Report on Lattice QCD program on τ data for $(g-2)_\mu$

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for the RBC/UKQCD collaborations

Virtual mini workshop on tau decays, December 9th

a_μ ON THE LATTICE

Time-momentum representation [Bernecker, Meyer, '11]

$$G^\gamma(t) = \frac{1}{3}\sum_k \int d\vec{x}\, \langle j_k^\gamma(t,\vec{x}) j_k^\gamma(0)\rangle \quad \to \quad a_\mu = 4\alpha^2 \sum_t w_t G^\gamma(t)$$

$$\rho_{\mu\nu}(p) = \frac{1}{2\pi}\int d^4x\, e^{ipx}\, \langle 0| j_\mu^\gamma(x) j_\nu^\gamma(0)|0\rangle = (p_\mu p_\nu - g_{\mu\nu} s)\,\rho(s)\,,\ [p^2 = s]$$

e.g. restriction to 2π $\rho(s) \overset{2\pi}{=} \frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} |F_\pi(s)|^2$

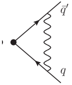
Lattice correlation functions

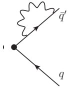
$G^\gamma(t) = \int ds\, \frac{\sqrt{s}}{2}\, e^{-\sqrt{s}t}\, \rho(s)$

fully inclusive: both channels and energy $s \in [0,\infty)$

Radiative corrections

Ward identity protects renormalization of EM current

$-Q_u Q_d \times$ 

$+ \, (Q_u^2 + Q_d^2) \times \frac{1}{2}$ 

Isospin decomposition of u, d current

$$j_\mu^\gamma = \frac{i}{6}\left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) + \frac{i}{2}\left(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d\right) = j_\mu^{(0)} + j_\mu^{(1)}$$

$$G_{II'}^\gamma \equiv \frac{1}{3}\sum_k \int d\vec{x} \langle j_k^{(I)}(x)\, j_k^{(I')}(0)\rangle$$

Examine low-energy regime

$j_k^{(0)}$ couples to 3π; $j_k^{(1)}$ couples to 2π

$G_{01}^\gamma \rightarrow$ mixing between 3π and $2\pi \rightarrow \rho - \omega$ mixing

$G_{11}^\gamma \rightarrow$ QED and SIB corrections to 2π (and 4π)

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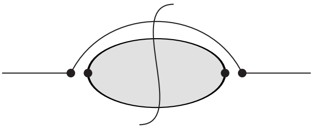
Hadronic τ decays

Fermi theory



$$d\Gamma = \frac{1}{4m} d\Phi_q \sum_f d\Phi_f \sum_{\text{spin}} |\mathcal{M}_f|^2$$

$$= \frac{1}{4m} d\Phi_q \frac{G_F^2 |V_{ud}|^2}{2} \mathcal{L}_{\mu\nu}(P,q)\, \rho^{\mathsf{w}}_{\mu\nu}(p)$$

Charged spectral density isospin limit $= \rho^{\mathsf{w},0}$ $\left[d\Phi_q = \frac{d^3 q}{(2\pi)^3 2\omega_q} \right]$

$$\frac{d\Gamma(s)}{ds} = G_F^2 |V_{ud}|^2 \frac{m^3}{16\pi^2} \left(1 + \frac{2s}{m^2}\right)\left(1 - \frac{s}{m^2}\right)^2 \rho^{\mathsf{w},0}(s)$$

$$= G_F^2 |V_{ud}|^2 \frac{m^3}{16\pi^2}\, \kappa(s)\, \rho^{\mathsf{w},0}(s)$$

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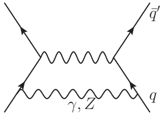
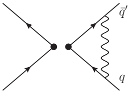
Short-distance effects

[Sirlin '82][Marciano-Sirlin '88][Brateen-Li '90]

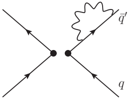


$$\frac{\alpha}{\pi}\left[(4Q_{\mathrm{u}} - Q_{\mathrm{d}})\log\frac{m_W}{m_\tau} + \ldots\right] \to \text{EFT } m_W \to \mu$$

"Hadronic" side $-\frac{\alpha}{2\pi}(Q_{\mathrm{u}} - Q_{\mathrm{d}})^2 \log(\mu/m_\tau)$



$\frac{\alpha}{\pi} Q_{\mathrm{u}} Q_{\mathrm{d}} \log\frac{\mu}{m_\tau} + \ldots$



$-\frac{\alpha}{2\pi}(Q_{\mathrm{u}}^2 + Q_{\mathrm{d}}^2)\log\frac{\mu}{m_\tau} + \ldots$

Adding all terms + τ self-energy $\to \frac{2\alpha}{\pi}\log\frac{m_W}{m_\tau}$
effects of Z boson $m_W \to m_Z$ leading to S_{EW}

Resummation of large logs via Renorm. Group (see [Erler '02])
$\alpha^n \log^n m_Z^2$ and $\alpha\alpha_s^n \log^n m_Z^2$

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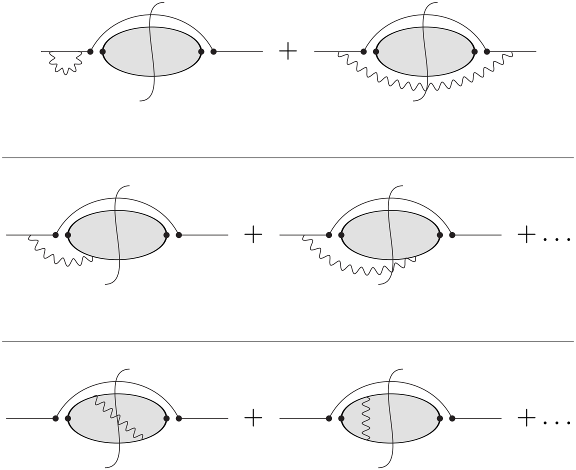
IR SAFETY



This separation is IR safe, but problematic in the UV sector

scheme, scale and (QED) gauge dependence

Roadmap

Goal: (several) intermediate to long Euclidean windows

- τ data competitive
- reduced sensitivity to high mult. channels ($a_\mu^W[4\pi] \approx 0.15 \times a_\mu^W[2\pi]$)
- reduced sensitivity to missing spectrum $[m_\tau, \infty)$

0. experimental data	exclusive 2π	inclusive (non-strange)
1. initial state	analytic	analytic
2. interference τ-hadrons	EFTs or dispersive	Lattice (*)

Hadronic spectral density w/ radiative corrs

$\frac{d\Gamma^{\rm exp}}{ds}$ - 1. - 2. "=" $(1+\alpha\delta Z)\rho^{\rm w,0}(s) + \delta\rho^{\rm w}(s)$

- scheme+scale depedent renormalization
- gauge-dependent terms, cancel w/ 1. and 2.

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Weak Euclidean correlator

[MB et al. in prep]

Introduce charged isovector currents $j_k^{(1,-)}(x) = \frac{i}{\sqrt{2}}(\bar{u}\gamma_\mu d)(x)$

$G_{11}^W = \frac{1}{3}\sum_k \int d^3x \, \langle 0 | j_k^{(1,+)}(t,\vec{x}) j_k^{(1,-)}(0) | 0 \rangle$

- contains full mass spectrum
- contains effects of both 2π and 4π channels
- requires renorm. on lattice, e.g. momentum scheme
  - mom. scheme $\neq \overline{\mathrm{MS}}$ so matching required

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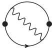
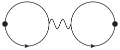
$\delta G_{11} = G_{11}^W - G_{11}^\gamma$ an interesting probe for IB effects

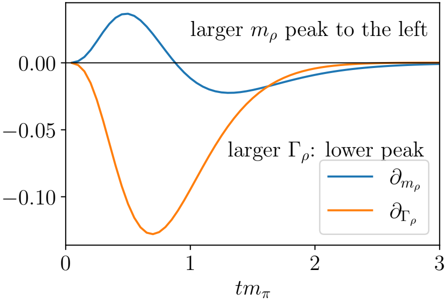
- isosymmetric limit $\delta G_{11} = 0$
- requires only  and 

From phenomenological point of view δG_{11} contains

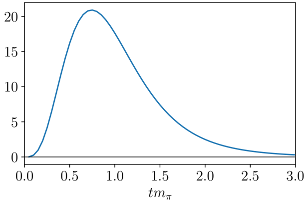
- pion mass splitting on phase space
- rho mass and width splitting
- neutral vs charged final state radiation

δG_{11} VS MODELS

Model spectral density as $\beta(s, m_\pi)^3 |F_\pi(s)|^2$ using Gounaris-Sakurai
shifts in m_ρ and Γ_ρ captured by δG_{11}



examine in Euclidean time effects of m_ρ and G_ρ



m_ρ shifts position of "HVP peak"; Γ_ρ shifts height of "HVP peak"

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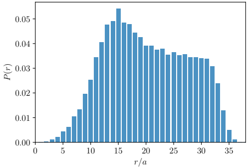
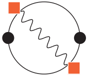
Status of rbc/ukqcd effort

$O(10^3)$ point sources
$\rightarrow O(10^6)$ pairs





Sampling strategy from stochastic point sources

Flagship ensembles at phys. pion mass
- 48I, 96I first datasets generated
- ongoing (blind) cross checks 2 groups
- generation of new data
  - larger volumes and heavier pion masses

Data complete QED+SIB (w/ unquenching) by end of January '25 $\rightarrow$ HVP program

Started renormalization program for τ
- coarse lattice spacing $a^{-1} = 1.73$ GeV

Inverse Laplace methods

Physical observables as integrals of spectral densities $P = \int ds\, \kappa(s)\, \rho(s)$

Euclidean correlator $C(t) = \int ds\, e^{-\sqrt{s}|t|} \rho(s) \frac{\sqrt{s}}{2}$

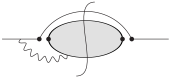
can we solve $P = \int dt\, f(t)\, C(t)$ for unknown f?

analytic structure of $\rho(s)$ vs analytic structure of $\kappa(s)$

$\rho(s)$ typically branch cuts, $\kappa(s)$ typically poles

Recently lots of activity in Lattice community

promising methods and interesting results

open a path to study inclusively 

Conclusions

Lattice QCD+QED program for $(g-2)_\mu$ from τ decays

- G^γ_{01} and G^γ_{00} required for complete IB corrections
- τ-hadron γ exchange from EFTs first, later LQCD
- paper spelling all this out in prep.

This talk: we propose δG_{11} as promising quantity for cross-checks

- only two diagrams, shorter time scales
- renorm. program started $\rightarrow \Gamma_\rho$ splitting
  - m_ρ splitting even w/o renorm.
- syst. errs. from $[m_\tau, \infty)$ and 4π to be addressed

Thanks for your attention