The goal of this minicourse is to make sense of this:

Definition:

A Calabi-Yau manifold is a compact, Kähler-polarised complex manifold with trivial canonical bundle.

Famous Example (appears in any discussion of String Theory)

Calabi-Yau Quintic Threefold:

Any smooth quintic hypersurface in $\mathbb{P}^4$ with Fubini-Study metric

e.g.: Fermat quintic hypersurface:

$$X = \{ [z_0 : \dots : z_4] \in \mathbb{P}^4 \mid z_0^5 + \dots + z_4^5 = 0 \}.$$

Plan:

§1. Complex manifolds

Definition, examples ($\mathbb{C}^n$, $\mathbb{C}^n/\Lambda$, $\mathbb{P}^n$, quintic 3fold)

§2. Canonical bundle

Holomorphic cotangent bundle, canonical bundle, examples, $\mathcal{O}_{\mathbb{P}^n}(k)$ bundles, adjunction formula

§3. Calabi-Yau manifolds

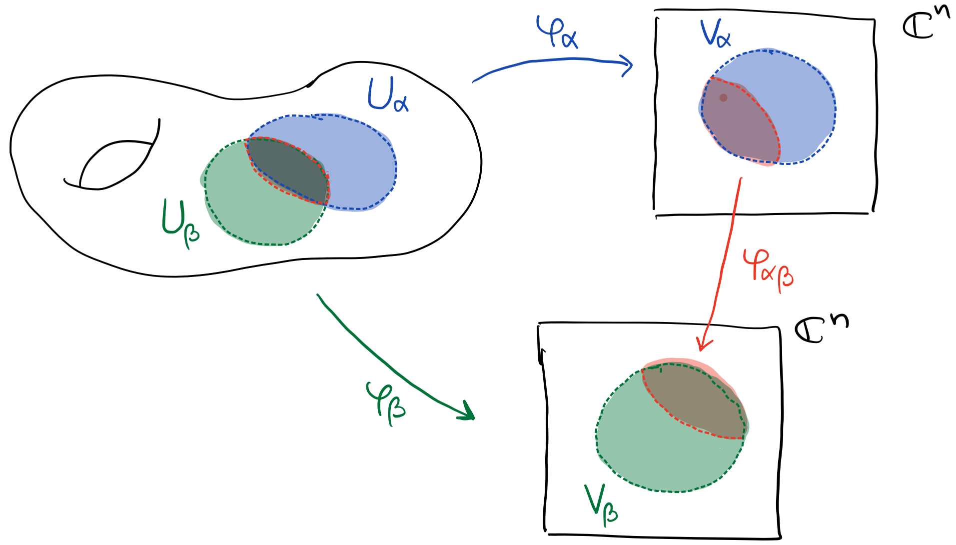
Hermitian structures, Kähler structures, Calabi-Yau definition, examples, Fubini-Study metric.

§4. Curvature and Physics

Ricci curvature, Einstein's equations, Calabi-Yau Theorem.

§1 Complex Manifolds

- Defn: Let X = topological space.
  An n-dimensional complex chart on X is a homeomorphism
  $$\varphi : U \xrightarrow{\sim} V$$
  where $U \subset X$ open subset and $V := \varphi(U) \subset \mathbb{C}^n$ open subset.



- Two complex charts $\varphi_\alpha : U_\alpha \to V_\alpha$ and $\varphi_\beta : U_\beta \to V_\beta$ are holomorphically compatible if the transition function
  $$\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$$
  is a holomorphism (i.e. holomorphic, invertible, inverse also holo).

- Defn: A complex manifold of (complex) dimension n is a (2nd-countable, Hausdorff, connected) topological space X equipped with a complex atlas := a collection of holomorphically compatible complex charts $\{\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{C}^n\}$ st $\bigcup_\alpha U_\alpha = X$.

Examples:

0 Euclidean space $\mathbb{C}^n$

- Standard global coordinate $\vec{z} = (z_1, \ldots, z_n)$;
  i.e. global chart $\varphi_0 : \mathbb{C}^n \to \mathbb{C}^n$ is given by $\varphi_0(\vec{z}) = \vec{z}$

- Note: always have the decomposition $z_k = x_k + i y_k$ where (x_k, y_k) real coordinates $\Rightarrow$ every cx mfld of dim n is in particular a real mfld of (real) dim $2n$.

- Given $\vec{w} = (w_1, \ldots, w_n)$ = another coordinate,
  $\varphi_1 : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ given by $\varphi_1(\vec{z}) = \vec{w}$

the transition function is $\varphi_{01}(\vec{z}) = \varphi_1 \circ \varphi_0^{-1}(\vec{z}) = \vec{w}(\vec{z})$

φ_0, φ_1 are compatible if $\vec{w}(\vec{z})$ is holomorphic.

e.g.: $\vec{w}(\vec{z}) = (z_1 + 1, \ldots, z_n + 1)$ ✓

$\vec{w}(\vec{z}) = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n)$ ✗

complex conjugate

② Complex Torus $\mathbb{T}_\Lambda = \mathbb{C}^n/\Lambda$

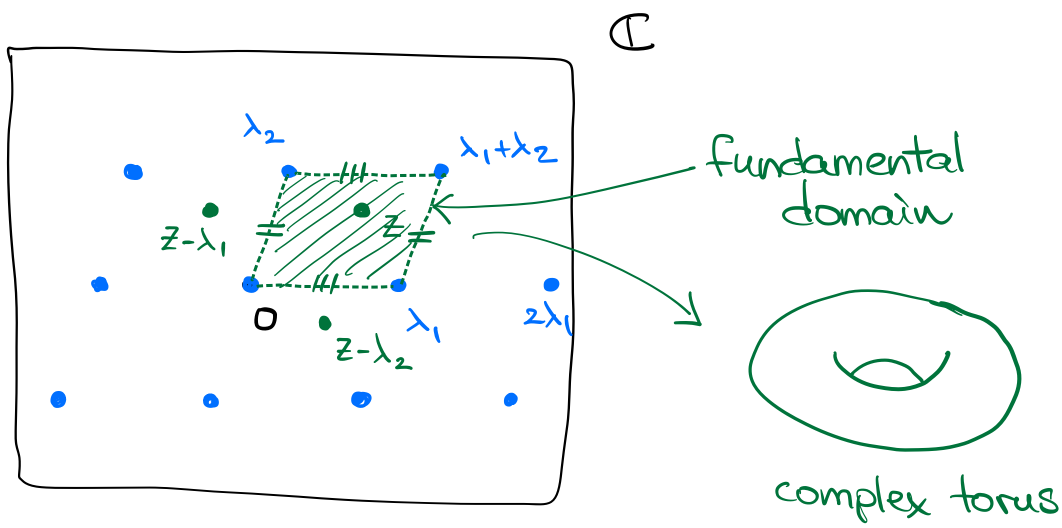
- Choose any $\lambda_1, \dots, \lambda_{2n} \in \mathbb{C}^n \setminus \{0\}$ linearly indep over $\mathbb{R}$.
- Consider lattice $\Lambda := \mathrm{span}_{\mathbb{Z}}(\lambda_1, \dots, \lambda_{2n}) \subset \mathbb{C}^n$

$$= \{ \vec{z} = m_1\lambda_1 + \dots + m_{2n}\lambda_{2n} \mid m_1, \dots, m_{2n} \in \mathbb{Z} \}$$

- Consider equivalence relation on $\mathbb{C}^n$

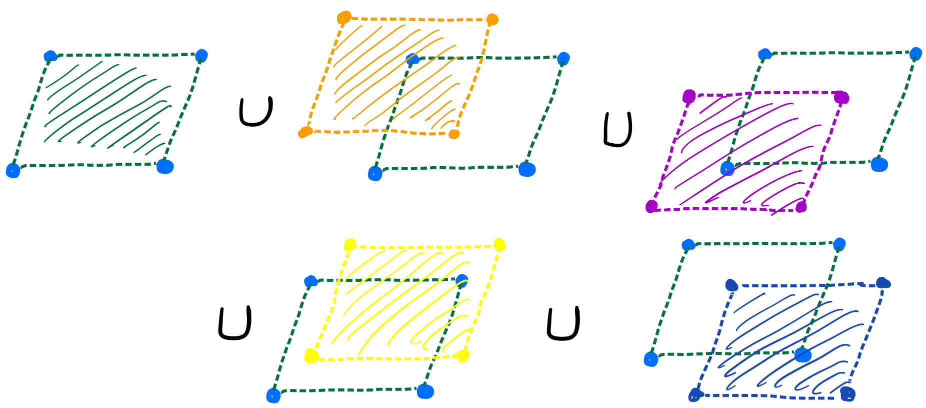
$$\vec{z} \sim \vec{z}' \;:\Leftrightarrow\; \vec{z} - \vec{z}' \in \Lambda.$$

- Define: $\mathbb{T}_\Lambda := \mathbb{C}^n/\Lambda := \mathbb{C}^n/\sim \; = \{ [\vec{z}] \mid z \in \mathbb{C}^n \}.$
- e.g. if $n=1$,



complex torus
/elliptic curve

- Topologically, $\mathbb{T}_\Lambda \simeq \mathbb{R}^{2n}/\Lambda \simeq (S^1)^{2n}$
- What are the complex charts? e.g.:



The important thing: transition functions are translations in $\mathbb{C}^n$
$\Rightarrow$ holomorphic

- Notice: $\mathbb{T}_\Lambda$ is compact.

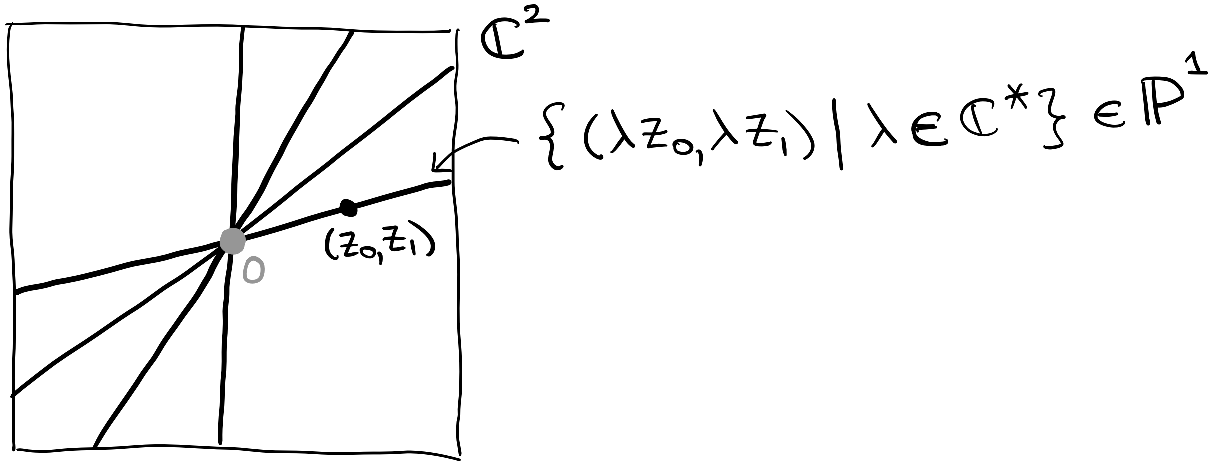
② Complex Projective Space $\mathbb{P}^n$

:= space of complex lines through the origin in $\mathbb{C}^{n+1}$:

$$\mathbb{P}^n := \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim$$

with $\vec{z} = (z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n) = \lambda \vec{z}$ for any $\lambda \in \mathbb{C}^*$.

- Equivalence classes $[\vec{z}] = [z_0 : \dots : z_n]$ = lines in $\mathbb{C}^{n+1}$:



- Observe: if $z_0 \neq 0$, then

$$(z_0, z_1, \dots, z_n) \sim \left(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)$$

$$\Rightarrow [z_0 : z_1 : \dots : z_n] = \left[1 : \frac{z_1}{z_0} : \dots : \frac{z_n}{z_0}\right]$$

- Famous open cover by affine charts:

$$U_i := \left\{ [z_0 : \dots : z_n] \in \mathbb{P}^n \mid z_i \neq 0 \right\}$$

- $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \Rightarrow \mathbb{P}^n = \bigcup_{i=0}^{n} U_i$

- $\varphi_i : U_i \xrightarrow{\sim} \mathbb{C}^n$

$$\begin{array}{ccc} [z_0 : \dots : z_i : \dots : z_n] & & (w_1, \dots, w_n) \in \mathbb{C}^n \\ \parallel & & \parallel \\ \left[\frac{z_0}{z_i} : \dots : \frac{z_{k-1}}{z_i} : 1 : \frac{z_{k+1}}{z_i} : \dots : \frac{z_n}{z_i}\right] & \longmapsto & \left(\underset{w_1}{\frac{z_0}{z_i}}, \dots, \underset{w_k}{\frac{z_{k-1}}{z_i}}, \underset{w_{k+1}}{\frac{z_{k+1}}{z_i}}, \dots, \underset{w_n}{\frac{z_n}{z_i}}\right). \end{array}$$

- transition function (e.g. $n=3$):

$$[Z_0:Z_1:Z_2]=\left[1:\frac{Z_1}{Z_0}:\frac{Z_2}{Z_0}\right]\xmapsto{\varphi_0}(w_1,w_2)=\left(\frac{Z_1}{Z_0},\frac{Z_2}{Z_0}\right)$$

$$\parallel\ \left[\frac{Z_0}{Z_1}:1:\frac{Z_2}{Z_1}\right]\xmapsto{\varphi_1}(\tilde{w}_1,\tilde{w}_2)=\left(\frac{Z_0}{Z_1},\frac{Z_2}{Z_1}\right)$$

$$(w_1,w_2)\xmapsto{\varphi_0^{-1}}[1:w_1:w_2]=\left[\frac{1}{w_1}:1:\frac{w_2}{w_1}\right]\xmapsto{\varphi_1}\left(\frac{1}{w_1},\frac{w_2}{w_1}\right)$$

$$\varphi_0(U_0\cap U_1)=\varphi_0(\{Z_0,Z_1\neq 0\})=\{w_1\neq 0\}$$

$$\varphi_1(U_0\cap U_1)=\varphi_1(\text{— '' —})=\{\tilde{w}_1\neq 0\}$$

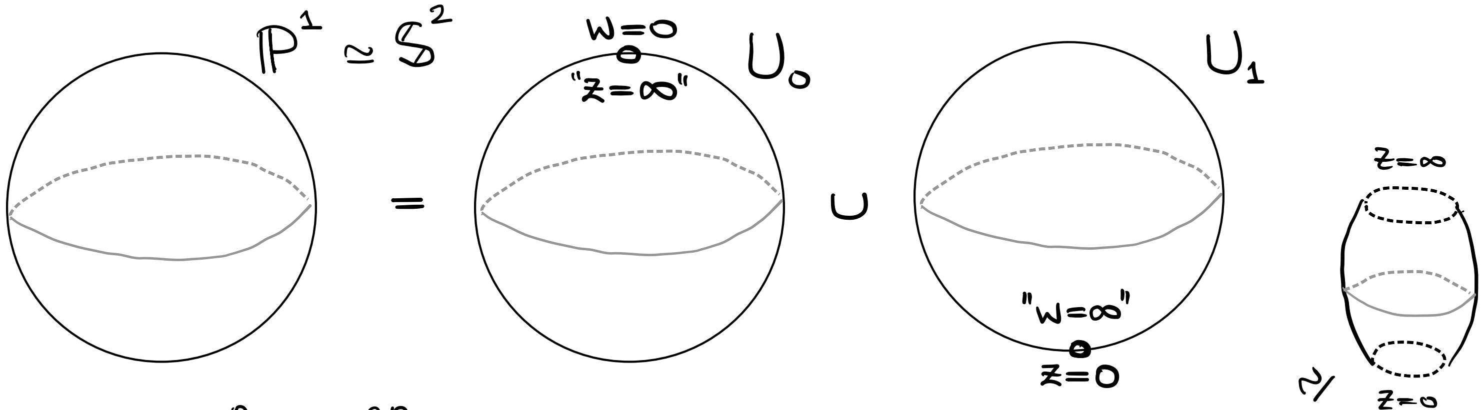
- Exercise: find φ_{ij} for arbitrary n.
- Note: $\mathbb{P}^n$ is compact
- In particular, on $\mathbb{P}^1=\{[Z_0:Z_1]\}$, we have two charts

$$U_0=\{Z_0\neq 0\}\cong\left\{z=\frac{Z_1}{Z_0}\in\mathbb{C}\right\}\cong\mathbb{C}$$

and
$$U_1=\{Z_1\neq 0\}\cong\left\{w=\frac{Z_0}{Z_1}\in\mathbb{C}\right\}\cong\mathbb{C}$$

with transition $w=1/z$ over $U_0\cap U_1\cong\mathbb{C}^*$.

$\mathbb{P}^1\simeq S^2$



$\cong$

$U_0\cap U_1\cong\mathbb{C}^*$

- note: $\mathbb{P}^n\not\simeq S^{2n}$ for $n\geq 2$.
- In fact: S^m cannot be a cx mfld for any m except $m=2$ and 6
- Famous open problem (Hopf problem): Can S^6 be a cx mfld?

3 Quintic Threefold in $\mathbb{P}^4$ (Fermat quintic)

Consider subset of $\mathbb{P}^4$ given as zero locus of a homogeneous polynomial:

$$X := \left\{ [z_0 : \cdots : z_4] \in \mathbb{P}^4 \;\middle|\; \underbrace{z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5}_{P(\vec{z})} = 0 \right\} \subset \mathbb{P}^4$$

Notice:

- well-defined b/c P is homogeneous: $P(\lambda \vec{z}) = \lambda P(\vec{z}) \quad \forall \lambda \in \mathbb{C}^*$
- X = cx mfld b/c can use (Holomorphic) Imp FT:

$$\frac{\partial P}{\partial \vec{z}} = \begin{bmatrix} 5z_0^4 & 5z_1^4 & 5z_2^4 & 5z_3^4 & 5z_4^4 \end{bmatrix}$$

for any $[\vec{z}] \in \mathbb{P}^4$, at least one z_i is nonzero.

- X is compact.

§2 Canonical Bundle

- On $X = \mathbb{C}^n$ with coordinates $(z_1, \dots, z_n)$, just like in real diff. geom. have holomorphic differentials $dz_1, \dots, dz_n$.

- They span the holomorphic cotangent space at every $p = \vec{z} \in \mathbb{C}^n$

$$T_p^*X := \mathrm{span}_{\mathbb{C}}\{dz_1, \dots, dz_n\} = \{ t_1 dz_1 + \dots + t_n dz_n \mid \vec{t} \in \mathbb{C}^n \}$$

- Thinking of $\mathbb{C}^n$ as $X_{\mathbb{R}} := \mathbb{R}^{2n}$ with $z_k = x_k + iy_k$ and ctgt space

$$T_p^* X_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}(dx_1, dy_1, \dots, dx_n, dy_n),$$

we have: $$dz_k := dx_k + i dy_k \in T_p^* X_{\mathbb{R}} \otimes \mathbb{C} =: \Lambda^1_p X$$
$$T_p^* X_{\mathbb{R}} \otimes \mathbb{C} = \mathrm{span}_{\mathbb{C}}\{dx_1, dy_1, \dots, dx_n, dy_n\}$$

- Notice: $T_p^* X \subsetneq T_p^* X_{\mathbb{R}} \otimes \mathbb{C}$

$$T_p^* X = \mathrm{span}_{\mathbb{C}}\{dx_1 + i dy_1, \dots, dx_n + i dy_n\} =: \Lambda_p^{1,0} X$$

complementary subspace $$\mathrm{span}_{\mathbb{C}}\{dx_1 - i dy_1, \dots, dx_n - i dy_n\} =: \Lambda_p^{0,1} X$$
$$\mathrm{span}_{\mathbb{C}}\{dx_1 - i dy_1, \dots, dx_n - i dy_n\} = \mathrm{span}_{\mathbb{C}}\{d\bar{z}_1, \dots, d\bar{z}_n\}$$

$$\Rightarrow \quad \Lambda^1_p X = \Lambda^{1,0}_p X \oplus \Lambda^{0,1}_p X$$

we will return to this decomposition

- T_p^*X are fibres of the <u>holomorphic cotangent bundle</u>:

$$\begin{tikzcd} \coprod_{p\in X} T_p^*X =: T^*X \arrow[d] & T_p^*X \arrow[l, hook'] \arrow[d] \\ \mathbb{C}^n & \{p\} \arrow[l, hook'] \end{tikzcd}$$

holomorphic vector bundle with fibres $T_p^*X \cong \mathbb{C}^n$

- <u>Holomorphic differential one-forms</u> are the holomorphic sections:

$$\Omega^1_X = \left\{ \begin{array}{l}\text{local}\\ \text{holo}\\ \text{sections}\end{array} \quad \omega \left(\begin{array}{c} T^*X \\ \downarrow \\ X \end{array} \right. \right\} =: \Gamma_{\mathrm{loc}}(T^*X)$$

$$= \left\{ \omega = f_1(\vec{z})\,dz_1 + \ldots + f_n(\vec{z})\,dz_n \right\}$$

holomorphic functions $\quad f_i : U \subset X \longrightarrow \mathbb{C}$

$$=: \mathrm{span}_X \{ dz_1, \ldots, dz_n \}$$

- Holomorphic two-forms:

$$\Omega^2_X := \Big\{ \omega = \sum_{i<j} \underbrace{f_{ij}(\vec{z})}_{\text{holo}}\, dz_i \wedge dz_j \Big\}$$

$$= \mathrm{span}_X \{ dz_i \wedge dz_j \mid i<j \}$$

$$= \Omega^1_X \wedge \Omega^1_X = \Lambda^2 \Omega^1_X \quad \left(2^{\text{nd}} \text{ exterior power of } \Omega^1_X \right)$$

$$= \Gamma_{\mathrm{loc}}\left(\Lambda^2 T^*X \right)$$

where $\Lambda^2 T_p^*X = \mathrm{span}_{\mathbb{C}} \{ dz_i \wedge dz_j \mid i<j \} \cong \mathbb{C}^{(n^2-n)/2}$

$\vdots$

- $\Omega^n_X := \Lambda^n \Omega^1_X$

$$= \{\omega = \underbrace{f(\vec{z})}_{\text{holo}} dz_1 \wedge \dots \wedge dz_n\}$$

$$= \operatorname{span}_X \{dz_1 \wedge \dots \wedge dz_n\}$$

$$= \Gamma_{loc}(\Lambda^n T^* X)$$

where $\Lambda^n T_p^* X = \operatorname{span}_{\mathbb{C}}\{dz_1 \wedge \dots \wedge dz_n\} \cong \mathbb{C}$.

- Ω^n_X is (sections of) a line bundle over X
- Ω^n_X is the top exterior power of Ω^1_X b/c $\Omega^{n+1}_X = \{0\}$

$\Rightarrow \Omega^n_X$ is called the canonical bundle of $X = \mathbb{C}^n$

- notation: $\Omega^{top}_X = \Omega_X = \omega_X = K_X$

- $\Omega_X = \Omega_{\mathbb{C}^n}$ admits a nowhere-vanishing global section

$$\omega = dz_1 \wedge \dots \wedge dz_n \in H^0(X, \Omega_X).$$

called a (holomorphic) volume form on X.

- If $\vec{w} = (w_1, \dots, w_n)$ another coordinate on $\mathbb{C}^n$, then

$$dw_1 \wedge \dots \wedge dw_n = \underbrace{\det\left(\frac{d\vec{w}}{d\vec{z}}\right)} \cdot dz_1 \wedge \dots \wedge dz_n$$

Jac.det is nowhere-vanishing

- More generally, any X = cx mfld has a holo ctgt bundle

$$\Omega^1_X = \left\{ \begin{smallmatrix}\text{local}\\ \text{hol}\\ \text{sections}\end{smallmatrix} \quad T^*X \longrightarrow X \right\}$$

- <u>Defn</u>: The <u>canonical bundle</u> of X is the line bundle

$$\Omega_X := \Lambda^n \Omega^1_X \qquad \text{where } n = \dim X.$$

- <u>Defn</u>: We say Ω_X is <u>trivial</u> if it admits a nowhere-vanishing global section; i.e. if X admits a *holomorphic* volume form.

- If (U, φ) = cx chart on X, then

$$\begin{tikzcd} T^*X \arrow[d] & T^*U \arrow[l, hook'] \arrow[d] & T^*\mathbb{C}^n|_V \arrow[l, "\sim"', "\varphi^*"] \arrow[d] \arrow[r, hook] & T^*\mathbb{C}^n \arrow[d] \\ X & U \arrow[l, hook'] \arrow[r, "\sim", "\varphi"'] & V \arrow[r, hook] & \mathbb{C}^n \end{tikzcd}$$

i.e. $\Omega^1_U = \Omega^1_X|_U = \Gamma_{loc}(T^*U) = \mathrm{span}_U \{\omega_1, \dots, \omega_n\}$

where $\omega_i := \varphi^*(dz_i)$

- If $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ two charts on X, then

? $\quad \Omega^1_{U_\alpha}\Big|_{U_\alpha \cap U_\beta} \xleftarrow{\sim} \Omega^1_{U_\beta}\Big|_{U_\alpha \cap U_\beta}$

- <u>Defn</u>: m_p := ideal of all hol. fns vanishing at p

Then $T^*_pX := m_p / m_p^2$

$$\Omega^1_X := \Delta^*(\mathcal{I}_\Delta / \mathcal{I}_\Delta^2)$$

Examples:

(0.) For $X = \mathbb{C}^n$:

$$\Omega_X = \operatorname{span}_X \{dz_1 \wedge \ldots \wedge dz_n\} \cong \mathcal{O}_X \quad \text{(trivial)}$$

(1.) For $X = \mathbb{T}_\Lambda = \mathbb{C}^n / \Lambda$:

The volume form $dz_1 \wedge \ldots \wedge dz_n$ on $\mathbb{C}^n$ is translation-invariant $\Rightarrow$ descends to a volume form on X:

$$\omega = dz_1 \wedge \ldots \wedge dz_n \in H^0(X, \Omega_X)$$

$\Rightarrow \Omega_X \cong \mathcal{O}_X$ for any Λ.

(2.) For $X = \mathbb{P}^n$:

- First, study the case $X = \mathbb{P}^1 = \{[z_0 : z_1]\}$

  Here, $\Omega_X = \Omega^1_X$.
- Recall affine charts:

$$U_0 = \{z_0 \neq 0\} \cong \left\{ z = \frac{z_1}{z_0} \in \mathbb{C} \right\} \cong \mathbb{C}$$

$$U_1 = \{z_1 \neq 0\} \cong \left\{ w = \frac{z_0}{z_1} \in \mathbb{C} \right\} \cong \mathbb{C}$$

with transition function $z = 1/w$.

- So: $\Omega^1_{U_0} = \operatorname{span}_{U_0}\{dz\}$ and $\Omega^1_{U_1} = \operatorname{span}_{U_1}\{dw\}$

$$\Omega^1_{U_0}\Big|_{U_0 \cap U_1} \xrightarrow{\sim} \Omega^1_{U_1}\Big|_{U_0 \cap U_1}$$

$$dz \longmapsto \frac{dz}{dw} \cdot dw = \frac{d}{dw}\left(\frac{1}{w}\right) \cdot dw = -\frac{dw}{w^2}$$

$\Rightarrow$ dz has a pole of order 2 at $z=\infty$.

$\Rightarrow$ dz does not define a global non-vanishing hol sec of $\Omega_{\mathbb{P}^1}$

$\Rightarrow$ dz does not define a volume form on $\mathbb{P}^1$

- Does $\mathbb{P}^1$ admit a volume form?

Suppose $\omega \in H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1})$ is a volume form.
Then in the two affine charts, we have:

$$\omega(z) = f(z)\,dz \qquad \text{and} \qquad \omega(w) = g(w)\,dw$$

where f, g = nowhere-vanishing entire hol. fns, related over $U_0 \cap U_1$ by

$$f(z)\,dz = g(w(z))\,dw(z) = -g(1/z)\,\frac{dz}{z^2}$$

i.e. $$f(z) = -g(1/z)\cdot\frac{1}{z^2}$$

For simplicity, assume they have Taylor expansions with inf. radius of convergence

$$\sum_{k=0}^{\infty} f_k\cdot z^k = -\sum_{k=0}^{\infty} g_k\cdot\left(\frac{1}{z}\right)^k\cdot\frac{1}{z^2}$$

$\Rightarrow$ $f_0, f_1, \ldots = 0$ and $g_0, g_1, \ldots = 0$

$\Rightarrow$ $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = \{0\}$

$\Rightarrow$ $\mathbb{P}^1$ does not admit a volume form

$\Rightarrow$ $\Omega_{\mathbb{P}^1}$ is not trivial!

- We have also shown that $\Omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$.

<u>Defn</u>: For any $k \in \mathbb{Z}$, the <u>line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$</u> is the lb obtained by gluing trivial line bundles over U_0 and U_1 using the transition function $g_{01} = z^k = (z_1/z_0)^k$

i.e. a section of $\mathcal{O}_{\mathbb{P}^1}(k)$ is given by two functions

$$s_0 : U_0 \longrightarrow \mathbb{C} \quad \text{and} \quad s_1 : U_1 \longrightarrow \mathbb{C}$$

related over $U_0 \cap U_1$ by $\quad s_0(z) = z^k s_1(1/z)$.

$$\begin{array}{ccc} \| & & \| \\ \sum_{i=0}^{\infty} a_i \cdot z^i & = & z^k \sum_{i=0}^{\infty} b_i z^{-i} \end{array}$$

$\Rightarrow$ $a_i = b_i = 0 \quad \forall\, i > k$ and $a_0 = b_k$, $a_1 = b_{k-1}, \ldots$

$\Rightarrow$ a global section of $\mathcal{O}_{\mathbb{P}^1}(k)$ is a polynomial $\sum_{i=0}^{k} a_i z^i = \sum_{i=0}^{k} a_i \left(\frac{z_1}{z_0}\right)^k$

$$\Rightarrow \boxed{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) = \left\{ \begin{array}{l} \text{homogeneous polynomials} \\ \text{of degree } k \text{ in variables } z_0, z_1 \end{array} \right\} \cong \mathbb{C}^{k+1}}$$

$\Rightarrow$ only $\mathcal{O}_{\mathbb{P}^1}(k=0) = \mathcal{O}_{\mathbb{P}^1}$ admits a global <u>nonvanishing</u> section.

- <u>Defn</u>: Over $X = \mathbb{P}^n$, for any $k \in \mathbb{Z}$, the <u>line bundle $\mathcal{O}_{\mathbb{P}^n}(k)$</u> is the line bundle obtained by gluing trivial line bundles over each affine chart U_i using the transition map

$$g_{ij} := (z_j / z_i)^k.$$

- $$\boxed{H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \left\{ \begin{array}{l} \text{homogeneous polynomials} \\ \text{of degree } k \text{ in } n+1 \text{ variables} \end{array} \right\} \cong \mathbb{C}^{\binom{k+n}{n}}}$$

- Important property : $\boxed{\mathcal{O}_{\mathbb{P}^n}(k_1) \otimes \mathcal{O}_{\mathbb{P}^n}(k_2) \cong \mathcal{O}_{\mathbb{P}^1}(k_1+k_2)}$

- A local section of $\Omega_{\mathbb{P}^n}$ over any $U_i = \{z_i \neq 0\} \cong \mathbb{C}^n_{w_1 \dots w_n} \subset \mathbb{P}^n$ is

$$\omega_i = f_i(\vec{w})\, dw_1 \wedge \dots \wedge dw_n$$

e.g. for $n=2$: $\varphi_{01}(w_1, w_2) = (\tilde{w}_1, \tilde{w}_2) = \left(\frac{1}{w_1}, \frac{w_2}{w_1}\right)$

$$\Rightarrow d\tilde{w}_1 \wedge d\tilde{w}_2 = d\left(\frac{1}{w_1}\right) \wedge d\left(\frac{w_2}{w_1}\right) = -\frac{1}{w_1^3}\, dw_1 \wedge dw_2$$

for $n=3$: $\varphi_{01}(w_1, w_2, w_3) = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) = \left(\frac{1}{w_1}, \frac{w_2}{w_1}, \frac{w_3}{w_1}\right)$

$$[z_0 : z_1 : z_2 : z_3] = \left[1 : \frac{z_1}{z_0} : \frac{z_2}{z_0} : \frac{z_3}{z_0}\right] \longmapsto (w_1, w_2, w_3) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right)$$

$$= \left[\frac{z_0}{z_1} : 1 : \frac{z_2}{z_1} : \frac{z_2}{z_1}\right] \longmapsto (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) = \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}, \frac{z_3}{z_1}\right)$$

$$(w_1, w_2, w_3) \longmapsto [1 : w_1 : w_2 : w_3] = \left[\frac{1}{w_1} : 1 : \frac{w_2}{w_1} : \frac{w_3}{w_1}\right] \longmapsto \underbrace{\left(\frac{1}{w_1}, \frac{w_2}{w_1}, \frac{w_3}{w_1}\right)}_{= (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)}$$

$$\Rightarrow d\tilde{w}_1 \wedge d\tilde{w}_2 \wedge d\tilde{w}_3 = d\left(\frac{1}{w_1}\right) \wedge d\left(\frac{w_2}{w_1}\right) \wedge d\left(\frac{w_3}{w_1}\right)$$

$$= -\frac{1}{w_1^4}\, dw_1 \wedge dw_2 \wedge dw_3$$

$\boxed{\underline{\text{Lem}}: \ \Omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)}$

<u>Exe</u>: prove this lemma

$\Rightarrow \Omega_{\mathbb{P}^n}$ is <u>not</u> trivial for any n

$\Rightarrow \mathbb{P}^n$ does not admit a volume form!

3. For $X = \{ z_0^5 + \cdots + z_4^5 = 0 \} \subset \mathbb{P}^4$

Fact 1 (Adjunction Formula):

If $X \subset Y$ smooth hypersurface, then

$$\Omega_X \cong \left(\Omega_Y \otimes \mathcal{O}_Y(X)\right)\big|_X$$

Fact 2: If $Y = \mathbb{P}^n$ and X = hypersurface of degree d, then

$$\mathcal{O}_Y(X) \cong \mathcal{O}_{\mathbb{P}^n}(d)$$

So: if X = sm. hypersurf of deg d, then

$$\Omega_X \cong \left(\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d)\right)\big|_X$$

$$\cong \left(\mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbb{P}^n}(d)\right)\big|_X$$

$$\cong \mathcal{O}_{\mathbb{P}^n}(d-n-1)\big|_X$$

$\hookrightarrow$ trivial $\Leftrightarrow$ $d = n+1 = 4+1 = 5$ $\rightarrow$ quintic hypersurface.

$\Rightarrow$ our X admits volume form i.e. Ω_X is trivial!

§3 Calabi-Yau Manifolds

- Recall: if X cx mfld of dim n, then the complexified real cotangent space of the underlying real mfld $X_{\mathbb{R}}$,

$$\Lambda^1_p X := T^*_p X_{\mathbb{R}} \otimes \mathbb{C} \qquad \forall p \in X$$

decomposes as

$$\Lambda^1_p X = \Lambda^{1,0}_p X \oplus \Lambda^{0,1}_p X$$

where $\forall p \in X$ and local coords $\vec{z}$ near p,

$$\Lambda^{1,0}_p X = \operatorname{span}_{\mathbb{C}}\{dz_1, \dots, dz_n\}$$

$$\Lambda^{0,1}_p X = \operatorname{span}_{\mathbb{C}}\{d\bar{z}_1, \dots, d\bar{z}_n\}$$

$$\Rightarrow \Lambda^1 X := \coprod_{p \in X} \Lambda^1_p X = \Lambda^{1,0} X \oplus \Lambda^{0,1} X := \coprod_{p \in X} \left(\Lambda^{1,0}_p X \oplus \Lambda^{0,1}_p X\right)$$

- Look at 2^{nd} exterior power:

$$\begin{aligned} \Lambda^2_p X &= \left(\Lambda^{1,0}_p \oplus \Lambda^{0,1}_p\right) \wedge \left(\Lambda^{1,0}_p \oplus \Lambda^{0,1}_p\right) \\ &= \left(\Lambda^{1,0}_p \wedge \Lambda^{1,0}_p\right) \oplus \left(\Lambda^{1,0}_p \wedge \Lambda^{0,1}_p\right) \oplus \left(\Lambda^{0,1}_p \wedge \Lambda^{0,1}_p\right) \\ &=: \Lambda^{2,0}_p \oplus \Lambda^{1,1}_p \oplus \Lambda^{0,2}_p \end{aligned}$$

$$\Rightarrow \Lambda^2 X = \Lambda^{2,0} X \oplus \Lambda^{1,1} X \oplus \Lambda^{0,2} X$$

CY geometry lives here

Definitions:

- A hermitian metric on a cx mfld X is a C^∞-section

$$h \in C^\infty(X_{\mathbb{R}}, \Lambda^{1,0}X \otimes \Lambda^{0,1}X)$$

locally given by

$$h \underset{\text{loc}}{=} h(\vec{z}, \vec{\bar{z}}) = \sum_{i,j} h_{ij}(\vec{z}, \vec{\bar{z}})\, dz_i \otimes d\bar{z}_j$$

where (h_{ij}) is a smoothly varying pos-definite hermitian matrix.
(i.e. $(h_{ij})^\dagger = (h_{ij})$)

- The fundamental form of (X, h) is a $(1,1)$-form

$$\omega \in C^\infty(X_{\mathbb{R}}, \Lambda^{1,1}X)$$

defined in local coords $(\vec{z}, \vec{\bar{z}})$ by

$$\omega = \omega(\vec{z}, \vec{\bar{z}}) := \frac{i}{2} \sum_{i,j} h_{ij}(\vec{z}, \vec{\bar{z}})\, dz_i \wedge d\bar{z}_j.$$

- The fundamental form ω is closed (i.e. $d\omega = 0$) if (h_{ij}) satisfies the system of PDEs

$$\frac{\partial h_{ij}}{\partial z_k} = \frac{\partial h_{kj}}{\partial z_i} \quad \text{and} \quad \frac{\partial h_{ij}}{\partial \bar{z}_k} = \frac{\partial h_{kj}}{\partial \bar{z}_i}$$

$$d\omega = \frac{i}{2} \sum_{i,j,k} \left(\frac{\partial h_{ij}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j + \frac{\partial h_{ij}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_i \wedge d\bar{z}_j \right)$$

- A Kähler metric h on a cx mfld X is a hermitian metric on X with closed fundamental form ω (Kähler form); i.e. $d\omega = 0$.

- A Kähler cx mfld is a pair (X, ω).

- Since $d\omega=0$, a Kähler form determines a cohomology class
$$[\omega] \in H^2(X_{\mathbb{R}}, \mathbb{R})$$
called <u>Kähler class</u>.

- If $[\omega]=[\omega']$, then $\omega' = \omega + i\partial\bar{\partial} f$
for some smooth real-valued function $f: X_{\mathbb{R}} \to \mathbb{R}$; i.e.
$$\omega'(\vec{z}, \vec{\bar{z}}) = \frac{i}{2} \sum_{i,j} \left(h_{ij} + 2\frac{\partial^2 f}{\partial z_i \cdot \partial \bar{z}_j} \right) dz_i \wedge d\bar{z}_j$$

- A <u>Kähler-polarised</u> cx mfld is a pair $(X, [\omega])$.

- A <u>Calabi-Yau manifold</u> is a compact Kähler-polarised cx mfld $(X, [\omega])$ with trivial canonical bundle; i.e. $\Omega_X \cong \mathcal{O}_X$.

Examples:

⓪ For $X = \mathbb{C}^n$:

- standard hermitian metric $h_0 = \sum_i dz_i \otimes d\bar{z}_i$
- $\omega_0 = \frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i$
- $d\omega_0 = 0 \Rightarrow (X, \omega_0)$ is Kähler
- $\Omega_X = \text{span}\{dz_1 \wedge \ldots \wedge dz_n\} \cong \mathcal{O}_X$
- But X is <u>not</u> compact

$\Rightarrow (\mathbb{C}^n, [\omega_0])$ <u>not</u> Calabi-Yau

① For $X = \mathbb{T}_\Lambda$

- std Kähler form ω_0 on $\mathbb{C}^n$ is translation-invariant

$\Rightarrow$ descends to the quotient $X = \mathbb{C}^n/\Lambda$

$\Rightarrow (\mathbb{T}_\Lambda, [\omega_0])$ is Kähler-polarised

- Recall: X is compact and $\Omega_X \cong \mathcal{O}_X$

$\Rightarrow (\mathbb{T}_\Lambda, [\omega_0])$ <u>is</u> CY

(2) For $X = \mathbb{P}^n$

- <u>Fubini-Study metric</u> :

  if $[Z_0 : Z_1 : \cdots : Z_n]$ = homog. coords on $\mathbb{P}^n$

  on an affine chart e.g. $U_0 = \{Z_0 \neq 0\}$,
  let $z_i := Z_i / Z_0$, and define

$$\omega_{FS} := \frac{i}{2} \partial \bar{\partial} \log\left(1 + |\vec{z}|^2\right) = \frac{i}{2} \sum_{i,j} \frac{\partial^2 \log(1+|\vec{z}|^2)}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$$

$$h_{FS} = \frac{i}{2} \sum \text{ —''— } dz_i \otimes d\bar{z}_j$$

- <u>FACT</u>: h_{FS} is hermitian and $d\omega_{FS} = 0$

$\Rightarrow (\mathbb{P}^n, [\omega_{FS}])$ is Kähler-polarised.

- Recall: $\mathbb{P}^n$ is compact
- <u>But</u>: $\Omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \not\cong \mathcal{O}_{\mathbb{P}^1}$

$\Rightarrow (\mathbb{P}^n, [\omega_{FS}])$ <u>not</u> CY

(3) For $X = \{Z_0^5 + \cdots + Z_4^5 = 0\} \subset \mathbb{P}^4$

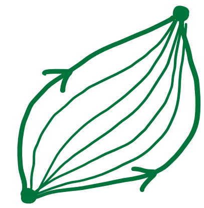
- $\omega := \omega_{FS}\big|_X$ and $h := h_{FS}\big|_X$

$\Rightarrow (X, [\omega])$ is Kähler-polarised

- X compact
- <u>Recall</u>: $\Omega_X \cong \mathcal{O}_X$

$\Rightarrow (X, [\omega])$ <u>is</u> CY

§4 Curvature and Physics



- Metric h on a mfld $\rightsquigarrow R(h)$ <u>curvature tensor</u>

$$\begin{array}{c} R(h) \\ \Big\downarrow \text{trace} \\ \mathrm{Ric}(h) \end{array}$$

<u>Ricci curvature</u>

- A metric h is <u>Ricci-flat</u> if $\mathrm{Ric}(h) = 0$.

- In physics, Ricci-flat metrics represent vacuum solutions of Einstein's field equations (w/o cosmological constant)

- <u>Calabi-Yau Theorem</u>:

  If $(X, [\omega])$ is CY, then $\exists!$ representative

$$\omega_{RF} \in [\omega]$$

  such that the corresponding Kähler metric h_{RF} is Ricci-flat.

- <u>Examples</u>:

  1. For $X = \mathbb{T}_\Lambda$, the std Kähler metric h_0 is Ricci-flat
  3. For $X = \{Z_0^5 + \dots + Z_4^5 = 0\} \subset \mathbb{P}^4$, the FS metric h_{FS} is <u>not</u> Ricci-flat. But CY Thm says $\exists!\ h_{RF}$ st $[\omega_{RF}] = [\omega_{FS}]$ and $\mathrm{Ric}(\omega_{RF}) = 0$.

This means there is a sm. real function $f : X_{\mathbb{R}} \to \mathbb{R}$ st

$$\omega_{RF} = \omega_{FS} + i\partial\bar\partial f$$

i.e. $$h_{RF}(\vec{Z}, \vec{\bar{Z}}) = \frac{i}{2} \sum_{i,j} \left((h_{FS})_{ij} + 2 \frac{\partial^2 f}{\partial Z_i \, \partial \bar{Z}_j} \right) dZ_i \otimes d\bar{Z}_j$$

<u>Huge difficulty</u>: f is extremely complicated and nonexplicit!