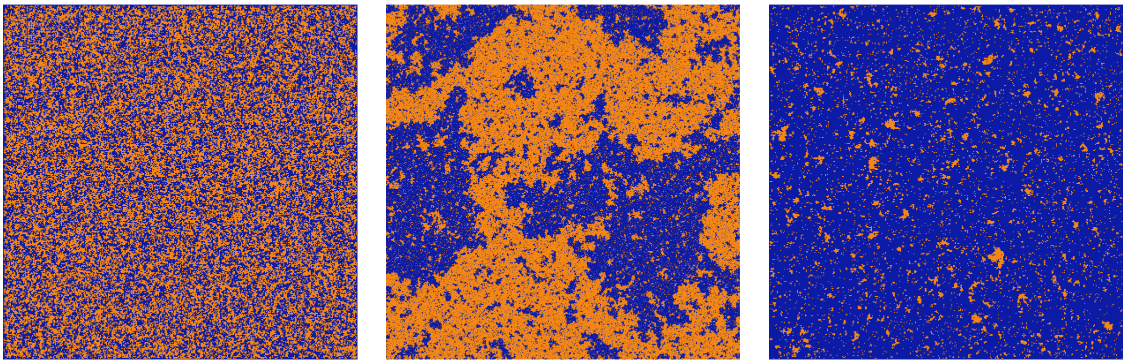
$$\frac{\partial}{\partial h^+}\Psi(\beta, h) = \mu^+_{\beta,h}(\sigma_0), \qquad \frac{\partial}{\partial h^-}\Psi(\beta, h) = \mu^-_{\beta,h}(\sigma_0).$$

Exercise 1.14. Show that $\mu^+_{\beta,h}(\sigma_0) = \mu^-_{\beta,h}(\sigma_0)$ implies $\mu^+_{\beta,h} = \mu^-_{\beta,h}$, and thus $|\mathcal{G}(\beta, h)| = 1$. *Hint: you only need to prove $\mu^+_{\beta,h}(n_V) = \mu^-_{\beta,h}(n_V)$ for all $V \Subset \mathbb{Z}^d$. Also, note that $\sum_{i\in V} n_i - n_V$ is an increasing function. Try to compare its expectation under $\mu^+_{\beta,h}$ and under $\mu^-_{\beta,h}$.*

1.6 Phase transitions

1.6.1 Definitions of phase transitions

Loosely speaking, a phase transition is an abrupt change of behaviour of a system as some parameters are varied through a certain threshold.



(a) High temperature (b) Curie temperature (c) Low temperature

Figure 1.1: Samples of the Ising model at 0 magnetic field, going from a disordered (paramagnetic) state to a globally ordered (ferromagnetic) state as temperature decreases.

The classical way to look at phase transitions is through *regularity properties of the pressure*. We say that (β, h) is

Not a phase transition point if Ψ is analytic in both variables at (β, h).

A first order transition point if Ψ is not differentiable at (β, h).

A critical point if Ψ is differentiable but not analytic at (β, h).

Another way to look at phase transitions is through the structure of $\mathcal{G}(\beta, h)$. In this case, one says that there is a *first order phase transition* at (β, h) if $|\mathcal{G}(\beta, h)| > 1$. It is a general feature that this definition of first order points is equivalent to the one using the pressure. To grasp higher order transitions, one needs to look further at the finer properties of the Gibbs measure.

1.6.2 Existence of Curie temperature

Note that $h \mapsto \Psi(\beta, h)$ being convex, it is almost everywhere differentiable. In particular, by Lemma 1.5.5 and the Exercise following it, one has that $|\mathcal{G}(\beta, h)| = 1$ for almost every h. Introduce the *spontaneous magnetisation*

$$m^*(\beta) = \lim_{h \searrow 0} \mu_{\beta,h}(\sigma_0) = \lim_{h \searrow 0} \frac{\partial}{\partial h} \Psi(\beta, h) = \frac{\partial}{\partial h^+} \Psi(\beta, h)\Big|_{h=0} = \mu^+_{\beta,0}(\sigma_0), \tag{1.9}$$

where the first two limits are taken along any sequence of h's such that $|\mathcal{G}(\beta, h)| = 1$, and $\mu_{\beta,h}$ denotes the unique infinite volume Gibbs measure.

Let $\mu^+_{\Lambda;\beta} \equiv \mu^+_{\Lambda;\beta,0}$. As the functions $\mu^+_{\Lambda;\beta}(\sigma_0)$ are non-decreasing in β (by GKS inequality), one has that $m^*(\beta) = \mu^+_\beta(\sigma_0)$ is non-decreasing in β. In particular, defining the *inverse Curie temperature*

$$\beta_c \equiv \beta_c(d) = \inf\{\beta \geq 0 : \ m^*(\beta) > 0\}, \tag{1.10}$$

one has

- for any $\beta < \beta_c$, Ψ is smooth at $(\beta, 0)$, $|\mathcal{G}(\beta, 0)| = 1$, and $m^*(\beta) = 0$;
- for any $\beta > \beta_c$, Ψ is non-differential at $(\beta, 0)$, $|\mathcal{G}(\beta, 0)| = +\infty$, and $m^*(\beta) > 0$.

In particular, the Ising model provides a "decent" model for the ferro/para-magnetic transition provided $\beta_c \in (0, +\infty)$.

1.6.3 No phase transition at high temperature

Theorem 1.6.1. *For $d \geq 1$, there is $\beta_0 > 0$ such that for all $0 \leq \beta < \beta_0$,*

$$\mu_\beta^+(\sigma_0) = m^*(\beta) = 0.$$

In particular,

- $\partial_{h^+}\Psi(\beta, 0) = \partial_{h^-}\Psi(\beta, 0)$ *and Ψ is differentiable in h at 0.*
- $\mu_\beta^+ = \mu_\beta^-$*, and there is therefore only one infinite volume Gibbs measure at $(\beta, 0)$.*

If moreover $d = 1$, $\beta_0 = +\infty$.

Proof. The proof of this result feature a first *graphical expansion* of the Ising model: the *high temperature expansion.* This expansion is based on the identity: for $\omega_i, \omega_j \in \{-1, 1\}$,

$$e^{\beta\omega_i\omega_j} = \cosh(\beta) + \omega_i\omega_j \sinh(\beta) = \cosh(\beta)(1 + \omega_i\omega_j \tanh(\beta)).$$

Denote

$$\bar{E}_\Lambda = \big\{\{i, j\} \subset \mathbb{Z}^d : \ |i - j| = 1, \{i, j\} \cap \Lambda \neq \varnothing\big\}.$$

For $\eta \subset \bar{E}_\Lambda$, denote

$$\deg_i(\eta) = \sum_{j\in\bar{\Lambda}:j\sim i} \mathbb{1}_{\{i,j\}\in\eta}$$

the degree of i in the graph $(\bar{\Lambda}, \eta)$. Define

$$\partial\eta = \{i \in \bar{\Lambda} : \ \deg_i(\eta) \text{ is odd}\}.$$

Then, one has the expansion

$$\begin{aligned}
\mu_{\Lambda;\beta}^+(\sigma_0) &= \frac{\sum_{\omega\in\Omega_\Lambda^+} \omega_0 \prod_{i\sim j} e^{\beta\omega_i\omega_j}}{\sum_{\omega\in\Omega_\Lambda^+} \prod_{i\sim j} e^{\beta\omega_i\omega_j}} \\
&= \frac{\cosh(\beta)^{|\bar{E}_\Lambda|} \sum_{\eta\subset\bar{E}_\Lambda} \sum_{\omega\in\Omega_\Lambda^+} \omega_0 \prod_{\{i,j\}\in\eta} \tanh(\beta)\omega_i\omega_j}{\cosh(\beta)^{|\bar{E}_\Lambda|} \sum_{\eta\subset\bar{E}_\Lambda} \sum_{\omega\in\Omega_\Lambda^+} \prod_{\{i,j\}\in\eta} \tanh(\beta)\omega_i\omega_j} \\
&= \frac{\sum_{\eta\subset\bar{E}_\Lambda} \tanh(\beta)^{|\eta|} \sum_{\omega\in\Omega_\Lambda^+} \prod_{i\in\bar{\Lambda}} \omega_i^{\deg_i(\eta)+\mathbb{1}_{i=0}}}{\sum_{\eta\subset\bar{E}_\Lambda} \tanh(\beta)^{|\eta|} \sum_{\omega\in\Omega_\Lambda^+} \prod_{i\in\bar{\Lambda}} \omega_i^{\deg_i(\eta)}}.
\end{aligned} \tag{1.11}$$

Now, the key (simple) observation is

$$\sum_{\omega\in\Omega_\Lambda^+} \prod_{i\in\bar{\Lambda}} \omega_i^{\deg_i(\eta)} = 2^{|\Lambda|}\mathbb{1}_{\partial\eta\subset\partial\Lambda}, \qquad \sum_{\omega\in\Omega_\Lambda^+} \prod_{i\in\bar{\Lambda}} \omega_i^{\deg_i(\eta)+\mathbb{1}_{i=0}} = 2^{|\Lambda|}\mathbb{1}_{0\in\partial\eta\subset\partial\Lambda}.$$

In particular, (1.11) yields

$$\mu_{\Lambda;\beta}^+(\sigma_0) = \frac{\sum_{\eta\subset\bar{E}_\Lambda:\,0\in\partial\eta\subset\partial\Lambda} \tanh(\beta)^{|\eta|}}{\sum_{\eta\subset\bar{E}_\Lambda:\,\partial\eta\subset\partial\Lambda} \tanh(\beta)^{|\eta|}}.$$

The next observation is that for any η with $0 \in \partial\eta \subset \partial\Lambda$, there is an edge-self-avoiding path from $0 \to \partial\Lambda$ using only edges in η. We define $\gamma_*(\eta)$ as a sequence of (directed) edges as follows.

1. Fix a total order of $\bar{E}_\Lambda$. Set $k=0$, $i_0=0$, and $\eta^{(0)}=\eta$.
2. Let $e_k=\{i_k,i_{k+1}\}$ be the smallest edge in $\eta^{(k)}$ having i_k as an endpoint.
3. Update $\eta^{(k+1)}=\eta^{(k)}\setminus\{i_k,i_{k+1}\}$.
4. If $i_{k+1}\notin\partial\Lambda$, update $k=k+1$, and repeat 2. to 4.. Else, go to 5..
5. Return $\gamma_*(\eta)=(i_0=0,i_1,\dots,i_{k+1})$.

Note that for a given path $\gamma:0\to\partial\Lambda$, having $\gamma_*(\eta)=\gamma$ implies that some edges (those smaller than the ones of γ but sharing an endpoint with one of the edges in γ) were never chosen in the exploration process, and therefore must have been not present in η. Denote $\bar{\gamma}$ the smallest set of edges such that $f(\eta)=\mathbb{1}_{\gamma_*(\eta)=\gamma}$ can be determined only by looking at whether each of the edges of $\bar{\gamma}$ is present in η or not. Then, noting that for η with $0\in\partial\eta\subset\partial\Lambda$, $\partial(\eta\setminus\gamma_*(\eta))\subset\partial\Lambda$,

$$\begin{aligned}\mu^+_{\Lambda;\beta}(\sigma_0)&=\sum_{\gamma:0\to\partial\Lambda}\tanh(\beta)^{|\gamma|}\frac{\sum_{\eta\subset\bar{E}_\Lambda\setminus\bar{\gamma}:\,\eta\subset\partial\Lambda}\tanh(\beta)^{|\eta|}}{\sum_{\eta\subset\bar{E}_\Lambda:\,\partial\eta\subset\partial\Lambda}\tanh(\beta)^{|\eta|}}\\&\le\sum_{\gamma:0\to\partial\Lambda}\tanh(\beta)^{|\gamma|},\end{aligned}$$

where sums over γ are over edge-self-avoiding paths. Finally, the number of edge-self-avoiding paths of length n is at most $2d(2d-1)^{n-1}$ ($2d$ choices for the first step, $2d-1$ for the next ones as backtrack is not allowed), we get

$$\begin{aligned}\mu^+_{\Lambda_L;\beta}(\sigma_0)&\le\sum_{n\ge L}\sum_{\gamma:0\to\partial\Lambda_L,\,|\gamma|=n}\tanh(\beta)^{|\gamma|}\\&\le\frac{2d}{2d-1}\sum_{n\ge L}\tanh(\beta)^n(2d-1)^n.\end{aligned}$$

Taking β small enough in $d\ge 2$ gives a term that goes to 0 with L, whilst when $d=1$,

$$\mu^+_{\Lambda_L;\beta}(\sigma_0)=\sum_{\gamma:0\to\partial\Lambda}\tanh(\beta)^{|\gamma|}\frac{\sum_{\eta\subset\bar{E}_\Lambda\setminus\bar{\gamma}:\,\eta\subset\partial\Lambda}\tanh(\beta)^{|\eta|}}{\sum_{\eta\subset\bar{E}_\Lambda:\,\partial\eta\subset\partial\Lambda}\tanh(\beta)^{|\eta|}}=\frac{2\tanh(\beta)^L}{1+\tanh(\beta)}.$$

□

1.6.4 First order phase transition at low temperature

Theorem 1.6.2. *For $d\ge 2$, there is $\beta_0<\infty$ such that for all $\beta>\beta_0$,*

$$\mu^+_\beta(\sigma_0)=m^*(\beta)>0.$$

In particular,

- $\partial_{h^+}\Psi(\beta,0)>0>\partial_{h^-}\Psi(\beta,0)$ *and Ψ is therefore not differentiable in h at 0.*
- $\mu^+_{\beta,0}\neq\mu^-_{\beta,0}$, *and there is therefore more than one (infinitely many) infinite volume Gibbs measure at $(\beta,0)$.*

Proof. Consider $d = 2$. Use the shorthand $\mu^+_{L;\beta} \equiv \mu^+_{\Lambda_L;\beta,0}$. The set Ω^+_Λ is in a one-to-one mapping with the set of *Peierls contours*: lines separating the $+1$'s from the -1's (see Figure 1.2). More formally, let Λ^*_L be the graph dual to Λ_L (see Figure 1.3). Then, a contour configuration is an edge-subgraph of Λ^*_L so that all vertices have even degrees. Denote $\mathcal{E}_L$ the set of such graphs.

- The spins of the Ising configuration are recovered from $\eta \in \mathcal{E}_L$ by setting them to be $\omega_i(\eta) = +1$ for $i \in \Lambda^c$, and $\omega_i(\eta) = (-1)^{\text{crossed}_i(\eta)}$ where $\text{crossed}_i(\eta)$ is the number of edges that are crossed by any fixed path going from Λ^c_L to i. This number can depend on the path, but its parity does not.

- The contour configuration is obtained from the spins by setting $\eta(\omega)$ to be the set of edges dual to edges of the form $\{i, j\}$ with $\omega_i\omega_j = -1$.

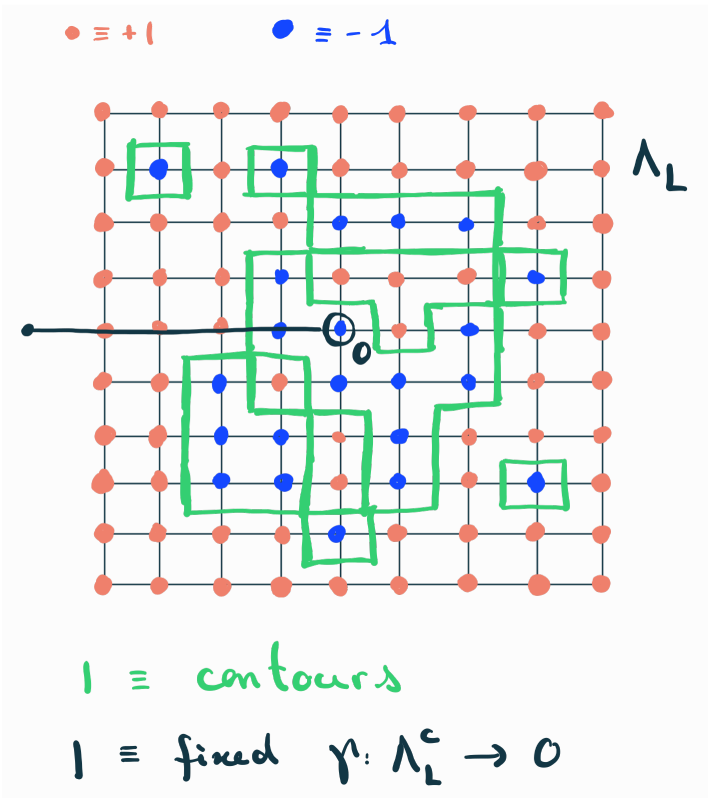


Figure 1.2: Peierls contours.

In particular, $\sigma_0(\omega) = (-1)^{\text{crossed}_0(\eta(\omega))}$. We obtain

$$\mu^+_{L;\beta}(\sigma_0) = \mu^+_{L;\beta}(\underbrace{\text{there is an even number of contours surrounding } 0}_{=:\text{Ev}_0})$$
$$- \mu^+_{L;\beta}(\underbrace{\text{there is an odd number of contours surrounding } 0}_{=:\text{Odd}_0}).$$

Note that $\text{Odd}_0 \cup \text{Ev}_0 = \Omega^+_{\Lambda_L}$, and that $\text{Odd}_0 \cap \text{Ev}_0 = \varnothing$. Also, remember that 0 is an even number, and that odd numbers are ≥ 1.

The idea is to use a *many-to-one principle* to compare the probability of the two events. We first define the *outermost contour surrounding* 0 *in* ω by looking at the

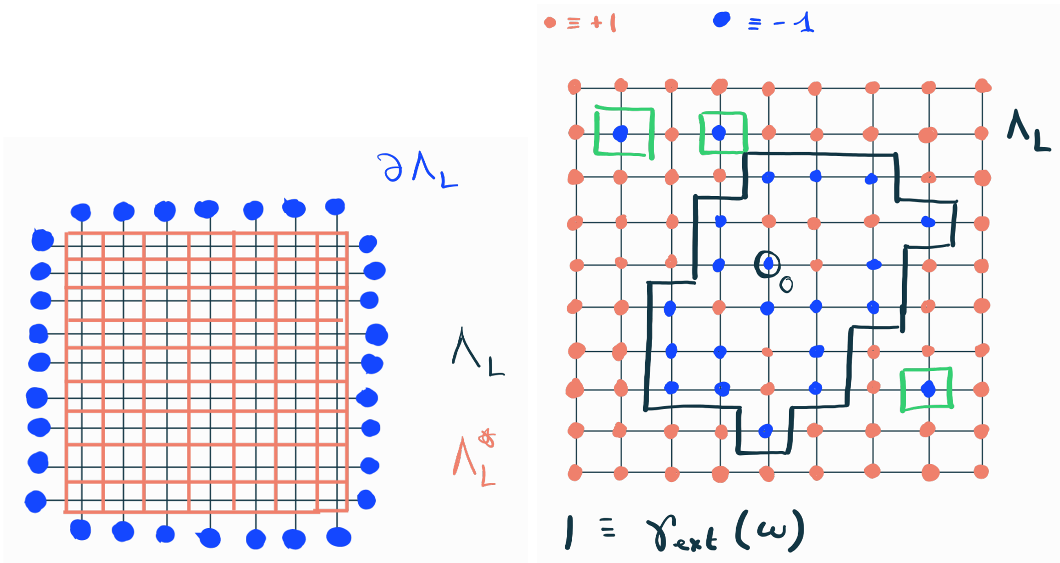


Figure 1.3: Planar duality and outermost contour surrounding 0 (in black). The green contours are the other parts of the boundary of the $+1$-connected component of Λ_L^c.

connected component of Λ^c in the graph obtained by deleting all sites that are -1 from $\mathbb{Z}^2$, and looking at the dual edges that form its boundary (see Figure 1.2). This gives a collection of contours. Denote $\gamma_{\text{ext}}(\omega)$ the only one that has 0 in its interior, see Figure 1.3 ($\gamma_{\text{ext}}(\omega) = \varnothing$ is allowed).

Define then $\psi : \text{Odd}_0 \to \text{Ev}_0$ as follows:

$$(\psi(\omega))_i = \begin{cases} \omega_i & \text{if } i \text{ is not in the interior of } \gamma_{\text{ext}}(\omega) \\ -\omega_i & \text{if } i \text{ is in the interior of } \gamma_{\text{ext}}(\omega) \end{cases}.$$

In words, ψ erases the outermost contour surrounding 0 in ω. Then

$$\mu^+_{L;\beta}(\text{Odd}_0) = \sum_{\omega\in\text{Odd}_0} \mu^+_{L;\beta}(\{\omega\})$$
$$= \sum_{\omega\in\text{Odd}_0} \sum_{\gamma \text{ surr. } 0} \mathbb{1}_{\gamma_{\text{ext}}(\omega)=\gamma} \sum_{\omega'\in\text{Ev}_0} \mathbb{1}_{\psi(\omega)=\omega'} \frac{\mu^+_{L;\beta}(\{\omega\})}{\mu^+_{L;\beta}(\{\omega'\})} \mu^+_{L;\beta}(\{\omega'\}),$$

where we multiplied by 1 in a clever way. Now, for ω, ω' such that $\psi(\omega) = \omega'$, and $\gamma_{\text{ext}}(\omega) = \gamma$, one has

$$\sum_{i\sim j\in\Lambda} \omega_i\omega_j - \sum_{i\sim j\in\Lambda} \omega'_i\omega'_j = \sum_{\{i,j\} \text{ dual to an edge in } \gamma} (-1-1) = -2|\gamma|.$$

Thus,

$$\frac{\mu^+_{L;\beta}(\{\omega\})}{\mu^+_{L;\beta}(\{\omega'\})} = \frac{e^{\beta\sum_{i\sim j\in\Lambda}\omega_i\omega_j}}{e^{\beta\sum_{i\sim j\in\Lambda}\omega'_i\omega'_j}} = e^{-2\beta|\gamma|}.$$

Using this, we get

$$\mu^+_{L;\beta}(\text{Odd}_0) = \sum_{\omega'\in\text{Ev}_0} \mu^+_{L;\beta}(\{\omega'\}) \sum_{\gamma \text{ surr. } 0} e^{-2\beta|\gamma|} \underbrace{\sum_{\omega\in\text{Odd}_0} \mathbb{1}_{\gamma_{\text{ext}}(\omega)=\gamma}\mathbb{1}_{\psi(\omega)=\omega'}}_{\leq 1}$$

where we used that the only thing needed to reconstruct ω from $\psi(\omega)$ is $\gamma_{\mathrm{ext}}(\omega)$. Now, the number of contours surrounding 0 with a fixed length n is at most $n4^n$ (the n accounts for the intersection of the contour with a fixed half-coordinate axis closest to 0, and the 4^n bounds the number of contours with length n starting from a given point). Thus,

$$\mu^+_{L;\beta}(\mathrm{Odd}_0) \leq \sum_{\omega' \in \mathrm{Ev}_0} \mu^+_{L;\beta}(\{\omega'\}) \sum_{n\geq 4} \sum_{\gamma \text{ surr. } 0\text{: } |\gamma|=n} e^{-2\beta n} \leq \mu^+_{L;\beta}(\mathrm{Ev}_0) \sum_{n\geq 4} n e^{-2\beta n} 4^n.$$

For β large enough, we get $\mu^+_{L;\beta}(\mathrm{Odd}_0) \leq \frac{1}{2}\mu^+_{L;\beta}(\mathrm{Ev}_0)$. In this case, $\mu^+_{L;\beta}(\mathrm{Ev}_0) \geq \frac{2}{3}$ as

$$1 = \mu^+_{L;\beta}(\mathrm{Ev}_0) + \mu^+_{L;\beta}(\mathrm{Odd}_0) \leq \frac{3}{2}\mu^+_{L;\beta}(\mathrm{Ev}_0),$$

and therefore

$$\mu^+_{L;\beta}(\sigma_0) = \mu^+_{L;\beta}(\mathrm{Ev}_0) - \mu^+_{L;\beta}(\mathrm{Odd}_0) \geq \mu^+_{L;\beta}(\mathrm{Ev}_0)(1-\tfrac{1}{2}) \geq \tfrac{1}{3}.$$

This bound is uniform over L and therefore gives the wanted claim. □

1.7 A few further results without proof

In this section we describe (part of) what is currently known about the Ising model/lattice gas transition. We will not have time to prove these statements, but references are given for the interested reader.

1.7.1 Sharpness of the phase transition

Theorem 1.7.1. *For any $\beta < \beta_c(d)$, there is $c > 0$ such that for any L large enough,*

$$\mu^+_{\Lambda_L;\beta}(\sigma_0) \leq e^{-cL}.$$

This result was first proven in [2]. A new proof is provided in [9].

1.7.2 Analyticity properties of the pressure

The Ising model is one of the rare model (i.e.: the only one...) where one can understand the full *phase diagram* (i.e.: one can identify the analytic properties of the pressure at *any* point (β, h)).

The first result in this direction is the famous Lee-Yang Theorem.

Theorem 1.7.2 (Lee-Yang)**.** *The partition functions $Z^+_{\Lambda;\beta,h}$ are non-zero for any $h \in \mathbb{C}$ with $\mathrm{Re}(h) \neq 0$. In particular, $h \mapsto \Psi(\beta,h)$ is analytic in h in the half spaces $\{\mathrm{Re}(h) > 0\}$ and $\{\mathrm{Re}(h) < 0\}$, and there is no first order phase transition whenever $\mathrm{Re}(h) \neq 0$.*

The first proof of this result is due to Lee and Yang [15, 21]. Asano gave an alternative proof introducing the so-called *Asano contraction method*, which was further refined by Ruelle. See [6, 19]. See also [10] for a pedagogical presentation.

Nowadays, the full picture is available.

Theorem 1.7.3. *$(\beta, h) \mapsto \Psi(\beta, h)$ is real-analytic everywhere but on the line $h = 0, \beta \geq \beta_c$.*

This was proved in [17] relying on several previous works on fine properties of the Ising model.

1.7.3 Structure of the set of Gibbs measures

A wealth of general properties of the set of Gibbs measure is available in more larger generalities than the Ising model. The main source is [12].

In dimension 2, a remarkable phenomenon occurs, and it turns out that *all* Gibbs measure are convex combination of the μ^+, μ^- measures we constructed.

Theorem 1.7.4. *For any $\beta \geq 0$, $h \in \mathbb{R}$, $\mathcal{G}(\beta, h) = \{t\mu^+_{\beta,h} + (1-t)\mu^-_{\beta,h} : \ t \in [0,1]\}$.*

See [14, 1] for the original proofs, and [13] for an elegant simpler alternative proof.

1.7.4 The Curie temperature

The main result available at $(\beta_c, 0)$ is the following.

Theorem 1.7.5. *For $d \geq 2$, $m^*(\beta_c) = 0$. In particular, $|\mathcal{G}(\beta_c, 0)| = 1$.*

The proof of the case $d = 2$ was given by Yang, relying on previous work of Onsager, through integrability. The case $d \geq 4$ follows from the work of Aizenman and Fernàndez, [5], and the case $d = 3$ from the "recent" paper of Aizenman, Duminil-Copin, and Sidoravicius, [4].

Another remarkable result concerns the $d = 2$ case. In this case, combinatorial miracles allows to compute the value of $\beta_c(2)$.

Theorem 1.7.6. $\beta_c(2) = \frac{1}{2}\ln\big(1 + \sqrt{2}\big)$.

The first proof of this result is due to Onsager, [16] using integrability. One can "guess" this result by looking at the low and high temperature expansion of the Ising model, and noting that they provide the same combinatorial object. This realization is due to Kramer and Wannier and is called *Kramer-Wannier duality*.

Chapter 2

Scaling Limits

We only discuss extremely briefly and informally this topic. More details and references can be found in [10, 11].

2.1 Scaling limits: general idea

In taking scaling limits, we do the reverse operation than when taking thermodynamic limit: we fix some volume $V \subset \mathbb{R}^d$ (say $[-1,1]^d$ or $\mathbb{R}^d$), fit a grid with small meshsize approximating the volume, put our Ising model on that graph, and take the mesh to 0. The resulting object should be "a random function from V to $\{-1,1\}$" but it is not a very well defined notion. Instead, we shift our viewpoint and see the Ising model as a *random distribution*. To simplify the discussion, we work on the volume $\mathbb{R}^d$, and fix some $\mu \in \mathcal{G}(\beta, h)$.

We first fix some local observable $\mathcal{O} : \Omega \to \mathbb{R}$ (e.g.: $\mathcal{O} = \sigma_0$ or $\mathcal{O} = \sum_{k=1}^d \sigma_0\sigma_{e_k} - \mu(\sigma_0\sigma_{e_k})$), and define $\theta_x\mathcal{O}$ its translate by x (e.g: if $\mathcal{O} = \sigma_0$, $\mathcal{O}_x = \sigma_x$). Then, for $\alpha \in \mathbb{R}$, we define for any $\delta > 0$,

$$\mathcal{O}_x^\delta : \{-1,1\}^{\delta\mathbb{Z}^d} \to \mathbb{R}, \quad \mathcal{O}_x^\delta(\omega) = \delta^\alpha \mathcal{O}_{x/\delta}(\omega).$$

We then view $\mathcal{O}^\delta(\omega) = (\mathcal{O}_x^\delta(\omega))_{x\in\delta\mathbb{Z}^d}$ as a distribution by setting for any $f : \mathbb{R}^d \to \mathbb{R}$ infinitely differentiable:

$$\langle \mathcal{O}^\delta(\omega), f\rangle \coloneqq \sum_{x\in\delta\mathbb{Z}^d} \mathcal{O}_x^\delta(\omega) f(x).$$

Note that this is formally equivalent to view $\mathcal{O}^\delta(\omega)$ as the signed measure

$$\mathcal{O}^\delta(\omega) = \sum_{x\in\delta\mathbb{Z}^d} \mathcal{O}_x^\delta(\omega)\delta_x$$

with δ_x the Dirac mass at x. In particular, under μ, $\mathcal{O}^\delta$ is a random distribution. The convergence towards a scaling limit is the existence of a probability measure Q on the a suitable distribution space such that

$$\mu\big(\langle \mathcal{O}^\delta, f\rangle\big) \xrightarrow{\delta\to 0} \int dQ(\nu)\langle \nu, f\rangle,$$

for all test functions f (for example, all compactly supported infinitely differentiable functions). The number $\alpha = d - \Delta$, with Δ the *scaling dimension* of the field, has to be chosen properly depending on $\beta, h, \mathcal{O}$ in order to get a convergence.

2.2 High temperature limit: white noise

A first example is the scaling limit of the Ising model with $\beta < \beta_c$, $h = 0$, $\mathcal{O} = \sigma_0$, the *magnetisation field*. There, a result of Newman gives that for $\alpha = \frac{d}{2}$, $\mathcal{O}^\delta$ converges towards a white noise in the limit (formally, each point of $\mathbb{R}^d$ gets an independent value, and all are identically distributed and Gaussian). Another way to phrase this is that the empirical magnetisation in a hyper-cubic box satisfies a central limit theorem.

A similar result holds for any $\mathcal{O}$, and any (β, h) with either $\beta < \beta_c$ or $h \neq 0$.

2.3 Low temperature limit: Wulff construction

When $\beta > \beta_c$, the most interesting phenomena occurs in the canonical ensemble. Look at $\mu^+_{\Lambda_L;\beta,M_L}$ with $\frac{M_L}{|\Lambda_L|} \xrightarrow{L\to\infty} m \in (-m^*(\beta), m^*(\beta)$. Then, scale the graph to have mesh $\delta = 1/L$, and look at $\mathcal{O} = \sigma_0$, $\alpha = d$ (so that we are effectively doing local empirical averages). This gives a sequence of random distributions on the volume $[-1, 1]^d$. The scaling limit will be a "function" taking value $-m^*(\beta)$ in some random translate of a deterministic shape, the *Wulff shape*, and value $+m^*(\beta)$ outside.

This construction has first been performed rigorously in dimension 2 by Dobrushin, Kotecky, and Shlosman in [8], and in dimension ≥ 3 by Bodineau in [7].

2.4 Critical point

At the critical point, some additional symmetries emerge in the scaling limit, and the scaling limit of, say, the magnetisation field is expected to be conformally invariant at $(\beta_c, 0)$. In dimensions greater or equal to 4, one can identify the scaling limit, see the work of Aizenman and Duminil-Copin [3], references therein and subsequent works.

When $d = 3$, nothing in known rigorously (and there remain challenges even for physicists).

When $d = 2$, the status is more satisfactory due to many integrability properties of the Ising model on the square lattice.

2.4.1 Dimension 2

In dimension 2, the set of conformal symmetries is much larger. One can therefore classify the possible "random distributions" that can be conformally invariant. The interested reader is invited to have a look at [11] for a full introduction to the topic. We mention here a few known results about the $2D$ Ising model at $(\beta_c, 0)$.

Spin field

Taking the observable $\mathcal{O} = \sigma_0$, one has that the correct exponent is $\alpha = \frac{15}{8}$, the scaling limit exists and is conformally invariant.

Energy field

Taking the observable $\mathcal{O} = \sigma_0\sigma_{e_1} + \sigma_0\sigma_{e_2} - \mu(\sigma_0\sigma_{e_1} + \sigma_0\sigma_{e_2})$, one has that the correct exponent is $\alpha = 1$, the scaling limit exists and is conformally invariant.

SLE/CLE approach

Another route towards conformal invariance is to look at the Peierls contours between $+1$ and -1. Keeping only the one that are "'large enough", one can construct a limit of this "random collection of random loops": the CLE measures. This model is a random collection of random curves in $\mathbb{R}^2$ that is conformally invariant.

Appendix A

Probability theory

The modern mathematical foundation of probability theory, formalized by A.N. Kolmogorov in 1933, treats probability strictly within the framework of measure theory. By abandoning intuitive but ill-defined notions of "chance," probability becomes the study of specialized measure spaces where the total mass is normalized to unity.

A.1 Probability Spaces

Definition A.1.1 (Probability Space)**.** A probability space is a triplet $(\Omega, \mathcal{F}, P)$ where

- Ω is the *sample space*, an arbitrary non-empty set representing all possible outcomes.
- $\mathcal{F}$ is a σ-algebra on Ω; i.e., a collection of subsets of Ω containing Ω, closed under complementation and finite or countable unions.
- P is a *probability measure* on $(\Omega, \mathcal{F})$: a function $P : \mathcal{F} \to [0, +\infty)$ satisfying

  Normalization: $P(\Omega) = 1$, $P(\varnothing) = 0$.

  Countable Additivity: For any sequence of **pairwise disjoint** sets $A_1, A_2, \dots \in \mathcal{F}$,

$$P\big(\cup_{n\geq 1} A_n\big) = \sum_{n=1}^{\infty} P(A_n).$$

The pair $(\Omega, \mathcal{F})$ is called a *measurable space.* If the condition $P(\Omega) = 1$ is dropped, P is called a *measure.*

Example A.1.1 (Finite or countable sample space)**.** On a finite or countable set Ω, one usually considers $\mathcal{F} = \mathcal{P}(\Omega) := \{A : A \subset \Omega\}$. Then, probability measures can be specified by a *probability vector/mass function*, $p : \Omega \to [0, 1]$ with $\sum_{\omega\in\Omega} p(\omega) = 1$. Then, the probability measure associated to p is

$$P_p(A) = \sum_{\omega\in A} p(\omega).$$

Example A.1.2 (Borel spaces)**.** If $(\Omega, \mathcal{T})$ is a topological space (e.g.: $\Omega = \mathbb{R}^d$ with the usual topology), one can define the *Borel σ-algebra* on $(\Omega, \mathcal{T})$ by $\mathcal{B}(\Omega, \mathcal{T}) = \cap_{\mathcal{F}\supset\mathcal{T}}\mathcal{F}$

where the intersection ranges over all σ-algebras over Ω. It is easy to check that this is a σ-algebra. By construction, it is the smallest contains the open sets.

Notation. When $(\Omega, \mathcal{T})$ is $\mathbb{R}^d$ with the usual topology, we will write $\mathcal{B}(\mathbb{R}^d)$ for the associated Borel σ-algebra.

Probability measures enjoy various useful properties.

Theorem A.1.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, all of the following hold.*

1. $P(\varnothing) = 0$, $P(\Omega) = 1$, *and for any event* A, $P(\Omega \setminus A) = 1 - P(A)$.
2. *If two events* A, B *are such that* $A \subset B$, $P(B) = P(A) + P(B \setminus A)$. *In particular,* $P(A) \leq P(B)$.
3. *For two events* A, B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
4. *Finite and countable σ-additivity: if I is a finite or countable set, and $A_i \in \mathcal{F}$, $i \in I$ are events such that $A_i \cap A_j = \varnothing$ for $i \neq j$, then*
$$P\big(\cup_{i \in I} A_i\big) = \sum_{i \in I} P(A_i).$$
5. *Finite and countable σ-sub-additivity: if I is a finite or countable set, and $A_i \in \mathcal{F}$, $i \in I$ are events, then*
$$P\big(\cup_{i \in I} A_i\big) \leq \sum_{i \in I} P(A_i).$$
6. *Monotone convergence: if $A_1, A_2, \dots$ are events such that either $A_i \subset A_{i+1}$ for all i's, or $A_i \supset A_{i+1}$ for all i's, then*
$$\lim_{n \to \infty} P(A_n) = P\big(\lim_{n \to \infty} A_n\big).$$

A.2 Random Variables as Measurable Functions

In rigorous terms, a random variable is neither random nor a variable; it is a deterministic function mapping the sample space to a state space, typically $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition A.2.1 (Random Variables)**.** Let $(\Omega, \mathcal{F}), (\mathcal{X}, \mathcal{E})$ be a measurable spaces. A function $X : \Omega \to \mathcal{X}$ is a ($\mathcal{X}$-valued) **random variable** if it is $(\mathcal{F}, \mathcal{E})$-measurable, meaning that for every set $B \in \mathcal{E}$,
$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$
If P is a probability measure on $(\Omega, \mathcal{F})$, the *law of X under P* is the probability measure on $(\mathcal{X}, \mathcal{E})$ given by $P \circ X^{-1}$.

We will often look at real random variables. When nothing is said explicitly, a random variable $X : \Omega \to \mathbb{R}^d$ is a $(\mathcal{F}, \mathcal{B}(\mathbb{R}^d))$-measurable function.

Example A.2.1 (Finite or countable sample space)**.** On $(\Omega, \mathcal{P}(\Omega))$ with Ω finite or countable, all functions are measurable, and so any function is a random variable.

A.3 Integration and Expectation

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The expectation of a real random variable $X : \Omega \to \mathbb{R}$ is its Lebesgue integral with respect to the probability measure P:

$$E_P(X) = \int_\Omega dP(\omega) X(\omega) \equiv \int dP\, X$$

The construction is done in a three-step program:

Simple functions: for indicator functions, define $E_P(\mathbb{1}_A) \equiv \int_\Omega dP(\omega) f(\omega) \mathbb{1}_A(\omega) = P(A)$. By linearity, for a simple function $g = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$, we define:

$$E_P(g) = \int_\Omega dP(\omega) g(\omega) = \sum_{i=1}^n c_i P(A_i).$$

Non-negative measurable functions: For any measurable function $X \geq 0$, the expectation is defined via an "approximation from below" by simple functions bounded above by X:

$$E_P(X) = \sup\{E_P(g) : 0 \leq g \leq X,\, g \text{ is simple}\}.$$

General measurable functions: we split $X = X^+ - X^-$, where $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$. If $E_P(X^+) < \infty$ or $E_P(X^-) < \infty$, we define:

$$E_P(X) = E_P(X_+) - E_P(X_-).$$

If $E_P(|X|) < \infty$, we say that X is P-integrable, and write $L^1(\Omega, \mathcal{F}, P)$ the set of real, P-integrable functions.

Example A.3.1 (Finite or countable sample space)**.** On a finite or countable set Ω (equipped with $\mathcal{P}(\Omega)$), if P_p is the probability measure associated to the mass function p, one has

$$E_{P_p}(X) = \sum_{\omega \in \Omega} p(\omega) X(\omega).$$

A.3.1 Convergence Theorems for expectations

The power of measure-theoretic probability lies in large part in its convergence theorems, which allow the interchange of limits and integrals under much broader conditions than what can be done using, for example, Riemann calculus.

Theorem A.3.1 (Monotone Convergence Theorem - MCT)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. If $0 \leq X_1 \leq X_2 \leq \dots$ is a sequence of non-negative random variables converging almost surely[a] to X, then:*

$$\lim_{n \to \infty} E_P(X_n) = E_P(X).$$

[a] $X_n(\omega) \xrightarrow{n \to \infty} X(\omega)$ for all $\omega \in D$ with $D \in \mathcal{F}$ a set such that $P(D) = 1$.

Theorem A.3.2 (Dominated Convergence Theorem - DCT)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X_n : \Omega \to \mathbb{R}$, $n \geq 1$, be a sequence of random variables such that $X_n \to X$ almost surely. If there exists an integrable random variable Y ($E_P(Y) < \infty$) such that $|X_n| \leq Y$ almost surely[a] for all n, then X is integrable and:*

$$\lim_{n\to\infty} E_P(X_n) = E_P(X).$$

[a] $|X_n(\omega)| \leq Y(\omega)$ for $\omega \in D$ with $D \in \mathcal{F}$ a set such that $P(D) = 1$.

These theorems play a central role when dealing with infinite dimensional probability spaces.

A.3.2 Weak Convergence and Topology

When Ω is a metric space (often a Polish space), the space of all probability measures on Ω, $\mathcal{M}_1(\Omega)$, is typically endowed with the *topology of weak convergence* (which is in fact weak-$*$ convergence in the functional analytic sense, see Section B.5). A sequence of measures μ_n *converges weakly* to μ (denoted $\mu_n \xrightarrow{\text{weak}} \mu$) if:

$$\lim_{n\to\infty} \int_\Omega f\, d\mu_n = \int_\Omega f\, d\mu$$

for all bounded, continuous functions $f \in C_b(\Omega)$.

Appendix B

Distribution theory

B.1 Introduction and Motivation

The classical notion of a function is insufficient for many problems in partial differential equations (PDEs) and harmonic analysis. For instance, the Dirac delta "function" δ, defined intuitively by its sampling property $\int_{\mathbb{R}^d} \delta(x) f(x) dx = f(0)$, cannot be realized as a Radon measure without losing its character as a differential object, nor can it be defined as a pointwise function.

The theory of distributions, formalized by Laurent Schwartz in the late 1940s, resolves this by shifting the paradigm from *pointwise evaluation* to *functional evaluation* against a space of highly regular test functions. Put in words: instead of being able to look at the value of the object at a point, one can only observe *weighted average values* of the object over non-trivial portions of space. The regularity of the weight used in the weighted average then compensates for the irregularity of the considered object.

B.2 The Space of Test Functions $\mathcal{D}(\mathbb{R}^d)$

We denote by $C_c^\infty \equiv C_c^\infty(\mathbb{R}^d, \mathbb{R})$ the vector space of real valued infinitely differentiable functions with compact support[1]. To define a topology on C_c^∞, we first introduce a notion of convergence.

Definition B.2.1 (Convergence in C_c^∞)**.** A sequence of functions $f_1, f_2, \dots \in C_c^\infty$ is said to *converge to* $f \in C_c^\infty$ if there exists a fixed compact set $K \subset \mathbb{R}^d$ such that f_k is supported on K for all k's, and one has that for every multi-index $\alpha \in \mathbb{N}^d$, the sequence of derivatives[a] $D^\alpha f_k$ converges uniformly to $D^\alpha f$ on K, i.e.,

$$\lim_{k\to\infty} \sup_{x \in K} |D^\alpha f_k(x) - D^\alpha f(x)| = 0.$$

[a] Recall $D^\alpha \equiv \prod_{i=1}^d (\frac{\partial}{\partial x_i})^{\alpha_i}$.

The space C_c^∞ equipped with this locally convex topology is denoted by $\mathcal{D}(\mathbb{R}^d)$.

[1] Recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is *supported on* $D \subset \mathbb{R}^d$ if $f(x) = 0$ for all $x \notin D$.

B.3 Distributions

Definition B.3.1 (Distribution)**.** A *distribution* on $\mathbb{R}^d$ is a complex-valued continuous linear functional on $\mathcal{D}(\mathbb{R}^d)$. The space of all distributions is denoted by $\mathcal{D}'$:

$$\mathcal{D}' = \{u : \mathcal{D} \to \mathbb{C} : \ F \text{ is linear and continuous}\}.$$

In terms of sequential convergence, a linear functional $u : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}$ is a distribution if and only if for every compact $K \subset \mathbb{R}^d$, there exist constants $C > 0$ and $N \in \mathbb{N}$ such that

$$|\langle u, f\rangle| \leq C \sum_{|\alpha|\leq N} \sup_{x\in K} |D^\alpha f(x)|$$

for all $f \in C_c^\infty$ supported on K. Here, $\langle u, f\rangle$ denotes the dual pairing $(u, f) \mapsto u(f)$.

Example B.3.1 (Regular Distributions)**.** Any locally integrable function[2] $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ defines a regular distribution $T_g \in \mathcal{D}'$ via:

$$\langle T_g, f\rangle = \int_{\mathbb{R}^d} g(x) f(x)\, dx, \quad \forall f \in C_c^\infty.$$

The mapping $g \mapsto T_g$ is injective, allowing us to embed $L^1_{\text{loc}}(\mathbb{R}^d) \subset \mathcal{D}'$.

Example B.3.2 (Singular Distributions)**.** The Dirac delta distribution δ_a centred at $a \in \mathbb{R}^d$ is defined by:

$$\langle \delta_a, f\rangle = f(a).$$

This distribution is singular; it cannot be represented by any L^1_{loc} function.

Remark B.3.1. We consider here the "most regular" space of functions, $\mathcal{D}(\mathbb{R}^d)$. This gives us the largest space of distributions (i.e.: the least regular one can define). We could consider functions with less regularity, such as C_c^p for $p \geq 1$, leading to smaller space of distributions. Having a distribution in a smaller space means having a more regular distribution.

B.4 Calculus of Distributions

One of the main points behind the introduction of distribution theory is that every distribution is infinitely differentiable, and the differentiation operator is continuous with respect to the weak-$*$ topology on $\mathcal{D}'$.

Definition B.4.1 (Distributional Derivative)**.** Let $u \in \mathcal{D}'$. For any multi-index α, the distributional derivative $D^\alpha u$ is defined through integration by parts:

$$\langle D^\alpha u, f\rangle = (-1)^{|\alpha|}\langle u, D^\alpha f\rangle, \quad \forall f \in C_c^\infty,$$

where $|\alpha| = \sum_{i=1}^d |\alpha_i|$.

Since $f \in C_c^\infty \implies D^\alpha f \in C_c^\infty$, this definition is always well-defined and yields another distribution.

[2] Integrable over compact subsets of $\mathbb{R}^d$.

Remark B.4.1. If $u = T_f$ is the distribution associated to f a continuously differentiable function (as in Example B.3.1), its distributional derivative matches the classical derivative of f. However, functions with jump discontinuities yield singular structures under distributional differentiation (e.g., the Heaviside step function differentiates to the Dirac delta).

B.5 Structure and Convergence

The space $\mathcal{D}'$ is typically endowed with the *weak-$*$ topology*.

Definition B.5.1 (Weak-$*$ topology)**.** A sequence $u_j \to u$ in $\mathcal{D}'$ if and only if $\langle u_j, f\rangle \to \langle u, f\rangle$ for every $f \in \mathcal{D}(\mathbb{R}^d)$.

B.6 Probability measures as distributions

Consider $\mathcal{M}_1(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ (measures with total mass 1). They can be viewed as distributions in the following fashion. For $\mu \in \mathcal{M}_1^+(\mathbb{R}^d)$, define T_μ via

$$\langle T_\mu, f\rangle = \int_{\mathbb{R}^d} d\mu(x) f(x) = E_\mu(f).$$

This is a linear mapping, and it is $\mathbb{R}$-valued as $f \in \mathcal{D}$ is continuous and compactly supported, hence bounded. We need to show that this mapping is continuous in the sense that if $f_k \xrightarrow{k\to\infty} f$ in $\mathcal{D}(\mathbb{R}^d)$, then $\langle T_\mu, f_k\rangle \xrightarrow{k\to\infty} \langle T_\mu, f\rangle$. But this is an exercise: by definition of convergence in $\mathcal{D}(\mathbb{R}^d)$, there is K compact such that f_k converges uniformly over K to f, so for any $\epsilon > 0$, there is $k_\epsilon \geq 1$ such that $\sup_{x\in K} \sup_{k\geq k_\epsilon} |f_k(x) - f(x)| \leq \epsilon$. Thus, for any $k \geq k_\epsilon$,

$$\big|E_\mu(f_k) - E_\mu(f)\big| \leq \int_{\mathbb{R}^d} d\mu(x) |f_k(x) - f(x)| \leq \epsilon.$$

Note that we did note use the full power of probability measures: working with locally finite measures would have been sufficient! Moreover, we only used the fact that f was continuous and compactly supported, so the linear mapping extends to a larger set of test functions than C_c^∞. This is due to the fact that, in some sense, measures are the *next best thing* after functions in terms of regularity, but there are much wilder objects in the space $\mathcal{D}'$.

Appendix C

Solutions to some Exercises

Solution to Exercise 1.1. To lighten notations, we drop β, h from the notation. We check the identity on singleton event $\{\omega'\}$, the identity for more complicated events is obtained by summing over $\omega' \in A$. One has

$$\sum_{\omega\in\Omega} \mu_{\Lambda'}(\{\omega\})\mu_{\Lambda}^{\omega}(\{\omega'\}) = \sum_{\omega\in\Omega_{\Lambda'}^{\xi}} \frac{1}{Z_{\Lambda'}^{\xi}} e^{-\beta H_{\Lambda'}(\omega)+hM_{\Lambda'}(\omega)} \mathbb{1}_{\Omega_{\Lambda}^{\omega}}(\omega') \frac{1}{Z_{\Lambda}^{\omega}} e^{-\beta H_{\Lambda}(\omega')+hM_{\Lambda}(\omega')}. \tag{C.1}$$

Now, when $\omega' \in \Omega_{\Lambda}^{\omega}$,

$$\begin{aligned} -\beta H_{\Lambda'}(\omega) + hM_{\Lambda'}(\omega) &= \beta \sum_{\{i,j\}\subset\Lambda:\, i\sim j} \omega_i\omega_j + \beta \sum_{i\in\Lambda, j\in\Lambda^c:\, i\sim j} \omega_i\omega_j \\ &\quad + \beta \sum_{\{i,j\}\subset\Lambda'\cup\partial\Lambda'\setminus\Lambda:\, i\sim j} \omega_i\omega_j + hM_{\Lambda}(\omega) + M_{\Lambda'\setminus\Lambda}(\omega) \\ &= \beta \sum_{\{i,j\}\subset\Lambda:\, i\sim j} \omega_i\omega_j + \beta \sum_{i\in\Lambda, j\in\Lambda^c:\, i\sim j} \omega_i\omega_j' \\ &\quad + \beta \sum_{\{i,j\}\subset\Lambda'\cup\partial\Lambda'\setminus\Lambda:\, i\sim j} \omega_i'\omega_j' + hM_{\Lambda}(\omega) + M_{\Lambda'\setminus\Lambda}(\omega') \\ &= -\beta H_{\Lambda}(\omega_\Lambda\omega'_{\Lambda^c}) + hM_{\Lambda}(\omega_\Lambda\omega'_{\Lambda^c}) \\ &\quad + \beta \sum_{\{i,j\}\subset\Lambda'\cup\partial\Lambda'\setminus\Lambda:\, i\sim j} \omega_i'\omega_j' + hM_{\Lambda'\setminus\Lambda}(\omega'). \end{aligned}$$

So, still when $\omega' \in \Omega_{\Lambda}^{\omega}$,

$$\begin{aligned} &-\beta H_{\Lambda'}(\omega) + hM_{\Lambda'}(\omega) - \beta H_{\Lambda}(\omega') + hM_{\Lambda}(\omega') \\ &\qquad = -\beta H_{\Lambda}(\omega_\Lambda\omega'_{\Lambda^c}) + hM_{\Lambda}(\omega_\Lambda\omega'_{\Lambda^c}) + hM_{\Lambda'}(\omega') - \beta H_{\Lambda'}(\omega') \end{aligned} \tag{C.2}$$

Now, summing over $\omega \in \Omega_{\Lambda'}^{\xi}$ is the same as summing over $\omega_\Lambda \in \{-1,1\}^{\Lambda}$ and over $\omega_{\Lambda'\setminus\Lambda} \in \{-1,1\}^{\Lambda'\setminus\Lambda}$ and extending the configuration using ξ. Moreover, $Z_{\Lambda}^{\xi} = Z_{\Lambda}^{\xi'}$ and $\Omega_{\Lambda}^{\xi} = \Omega_{\Lambda}^{\xi'}$ for any ξ, ξ' such that $\xi_\Lambda = \xi'_\Lambda$. Using these, the notation $\Delta = \Lambda' \setminus \Lambda$, and (C.2) in (C.1), one gets

$$\sum_{\omega\in\Omega}\mu_{\Lambda'}(\{\omega\})\mu_{\Lambda}^{\omega}(\{\omega'\}) = \frac{1}{Z_{\Lambda'}^{\xi}}\sum_{\omega_\Lambda,\omega_\Delta}\mathbb{1}_{\Omega_\Lambda^{\omega_\Delta\xi_{\Delta^c}}}(\omega')\frac{1}{Z_\Lambda^{\omega_\Delta\omega'_{\Delta^c}}}e^{-\beta H_\Lambda(\omega_\Lambda\omega'_{\Lambda^c})+hM_\Lambda(\omega_\Lambda\omega'_{\Lambda^c})+hM_{\Lambda'}(\omega')-\beta H_{\Lambda'}(\omega')}$$

$$= \mu_{\Lambda'}^{\xi}(\{\omega'\})\sum_{\omega_\Lambda,\omega_\Delta}\mathbb{1}_{\omega'_\Delta=\omega_\Delta}\frac{1}{Z_\Lambda^{\omega_\Delta\omega'_{\Delta^c}}}e^{-\beta H_\Lambda(\omega_\Lambda\omega'_{\Lambda^c})+hM_\Lambda(\omega_\Lambda\omega'_{\Lambda^c})}$$

$$= \mu_{\Lambda'}^{\xi}(\{\omega'\})\sum_{\omega_\Lambda}\frac{1}{Z_\Lambda^{\omega'_\Delta\omega'_{\Delta^c}}}e^{-\beta H_\Lambda(\omega_\Lambda\omega'_{\Lambda^c})+hM_\Lambda(\omega_\Lambda\omega'_{\Lambda^c})} = \mu_{\Lambda'}^{\xi}(\{\omega'\})\frac{Z_\Lambda^{\omega'}}{Z_\Lambda^{\omega'}},$$

as wanted

Solution to Exercise 1.2. See [10, Lemma 3.19] (freely available online).

Solution to Exercise 1.3. This uses some non-trivial results in general topology and probability. First, by Tychonoff's Theorem, Ω is compact. One can moreover put a metric on it. Thus, the set of probability measures on Ω equipped with the product σ-algebra is sequentially compact by Prokhorov's Theorem. One can then take any fixed ξ and extract a converging subsequence from $\mu_{\Lambda_L;\beta,h}^{\xi}$, $L\geq 1$. Verifying the DLR condition for the limit is left to the reader.

Solution to Exercise 1.11. We deduce only the first part, the rest is a direct consequence of the correspondence between the $\mu_{J,\underline{h}}$ measure and the Ising measures.

Introduce

$$Z_{J,\underline{h}}(f) = \sum_{\omega\in\{-1,1\}^\Lambda} e^{\sum_{\{i,j\}\subset\Lambda} J_{ij}\omega_i\omega_j + \sum i\in\Lambda h_i\omega_i} f(\omega).$$

Let $\underline{h}\leq\underline{h}'$. Then, for any function f,

$$Z_{J,\underline{h}'}(f) = \sum_{\omega\in\{-1,1\}^\Lambda} e^{\sum_{\{i,j\}\subset\Lambda} J_{ij}\omega_i\omega_j + \sum i\in\Lambda h_i\omega_i} e^{\sum i\in\Lambda(h'_i-h_i)\omega_i} f(\omega) = Z_{J,\underline{h}}(fF_{\underline{h}'-\underline{h}})$$

where $F_a(\omega) = e^{\sum i\in\Lambda a_i\omega_i}$ is non-decreasing whenever $a\geq 0$.

In particular, for any f by a non-decreasing function,

$$\mu_{J,\underline{h}'}(f) = \frac{Z_{J,\underline{h}'}(f)}{Z_{J,\underline{h}'}(1)} = \frac{Z_{J,\underline{h}}(fF_{\underline{h}'-\underline{h}})Z_{J,\underline{h}}(1)}{Z_{J,\underline{h}}(1)Z_{J,\underline{h}}(F_{\underline{h}'-\underline{h}})}$$

$$= \frac{\mu_{J,\underline{h}}(fF_{\underline{h}'-\underline{h}})}{\mu_{J,\underline{h}}(F_{\underline{h}'-\underline{h}})} \geq \frac{\mu_{J,\underline{h}}(f)\mu_{J,\underline{h}}(F_{\underline{h}'-\underline{h}})}{\mu_{J,\underline{h}}(F_{\underline{h}'-\underline{h}})} = \mu_{J,\underline{h}}(f)$$

by the FKG inequality.

Solution to Exercise 1.14. First, as $\sigma_i = 2n_i - 1$, $\mu_{\beta,h}^{+}(\sigma_0) = \mu_{\beta,h}^{-}(\sigma_0)$ iff $\mu_{\beta,h}^{+}(n_0) = \mu_{\beta,h}^{-}(n_0)$. Now, for any $V\Subset\mathbb{Z}^d$, not that $\sum_{i\in V} n_i - n_V$ is non-decreasing. In particular, as $\mu_{\beta,h}^{+}$ stochastically dominates $\mu_{\beta,h}^{-}$,

$$\mu_{\beta,h}^{+}\Big(\sum_{i\in V} n_i - n_V\Big) \geq \mu_{\beta,h}^{-}\Big(\sum_{i\in V} n_i - n_V\Big).$$

Using linearity and re-arranging, we get

$$0 \leq \mu_{\beta,h}^{+}(n_V) - \mu_{\beta,h}^{-}(n_V) \leq \sum_{i\in V}\big(\mu_{\beta,h}^{+}(n_i) - \mu_{\beta,h}^{-}(n_i)\big) = |V|\big(\mu_{\beta,h}^{+}(n_0) - \mu_{\beta,h}^{-}(n_0)\big)$$

by translation invariance of the measures. This last term is 0 by assumption, so $\mu_{\beta,h}^{+}(n_V) = \mu_{\beta,h}^{-}(n_V)$ for all $V\Subset\mathbb{Z}^d$, and therefore $\mu_{\beta,h}^{+} = \mu_{\beta,h}^{-}$.

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