University of Amsterdam

European Research Council
Established by the European Commission

Classical Log Soft Graviton Theorem from BMS Superrotation

Sangmin Choi

Amplitudes 2025
Seoul National University

Based on 2403.13053, 2412.16142, 2412.16149 and WIP with Alok Laddha and Andrea Puhm

ERC STARTING GRANT HOLOHAIR 852386

Asymptotic symmetry

- What is an asymptotic symmetry?

$$\text{asymptotic symmetry} = \frac{\text{allowed gauge symmetry}}{\text{trivial gauge symmetry}}$$

- Aren't all gauge transformations trivial?
  No. If you follow the Noether procedure, you notice that there is a falloff condition on the parameter for the symmetry to be trivial.

- For gravity in asymptotically flat spacetimes, the asymptotic symmetry group is called the BMS group.
  The BMS group extends the Poincaré group with an angle-dependent time translation called *supertranslation*.
  Lorentz symmetry is generalized to *superrotations*.

- Asymptotic symmetries are symmetries of scattering processes:

$$\langle \text{out}|(Q^+ S - S Q^-)|\text{in}\rangle = 0$$

Asymptotic symmetries and soft theorems

Over the past decade, a remarkable correspondence between soft theorems and asymptotic symmetries has been uncovered:

A soft theorem is the Ward identity (or charge conservation) of an asymptotic symmetry.

$$\langle \text{out}|(Q^+ S - S Q^-)|\text{in}\rangle = 0 \qquad \Longleftrightarrow \qquad \text{soft theorem}$$

- LGT (or "superphaserotation") $\Longleftrightarrow$ Leading soft photon theorem
  [He, Mitra, Porfyriadis, Strominger]
- BMS supertranslations $\Longleftrightarrow$ Leading soft graviton theorem
  [He, Mitra, Lysov, Strominger]
- Divergent LGT $\Longleftrightarrow$ Tree-level subleading soft photon theorem
  [Campiglia, Laddha]
- BMS superrotations $\Longleftrightarrow$ Tree-level subleading soft graviton theorem
  [Kapec, Lysov, Pasterski, Strominger]
- This extends to higher dimensions [He, Mitra]
- $\cdots$

Asymptotic symmetries and soft theorems

Schematically, here's what happens (for superrotation):

The leading/subleading soft graviton theorems at tree-level are

$$M_{n+1}(\omega\hat{k},\epsilon) \overset{\omega\to 0}{=} \left(\frac{1}{\omega}S_{-1}(\hat{k},\epsilon) + S_0(\hat{k},\epsilon)\right) M_n$$

Compute the future/past superrotation charges via covariant phase space method. They split into hard and soft parts

$$Q^{\pm}[Y] = Q_H^{\pm}[Y] + Q_S^{\pm}[Y]$$

A judicious choice of the sphere vector field Y^A yields

$$-\langle Q_H^+ S - S Q_H^- \rangle = S_0 M_n, \qquad \langle Q_S^+ S - S Q_S^- \rangle = \lim_{\omega\to 0}(1+\omega\partial_\omega)M_{n+1}$$

The Ward identity is

$$\langle Q^+ S - S Q^- \rangle = 0 \quad \Longrightarrow \quad \lim_{\omega\to 0}(1+\omega\partial_\omega)M_{n+1} = S_0 M_n$$

This picks out the subleading soft graviton theorem in the soft expansion.

Log soft theorem

- In 2014, Bern, Davies and Nohle have shown that the subleading soft behavior of gluon and graviton amplitudes are modified by loop effects. Using dimensional regularization, the subleading soft factors were shown to receive one-loop corrections [Bern, Davies, Nohle '14]

$$S_0 \quad \overset{\text{1-loop}}{\rightarrow} \quad S_0 + \frac{1}{\epsilon} S_{0,\text{div}},$$

  and no further corrections at two loops and higher.

- In 2018, Laddha, Sahoo and Sen have shown using massive QED and massive scalars coupled to gravity in 4D, that there must be a non-analytic term in the soft behavior of photons and gravitons [Laddha, Sen '18] [Sahoo, Sen '18]

$$\lim_{\omega\to 0} M_{n+1}(\omega\hat{k},\epsilon) \overset{\omega\to 0}{=} \left(\frac{1}{\omega} S_{-1}(\hat{k},\epsilon) + \ln\omega\, S_{\ln}(\hat{k},\epsilon) + O(\omega^0)\right) M_n$$

  which is one-loop exact.

Log soft theorem

For external massive scalars coupled to gravity, the log soft factor is given by

$$S_{\ln} = S_{\ln}^{\text{cl}} + S_{\ln}^{\text{qn}},$$

$$S_{\ln}^{\text{cl}} = \frac{1}{8\pi} \sum_{\substack{a,b \\ a\neq b, \eta_a=\eta_b}} \frac{p_a^\rho \epsilon_{\rho\mu} k_\nu}{p_a.k} J_a^{\mu\nu} \left[(p_a.p_b) \frac{(1+\beta_{ab}^2)}{\beta_{ab}} \right] - \frac{i}{4\pi} \sum_{b,\eta_b=-1} k.p_b \sum_a \frac{\epsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k},$$

$$S_{\ln}^{\text{qn}} = -\frac{i}{16\pi^2} \sum_{\substack{a,b \\ a\neq b}} \frac{p_a^\rho \epsilon_{\rho\mu} k_\nu}{p_a.k} J_a^{\mu\nu} \left[(p_a.p_b) \frac{(1+\beta_{ab}^2)}{\beta_{ab}} \ln\left(\frac{1+\beta_{ab}}{1-\beta_{ab}} \right) \right]$$

$$- \frac{1}{8\pi^2} \sum_a \frac{\epsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \sum_b p_b.k \ln \frac{m_b^2}{(p_b.\hat{k})^2}$$

$S_{\ln}^{\text{cl}}$ is the classical factor, $S_{\ln}^{\text{qn}}$ is the quantum factor, $\beta_{ab} = \sqrt{1 - \frac{p_a^2 p_b^2}{(p_a.p_b)^2}}$ is the relative speed, and J_{ab} is the (orbital) angular momentum operator.

Log soft theorem

This log term is referred to as the *logarithmic soft theorem* — a new soft theorem!

The correspondence suggests that it is the Ward identity of an asymptotic symmetry.

Some earlier literature in this direction includes:

- Campiglia and Laddha 1903.09133
- Atul Bhatkar 1912.10229
- Donnay, Nguyen and Ruzziconi 2205.11477
- Agrawal, Donnay, Nguyen and Ruzziconi 2309.11220

In this talk, I would like to convince you that long-range interactions of photons and gravitons naturally give rise to the logarithmic soft theorem.

For concreteness, we consider superrotation in a theory of massive scalar particles coupled to gravity. A similar story follows for massive QED.

Superrotation on i^+

Massive particle trajectories asymptote to timelike infinities. A suitable resolution of timelike infinity is given by the hyperbolic coordinates,

$$\tau = \sqrt{t^2 - r^2}, \qquad \rho = \frac{r}{\sqrt{t^2 - r^2}},$$

where $\tau \to \infty$ while ρ is fixed.

A superrotation on i^+ is generated by the hyperboloid vector field $\bar{Y}^\alpha \partial_\alpha$ where

$$\bar{Y}^\alpha(y) = \int d^2\hat{q}\, G^\alpha_A(y;\hat{q}) Y^A(\hat{q}), \qquad (y^\alpha \in \{\rho, z, \bar{z}\})$$

where Y^A is the sphere vector field parametrizing superrotation on $\mathcal{I}^+$ and G^α_A is the AdS_3 vector bulk-to-boundary propagator in Euclidean signature.

Hard charge on i^+

To find the asymptotic fields at i^+, we solve the Einstein equations and the Klein-Gordon equation perturbatively starting from free fields.

Massive scalars give rise to sourced graviton at i^+ ($\kappa = \sqrt{32\pi G}$ is the coupling constant)

$$h_{\tau\tau}(\tau, y) \overset{\tau\to\infty}{=} \frac{1}{\tau}\overset{1}{h}_{\tau\tau}(y) + \cdots, \qquad \overset{1}{h}_{\tau\tau}(y) = \frac{m^2\kappa}{4\pi}\int d^3y'\, \frac{\sigma^2 - \frac{1}{2}}{\sqrt{\sigma^2 - 1}} b^\dagger(y')b(y')$$

Solutions to the KG equations develop a phase that diverges as $\ln\tau$,

$$\phi(\tau, \rho, \hat{x}) \overset{\tau\to\infty}{=} \exp(\tfrac{1}{2}i\kappa\overset{1}{h}_{\tau\tau}\ln\tau)\frac{e^{-im\tau}}{\tau^{3/2}}b(\rho\hat{x}) + (\text{subleading in } \tau) + \text{h.c.}$$

The hard superrotation charge diverges as $\ln\tau$, which we regulate by introducing an infrared scale Λ:

$$Q_H = \ln\Lambda^{-1}Q_H^{(\ln)} + O(\Lambda^0), \qquad Q_H^{(\ln)} = -\kappa m^2\int_{i^+} d^3y\, b^\dagger b\bar{Y}^\alpha\partial_\alpha\overset{1}{h}_{\tau\tau}.$$

This expression is “one-loop exact” in κ.

Soft charge on $\mathcal{I}^+$

Now we turn to the soft superrotation charge on $\mathcal{I}^+$.

Just as the scalar field obtains a divergent phase on i^+, on $\mathcal{I}^+$ the shear obtains a log-divergent phase due to graviton-graviton interactions which, after regularizing, leads to

$$Q_S = \int dud^2\hat{x}\, D_z^3 Y^z \left(-\frac{2}{\kappa^2}u\partial_u C^{\bar{z}\bar{z}} - \ln\Lambda^{-1}\frac{1}{\kappa}h^0_{rr}\partial_u C^{\bar{z}\bar{z}}\right) + \text{h.c.}$$

Also, due to the presence of massive matter fields at i^+, the shear develops a particular large-u behavior. By solving the Einstein equations at $\mathcal{I}^+_+$, one finds that

$$\text{powers of } \ln\tau \text{ in } T_{\mu\nu} \quad \Longrightarrow \quad \tfrac{1}{u}(\ln u)^n \text{ in } C_{zz} \text{ at } u\to\infty.$$

We use ϕ and $h_{\mu\nu}$ to compute $T_{\mu\nu}$: there is only one $\ln\tau$ in $T_{\tau\tau}$, $T_{\tau\alpha}$ and none in $T_{\alpha\beta}$ at leading order, from which it follows that

$$C_{zz}(u,z,\bar{z}) \overset{u\to\infty}{=} C^{(0)}_{zz}(z,\bar{z}) + \frac{1}{u}C^{(1)}_{zz}(z,\bar{z}) + \cdots$$

The $1/u$ term makes the u-integral of $u\partial_u C_{zz}$ divergent.

Soft charge on $\mathcal{I}^+$

We regulate the divergent integral using a large-time cutoff Λ^{-1}. By choosing a suitably large but finite $u_0 < \Lambda^{-1}$, we write

$$\begin{aligned}\int_{-\infty}^{\infty} du\, u\partial_u C_{zz} &= \left(\int_{-\Lambda^{-1}}^{-u_0} + \int_{u_0}^{\Lambda^{-1}}\right) du\, u\partial_u C_{zz} + \cdots \\ &= \ln\Lambda^{-1}\left(C_{zz}^{(0)+} - C_{zz}^{(0)-}\right) + \cdots \\ &= \ln\Lambda^{-1}\int_{-\infty}^{\infty} du\,(-\partial_u u^2\partial_u C_{zz}) + \cdots\end{aligned}$$

Thus the divergent part of the soft charge can be pulled out as

$$\begin{aligned}Q_S &= \ln\Lambda^{-1} Q_S^{(\ln)} + \cdots\,, \\ Q_S^{(\ln)} &= \int dud^2\hat{x}\, D_z^3 Y^z\left(\frac{2}{\kappa^2}\partial_u u^2\partial_u C^{\bar{z}\bar{z}} - \frac{1}{\kappa}h_{rr}^0\partial_u C^{\bar{z}\bar{z}}\right) + \text{h.c.}\end{aligned}$$

Again, this expression is "one-loop exact" in κ.

Log soft theorem as superrotation Ward identity

The total superrotation charge is "1-loop exact" in κ and has the expression

$$
\begin{aligned}
Q &= \ln \Lambda^{-1} \left[Q_H^{(\ln)} + Q_S^{(\ln)} \right] + \cdots \,, \\
Q_H^{(\ln)} &= -\kappa m^2 \int_{i^+} d^3 y \, b^\dagger b Y^\alpha \partial_\alpha \overset{1}{h}_{\tau\tau} , \\
Q_S^{(\ln)} &= \int du d^2 \hat{x} \, D_z^3 Y^z \left(\frac{2}{\kappa^2} \partial_u u^2 \partial_u C^{\bar{z}\bar{z}} - \frac{1}{\kappa} h^0_{rr} \partial_u C^{\bar{z}\bar{z}} \right) + \text{h.c.}
\end{aligned}
$$

The Ward identity $\langle \text{out} | (Q^+ S - S Q^-) | \text{in} \rangle = 0$ amounts to

$$
\lim_{\omega \to 0} \ln \Lambda^{-1} \partial_\omega \omega^2 \partial_\omega M_{n+1}(\omega \hat{k}, \epsilon) = \ln \Lambda^{-1} S_{\ln}^{\text{cl}} M_n + O(\Lambda^0)
$$

which is finite as the cutoff is removed $\Lambda \to 0$.

The Ward identity of superrotation with interactions is exactly the classical log soft graviton theorem. The projection $\lim_{\omega \to 0} \partial_\omega \omega^2 \partial_\omega$ kills all powers and picks out only $\ln \omega$!

Summary

The classical log soft graviton theorem is derived as the Ward identity of superrotation.

- We started from free fields and solved the equations of motion perturbatively in κ. This naturally develops logarithms in the solutions; no assumptions on falloffs were made.
- The corrections due to interactions make superrotation charge IR divergent, but with suitable regularization the Ward identity is finite.
- The finite Ward identity is exactly the log soft graviton theorem. The soft graviton insertion has a projection $\lim_{\omega\to 0}\partial_\omega \omega^2 \partial_\omega$ to $\ln\omega$, and the remaining terms organize to the classical log soft factor.
- The Einstein equations and the KG equations are solved to all orders in the coupling, and we observe only one-loop corrections. This is an independent derivation of the one-loop exactness of classical log soft theorem.