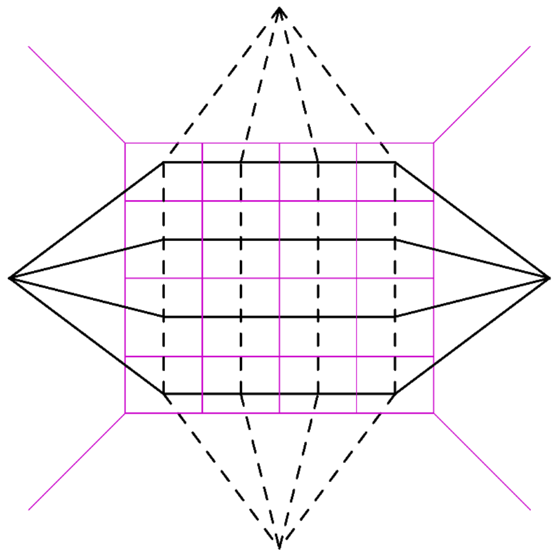
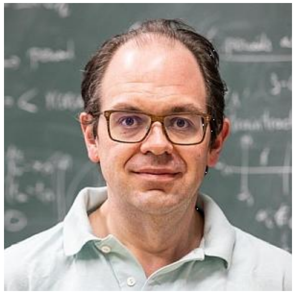
Antipodal (Self-)Duality in Planar N=4 SYM and Fishnet Theory



Lance Dixon (SLAC)

with Claude Duhr, 2502.00862



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NATIONAL ACCELERATOR LABORATORY

Outline & Summary

- Bizarre "Antipodal (Self) Duality" [ASD] for (some) form factors / amplitudes in Planar N=4 Super Yang-Mills theory
- All given by multiple polylogarithms with a Hopf algebra coaction whose maximal iteration is called the **symbol**
- A(S)D involves writing the **symbol** backwards (antipode), together with some **kinematic map**
- Antipode exchanges branch cuts and derivatives.
- **NEW:** A(S)D extends to some individual integrals – which are **also** amplitudes – four-point fishnet integrals
- Might help understand what physics underlies A(S)D

Reversing words

- Many words same backwards and forwards (palindromes):

ABBA

- Some also need a “letter map” (B⟷R, C ⟷ D):

ABRACADABRA

Planar N=4 SYM, testing ground for QCD amplitudes

- QCD's maximally supersymmetric cousin,
  N=4 super-Yang-Mills theory (SYM),
  gauge group $\mathrm{SU}(N_c)$, in large N_c (planar) limit

  Amplitudes = $\sum_i rational_i \times transcendental_i$

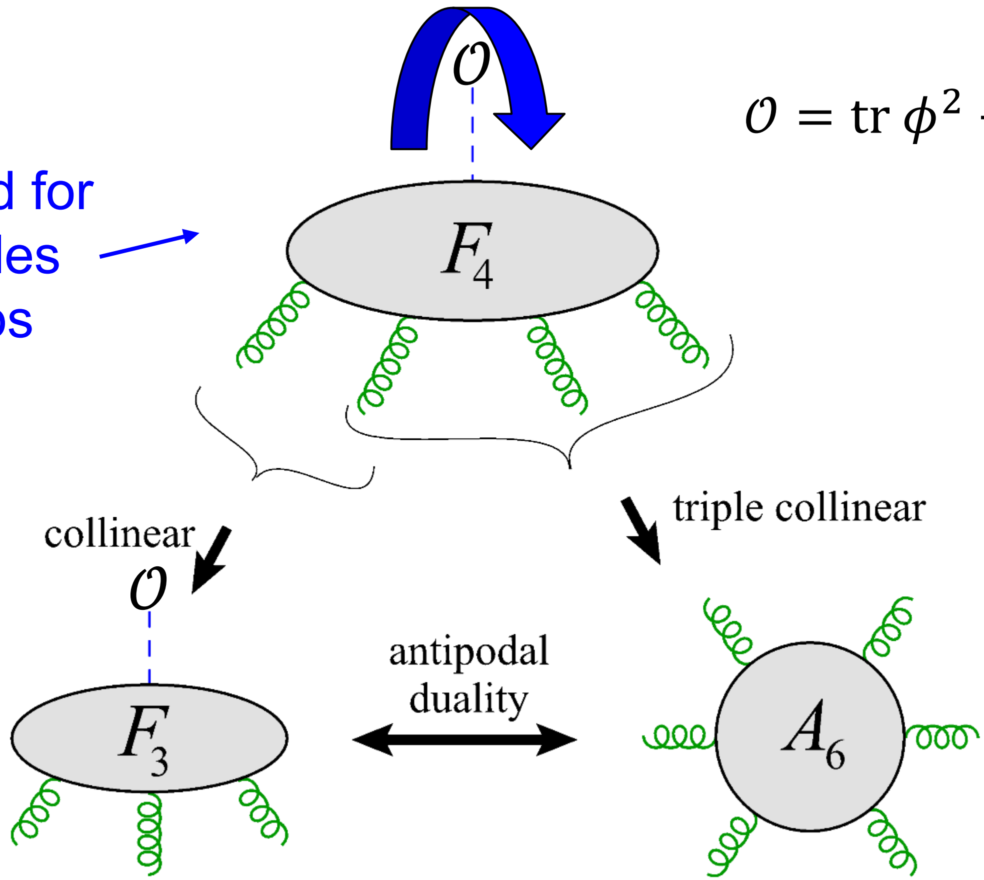
- For planar N=4 SYM, rational structure well understood, focus on transcendental functions.
- In processes we will look at today, the rational prefactors are very simple (tree amplitudes)
- Remove any infrared divergences by dividing by their known form (BDS-like ansatz)

Antipodal Self Duality
for a 4-point form factor in planar N=4 SYM

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410

$$\mathcal{O} = \mathrm{tr}\,\phi^2 + \cdots + \mathrm{tr}\,G_{\mu\nu}^{\mathrm{SD}} G^{\mu\nu\,\mathrm{SD}}$$

validated for 4 variables to 3 loops →



validated for 2 variables to 8 loops! →

antipode map reverses order of letters in symbol

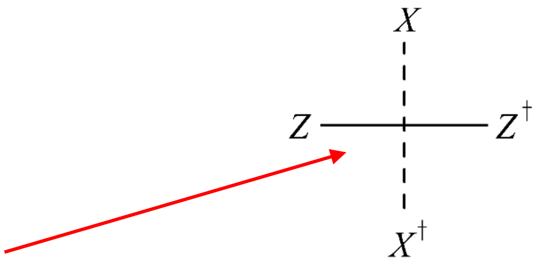
also requires a kinematic map

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

Strongly deformed planar N=4 SYM

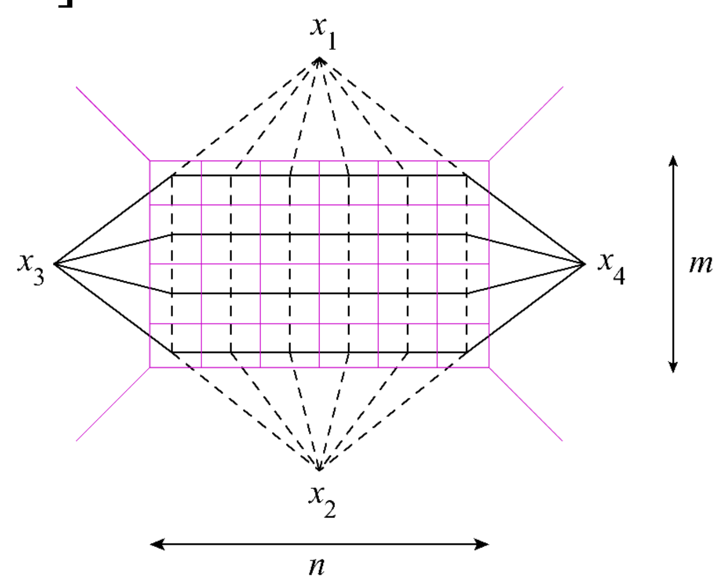
Gürdoğan, Kazakov, 1512.06704; Caetano, Gürdoğan, Kazakov, 1612.05895

- Deform planar N=4 SYM so that gauge coupling → 0 while N=1 superpotential gets very large
- Lose unitarity but keep integrability
- Simplest limit: only 2 complex scalars, X, Z with non-Hermitian coupling $\mathrm{Tr}[XZX^\dagger Z^\dagger]$



- Motivates 4-point fishnet integrals:

$$< \mathrm{Tr}[\,X^n(x_1)Z^m(x_3)X^{\dagger n}(x_2)Z^{\dagger m}(x_4)] > \; =$$



$$\propto \phi_{m,n}(u,v) = \; \phi_{m,n}(z,\bar{z})$$

$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \equiv \frac{z\bar{z}}{(1-z)(1-\bar{z})}\,, \quad v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \equiv \frac{u}{z\bar{z}}$$

Multiple polylogarithms (MPLs)

talks by Britto, Hartanto, Volovich, Yang, Bourjaily,…

- Characterize all form factors & amplitudes playing a role here
- At L loops, all results are weight $n = 2L$ MPLs, defined recursively as iterated integrals by

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

and $G\left(\vec{0}_n, x\right) = \frac{(\ln x)^n}{n!}$

MPL Hopf algebra

Goncharov; Brown; Goncharov, Spradlin, Vergu, Volovich; Duhr, Gangl, Rhodes

- Differential definition: $dF = \sum_{s_k \in \mathcal{L}} F^{s_k} \, d \ln s_k$
- Hopf algebra "co-acts" on space of MPLs,

  $\Delta: F \to F \otimes F$
- Derivative dF is one piece of Δ: $\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k$
- So we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F
- $\boldsymbol{s_k}$ **are letters in the symbol alphabet** $\mathcal{L}$

Iterate to get **symbol**

- Apply $\{n\text{-}1,1\}$ coaction iteratively:
- Define $\{n\text{-}2,1,1\}$ double coproducts, F^{s_k,s_j},

via derivatives of $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k \in \mathcal{L}} F^{s_k,s_j} \, d \ln s_k$$

- And so on for $\{n\text{-}m,1,\ldots,1\}$ m^{th} coproducts of F.
- Maximal iteration, n times for weight n function, is

the **symbol,** ["ln" is implicit in s_{i_k}]

$$\mathcal{S}[F] \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}} \, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers (often integers!)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example 1: Classical polylogarithms

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \mathrm{Li}_{n-1}(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

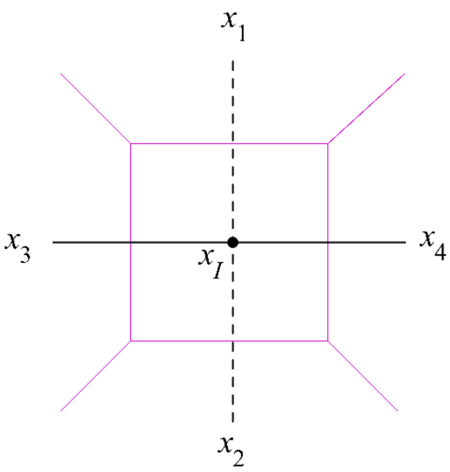
$$\frac{d}{dx} \mathrm{Li}_n(x) = \frac{\mathrm{Li}_{n-1}(x)}{x} = \mathrm{Li}_{n-1}(x) \frac{d \ln x}{dx}$$

- All are regular at $x = 0$, branch cut starts at $x = 1$.
- Iterated differentiation gives symbol:

$$\mathcal{S}[\mathrm{Li}_n(x)] = \mathcal{S}[\mathrm{Li}_{n-1}(x)] \otimes x$$
$$= \cdots = -(1-x) \otimes x \otimes x \ \ldots \otimes x$$

- Branch cut discontinuities displayed in first entry of symbol, e.g. clip off leading $(1-x)$ to compute discontinuity at $x = 1$.
- Derivatives computed from symbol by clipping last entry, multiplying by that $d\ln(\ldots)$. **Alphabet** $\mathcal{L} = \{x, 1-x\}$

Example 2



Bourjaily talk
Hodges (1977)
Bloch, Wigner (1978)

- First note that

$$\mathcal{S}[\ln x \ln y] = x \otimes y + y \otimes x$$

and $\ldots \otimes xy \otimes \cdots = \ldots \otimes x \otimes \cdots + \ldots \otimes y \otimes \cdots$

- Consider the 1 loop box integral [also known as the "(1,1) 4-point fishnet"] given by the Bloch-Wigner (single-valued) dilogarithm

$$\phi_{1,1} = 2[\underbrace{\mathrm{L}i_2(z)}_{-(1-z)\otimes z} - \mathrm{L}i_2(\bar{z})] + \ln(z\bar{z})\ln(\frac{1-z}{1-\bar{z}})$$

- Its symbol is $\mathcal{S}[\phi_{1,1}] = z\bar{z} \otimes \frac{1-z}{1-\bar{z}} - (1-z)(1-\bar{z}) \otimes \frac{z}{\bar{z}}$
- Symbol alphabet: $\mathcal{L} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$
- $\phi_{1,1}$ has an antipodal self-duality

ASD for (1,1) fishnet [box]

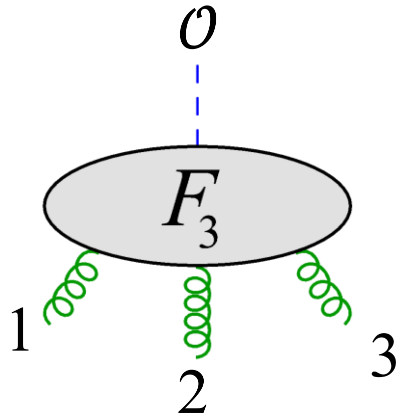
$$\mathcal{S}[\phi_{1,1}] = z\bar{z} \otimes \frac{1-z}{1-\bar{z}} - (1-z)(1-\bar{z}) \otimes \frac{z}{\bar{z}}$$

- Invariant under antipodal symmetry $\mathcal{S}$ (reverse letter order in symbol) combined with kinematic (letter) map:

$$\mathbf{C}: \quad z \to \bar{z} \qquad 1-z \to 1-\bar{z}$$
$$\bar{z} \to \frac{1}{z} \qquad 1-\bar{z} \to \frac{1}{1-z}$$

- Let $\hat{\mathbf{S}} \equiv \mathbf{C}\mathcal{S}$. By inspection:

$$\hat{\mathbf{S}}\,[\phi_{1,1}] = \phi_{1,1}$$



Three-gluon form factor

2 dimensionless variables u, v

$k_i^2 = 0 \quad s_{ij} = (k_i + k_j)^2 \qquad s_{123} = s_{12} + s_{23} + s_{31} = q^2$

$k_1 + k_2 + k_3 = -k_{\mathcal{O}}$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}} = 1 - u - v$$

- 6 letter symbol alphabet:

$$\mathcal{L} = \{ a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

- Symbols of form factor $F_3^{(L)}$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images:

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\ b\otimes d + \text{dihedral}$$

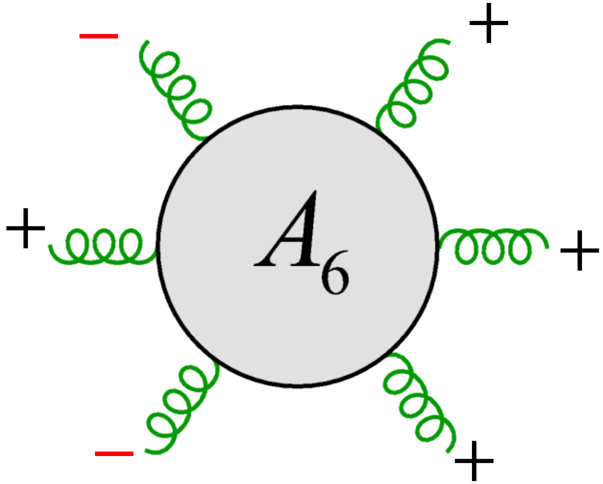
$$\mathcal{S}\left[F_3^{(2)}\right] = 4\ b\otimes d\otimes d\otimes d + 2\ b\otimes b\otimes b\otimes d + \text{dihedral}$$

Brandhuber, Travaglini, Yang, 1201.4170
LD, Gürdoğan, A. McLeod, M. Wilhelm, 2204.11901

known to 8 loops!

Six-gluon MHV amplitude

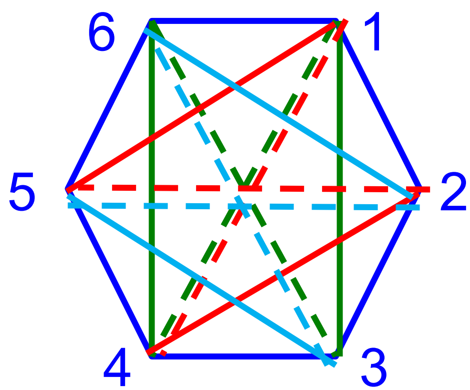
- Dual to Wilson hexagon, invariant under dual conformal transformations; it only depends on 3 dual conformal cross ratios, $\hat{u}, \hat{v}, \hat{w}$:



$$\hat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

$$\hat{v} = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$\hat{w} = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$



D_6 dihedral symmetry: cycle $(\mathrm{mod}\ 6)$ and flip, but it acts on $\hat{u}, \hat{v}, \hat{w}$ as $D_3 = S_3$

Parity-preserving surface: (lower-dimensional kinematics)

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

6-gluon symbol alphabet

talk by Volovich

- $\mathcal{L}_6 = \{\, \hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ → 1 for $\Delta = 0$

→ $\mathcal{L}'_6 = \{\, \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}},\ \hat{b} = \frac{\hat{v}}{\hat{w}u},\ \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Symbols of amplitude $A_6^{(L)}$ on $\Delta = 0$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images:

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b}\otimes\hat{d} + \text{dihedral}$$

$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b}\otimes\hat{d}\otimes\hat{d}\otimes\hat{d} + \frac{1}{2}\hat{b}\otimes\hat{b}\otimes\hat{b}\otimes\hat{d} + \text{dihedral}$$

$$\cdots$$

was known to 7 loops

Goncharov, Spradlin, Vergu, Volovich, 1006.5703, …, Caron-Huot, LD, Dulat, McLeod, von Hippel, 1903.10890

Antipodal duality (AD)

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

- Antipode map S, is a "coinverse" for the Hopf algebra.
- At symbol level, it simply reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

- Kinematic map in terms of underlying variables is:

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u+v+w=1$ to $\Delta = 0$ parity-preserving surface for A_6

Kinematic map on letters

$$\sqrt{\hat{a}} = d,\ \sqrt{\hat{b}} = e,\ \sqrt{\hat{c}} = f,\ \ \hat{d} = a,\ \hat{e} = b,\ \hat{f} = c$$

Works through 8 loops (even beyond symbol)!

LD, Liu, 2308.08199

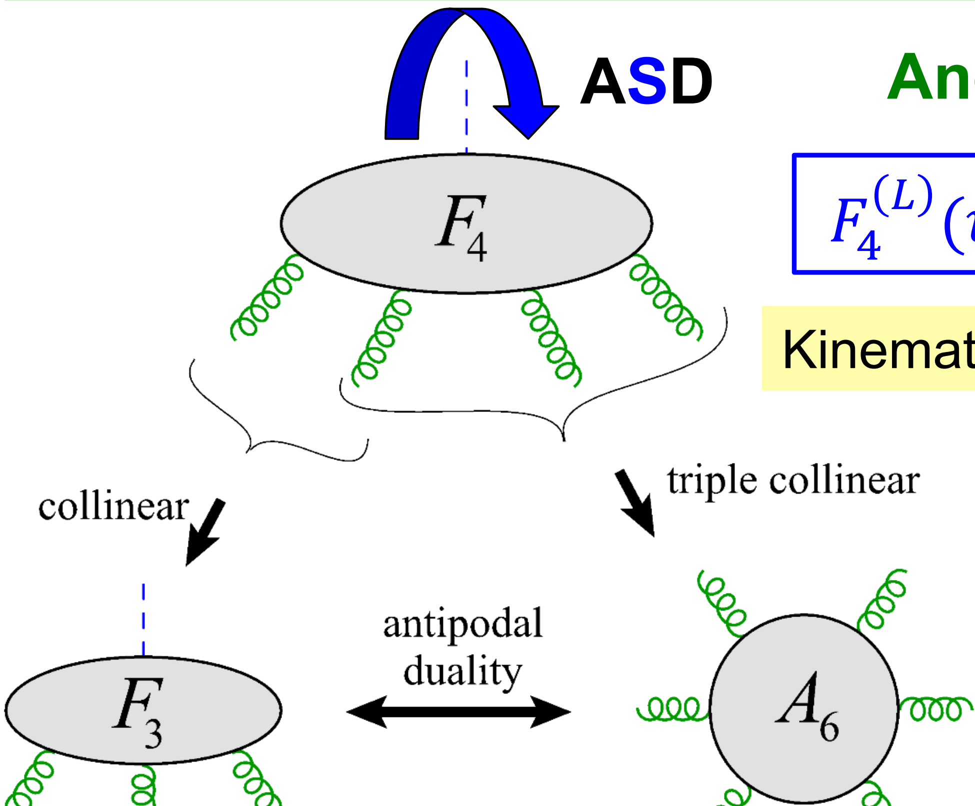
L	number of terms in symbol
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

(number of nonzero integers that match)

But why?!

Antipodal Self Duality

Given antipodal duality relating 2-collinear and 3-collinear limits of F_4, natural to search for self-duality of F_4 that holds for all parity-preserving bulk kinematics



$$F_3^{(L)}(u_i) = S[A_6^{(L)}(\hat{u}_i)]$$

And it's there!

$$F_4^{(L)}(u_i, v_i) = S[F_4^{(L)}(\mathbf{C}(u_i), \mathbf{C}(v_i))]$$

Kinematic map $\mathbf{C}$ simple in FFOPE variables:

$$\mathbf{C}: \quad T_2 \to \frac{T}{S}, \quad S_2 \to \frac{1}{TS}$$

$$T \to \sqrt{\frac{T_2}{S_2}}, \qquad S \to \sqrt{\frac{1}{T_2 S_2}}$$

$$F_2 = 1$$

Antipodal Self Duality Evidence by Counting

- 4-point form factor $F_4^{(L)}$ has palindromic {n,1,…,1} coproduct dimensions:

weight n	0	1	2	3	4	5	6	7	8
$L=2$	1	8	32	8	1	–	–	–	–
$L=3$	1	8	56	253	56	8	1	–	–
$L=4$	1	8	56	372	1730	372	56	8	1

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410 (rows $L=2$, $L=3$)

LD, Zhenjie Li, to appear → (row $L=4$)

- Only palindromic on a parity-preserving surface, or 3d kinematics:
- $\varepsilon_{\mu\nu\sigma\rho} k_1^\mu k_2^\nu k_3^\sigma k_4^\rho = 0$
- Only in "remainder function normalization"
- 93 letter alphabet → can only compute to 3 loops so far
- Palindromic behavior of {n,1,…,1} dimensions necessary but not sufficient for ASD

Higher-point form factors of $\mathcal{O}$?

- Two-loop MHV form factors $F_n^{(2)}$ computed from periodic Wilson loop   Z. Li, 2412.17974
- Especially remainder functions

  $R_n^{(2)} = F_n^{(2)} - \frac{1}{2}[F_n^{(1)}]^2$ for $n = 5,6$
- Go onto parity-preserving surface (3D kinematics), and count the number of independent letters in each symbol slot:

slot	1	2	3	4
$R_4^{(2)}$	8	24	24	8
$R_5^{(2)}$	15	70	70	15
$R_6^{(2)}$	24	150	150	24

Palindromic!

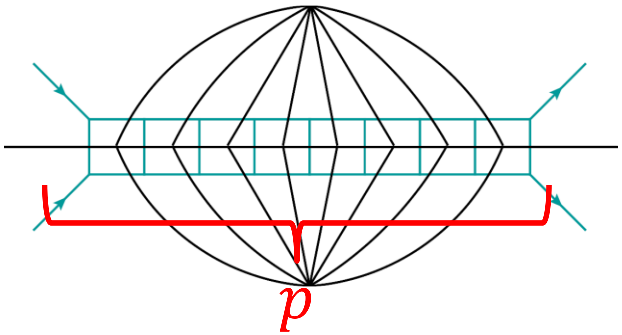
However, at least for $n = 5$, there does **not** appear to be a suitable kinematic map!   S. Xin

Antipodal self-duality for **integrals**?

Arkani-Hamed, Yuan, 1712.09991

- One loop n-gon integral
- $$I[Q] = \int \frac{d^n x\, \delta(1-\sum_i x_i)}{\sum_{i,j} Q_{ij} x_i x_j}$$
- Q_{ij} depends on kinematics
- Symbol antipode takes $Q \to Q^{-1}$
- However, if Q is for massless kinematics, Q^{-1} typically corresponds to internal masses
- So this occurrence of antipode is hard to interpret if particle masses are fixed…

How about ladder integrals?



Usyukina, Davydychev, PLB305, 136 (1993)

$$f_p(z,\bar{z}) = \sum_{j=p}^{j=2p} \frac{(p-1)!\, j!\, [-\ln(z\bar{z})]^{2p-j}}{(j-p)!\,(2p-j)!} [\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar{z})]$$

$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \equiv \frac{z\bar{z}}{(1-z)(1-\bar{z})}\,, \quad v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \equiv \frac{u}{z\bar{z}}$$

Alphabet: $\mathcal{L} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$

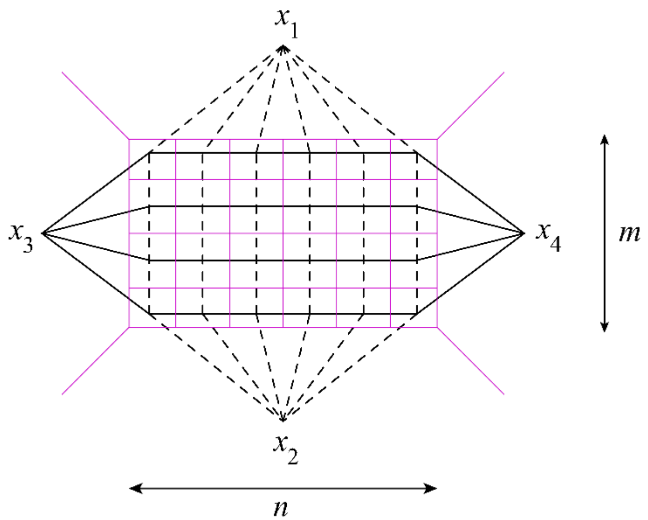
weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
(1, 1)	1	2	1														
(1, 2)	1	2	3	2	1												
(1, 3)	1	2	3	4	3	2	1										
(1, 4)	1	2	3	4	5	4	3	2	1								
(1, 5)	1	2	3	4	5	6	5	4	3	2	1						
(1, 6)	1	2	3	4	5	6	7	6	5	4	3	2	1				
(1, 7)	1	2	3	4	5	6	7	8	7	6	5	4	3	2	1		
(1, 8)	1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1

all **palindromic**

But closer inspection shows they **can't** be **self**-dual

Counting for 4-point fishnet integrals

Basso, LD, 1705.03545; Derkachov and Olivucci, 1912.07588, 2007.15049; Basso et al., 2105.10514



$$\propto \ \phi_{m,n} = \det_{1\leq i,j\leq m}(f_{n-m+i+j-1})$$

= Hankel determinant of ladders f_p

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
(2, 2)	1	2	3	4	6	4	3	2	1										
(2, 3)	1	2	3	4	6	8	10	10	10	7	4	2	1						
(3, 3)	1	2	3	4	6	8	11	14	17	20	17	14	11	8	6	4	3	2	1

palindromic only for $n = m$ – **square**!

The (1,1) [box] symbol

$$\mathcal{S}[\phi_{1,1}] = z\bar{z} \otimes \frac{1-z}{1-\bar{z}} - (1-z)(1-\bar{z}) \otimes \frac{z}{\bar{z}}$$

- It's invariant under antipodal symmetry S combined with the letter map:

$$\mathbf{C}: \quad z \to \bar{z} \qquad 1-z \to 1-\bar{z}$$
$$\bar{z} \to \frac{1}{z} \qquad 1-\bar{z} \to \frac{1}{1-z}$$

- Let $\hat{\mathbf{S}} \equiv \mathbf{C}S$. Then

$$\hat{\mathbf{S}}\,[\phi_{1,1}] = \phi_{1,1}$$

Square fishnet antipodal relation

LD, Duhr, 2502.00862

$$\hat{\mathbf{S}}[\phi_{m,m}] = \phi_{m,m}$$

- **Proof, Step 1**: Under $\hat{\mathbf{S}}$, the ladders transform as:

$$\tilde{f}_p \equiv \hat{\mathbf{S}}[f_p] = \sum_{k=0}^{p-1} \binom{p-1}{k} L^k \, f_{p-k} \qquad L = \ln z \ln \bar{z}$$

- Can derive, up to $(2\pi i)^2$ terms, by taking discontinuity around $z = 0$ (equivalent to acting with $z\partial_z + \bar{z}\partial_{\bar{z}}$ on other side of antipode) and using properties of single-valued functions.
- Can also derive completely using explicit antipode action on polylogs and combinatorial identities.

Proof, Step 2

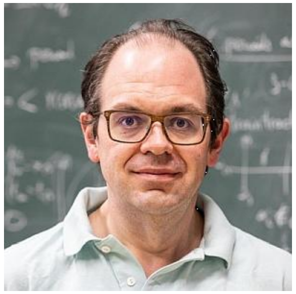
- Using $\tilde{f}_p \equiv \hat{\mathbf{S}}[f_p] = \sum_{k=0}^{p-1} \binom{p-1}{k} L^k f_{p-k}$,

  $\hat{\mathbf{S}}[\phi_{m,m}] = \det_{1\leq i,j\leq m}(\tilde{f}_{i+j-1})$

is a polynomial in L

- First compute $\partial_L \tilde{f}_p = (p-1)\tilde{f}_{p-1}$, then differentiate $\hat{\mathbf{S}}[\phi_{m,m}]$ by rows, expand by columns, to show that $\partial_L \hat{\mathbf{S}}[\phi_{m,m}] = 0$
- The surviving L-independent term is $\phi_{m,m}$

ASD "near" square fishnets?



in progress with Claude Duhr and Francesca Fernandes

- What other polynomials in ladder integrals f_p obey both the Steinmann relations and ASD, given the kinematic map $\mathbf{C}$ and hence $\tilde{f}_p \equiv \hat{\mathbf{S}}[f_p] = \sum_{k=0}^{p-1} \binom{p-1}{k} L^k f_{p-k}$?
- We find infinite families of solutions to these constraints, which are all linear combinations of determinants first defined in Coronado, 1811.03282
- See also: Caron-Huot, Coronado, 2106.03892; He, Jiang, Liu, Zhang, 2502.08871

Coronado determinants

$$M_{i_1,i_2,\cdots,i_n} = \begin{vmatrix} f_{i_1} & f_{i_2-1} & \cdots & f_{i_n-n+1} \\ f_{i_1+1} & f_{i_2} & \cdots & f_{i_n-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_1+n-1} & f_{i_2+n-2} & \cdots & f_{i_n} \end{vmatrix}$$

- Degree d polynomial from $d \times d$ determinant
- Labelled by increasing sequence of integers obeying $i_{k+1} \geq i_k + 2$
- We reverse the order of the columns which relabels (on diagonal still) by a decreasing sequence of integers $i_k \geq i_{k+1} \geq d$
- Subtract d from all entries: $N(i_1, \ldots, i_r) \equiv M_{i_1-d,\ldots,i_r-d}$

ASD transformations of N's

$$\tilde{N}(1) = N(1) + d\,L\,N(0)$$

$$\tilde{N}(1,1) = N(1,1) + (d-1)L\,N(0) + \frac{1}{2}d(d-1)L^2 N(0)$$

$$\tilde{N}(2) = N(2) + (d+1)L\,N(0) + \frac{1}{2}d(d+1)L^2 N(0)$$

- So $\boxed{(d-1)N(2) - (d+1)N(1,1)}$ is invariant!
- **Tip of the iceberg:** There is an infinite sequence (in d) of invariant combinations for every partition of every integer not containing 1. The coefficients involve polynomials related to elementary refinements of partitions (Hasse diagrams for Young's lattice)

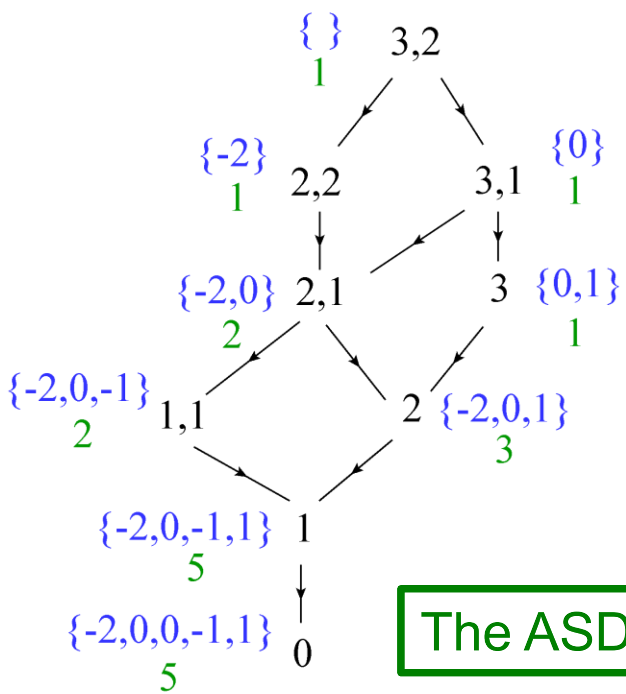
Another $\widetilde{N}$ example

$$
\begin{aligned}
\tilde{N}(3,2) = N(3,2) \\
&+ L\,(d+2)\,N(2,2) + L\,d\,N(3,1) \\
&+ \frac{L^2}{2!}\,2\,(d+2)d\,N(2,1) + \frac{L^2}{2!}\,d(d-1)\,N(3) \\
&+ \frac{L^3}{3!}\,2\,(d+2)(d+1)d\,N(1,1) + \frac{L^3}{3!}\,3\,(d+2)d(d-1)\,N(2) \\
&+ \frac{L^4}{4!}\,5\,(d+2)(d+1)d(d-1)\,N(1) \\
&+ \frac{L^5}{5!}\,5\,(d+2)\,(d+1)d^2(d-1)\,N(0)
\end{aligned}
$$

$$\widetilde{N}(\lambda) = \sum_{\mu \sqsubseteq \lambda} \frac{L^{n_\lambda - n_\mu}}{(n_\lambda - n_\mu)!} p(\lambda,\mu)\Pi(\lambda,\mu,d)N(\mu)$$

μ is an elementary refinement of λ

paths from λ to μ



$$\lambda = (\lambda_1, \ldots, \lambda_r), \qquad n_\lambda = \sum_i \lambda_i$$

$$\Pi(\lambda,\mu,d) = \prod_i (d - \kappa_i(\mu))$$

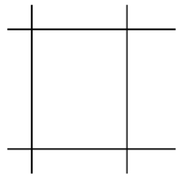
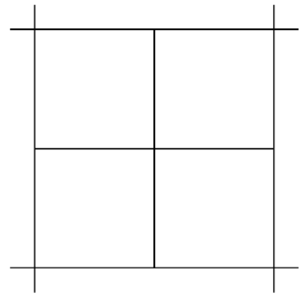
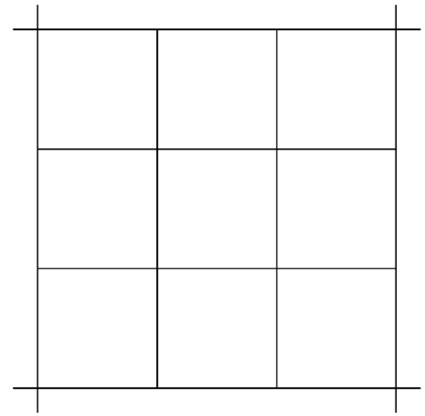
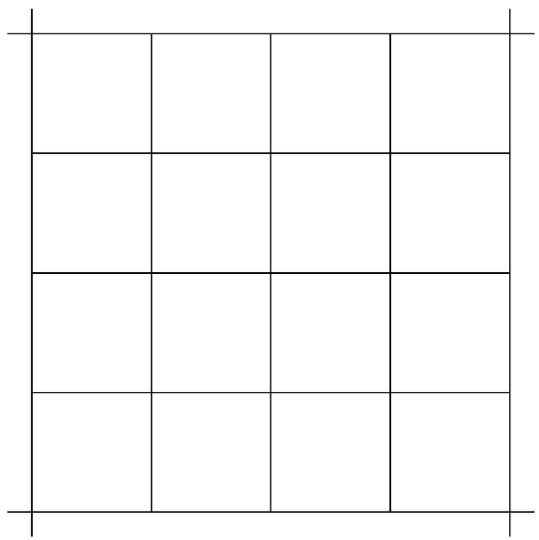
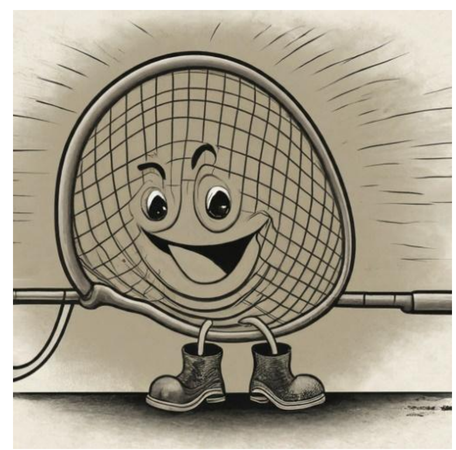
when $m \to m-1$, add the integer $(\#\geq m) - m$ to the set $\{\kappa_i\}$

The ASD invariant combinations can be described similarly

Summary & Open Questions

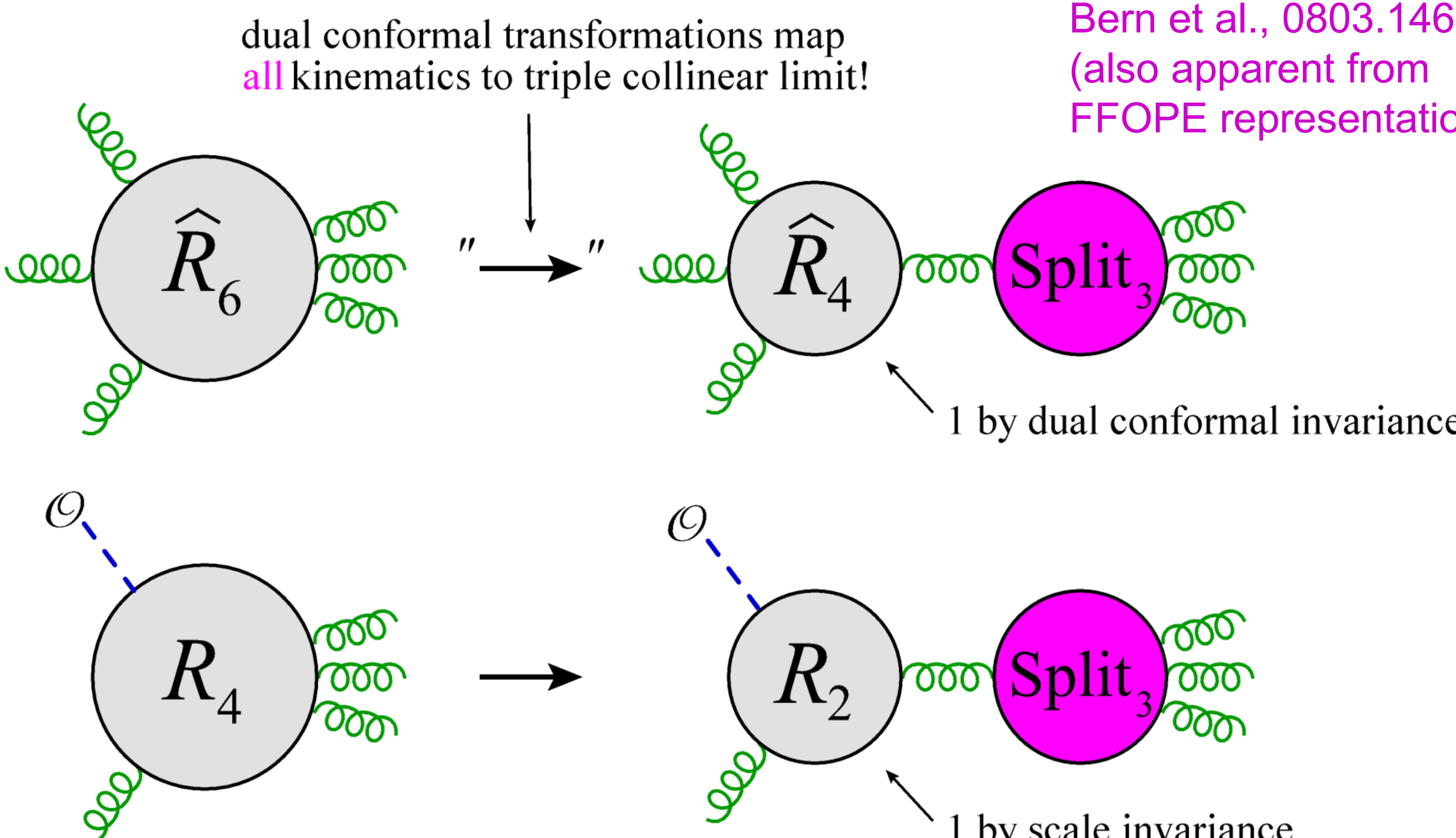
- 6-gluon amplitude ←→ 3-gluon form factor in planar N=4 SYM by strange new antipodal duality, swaps role of branch cuts and derivatives
- Embedded in 4-gluon form factor antipodal self-duality
- **Who ordered that?**
- Can now find at least antipodal self-duality in square 4d fishnet integrals (which have integrability-based representations)
- 3-dimensional kinematics seems to play a crucial role in all cases (parity preserving surfaces, or only 3 momenta). Why?
- Where else might it hold? 2d (elliptic) fishnet integrals?
  Duhr, Klemm, Loebert, Nega, Porkert, 2310.08625
- How much more can we exploit A(S)D to learn more about amplitudes, form factors, integrals in planar N=4 & beyond?

Extra Slides

Triple Collinear Limit of 4-point form factor → 6-gluon amplitude

Bern et al., 0803.1465;
(also apparent from
FFOPE representation)



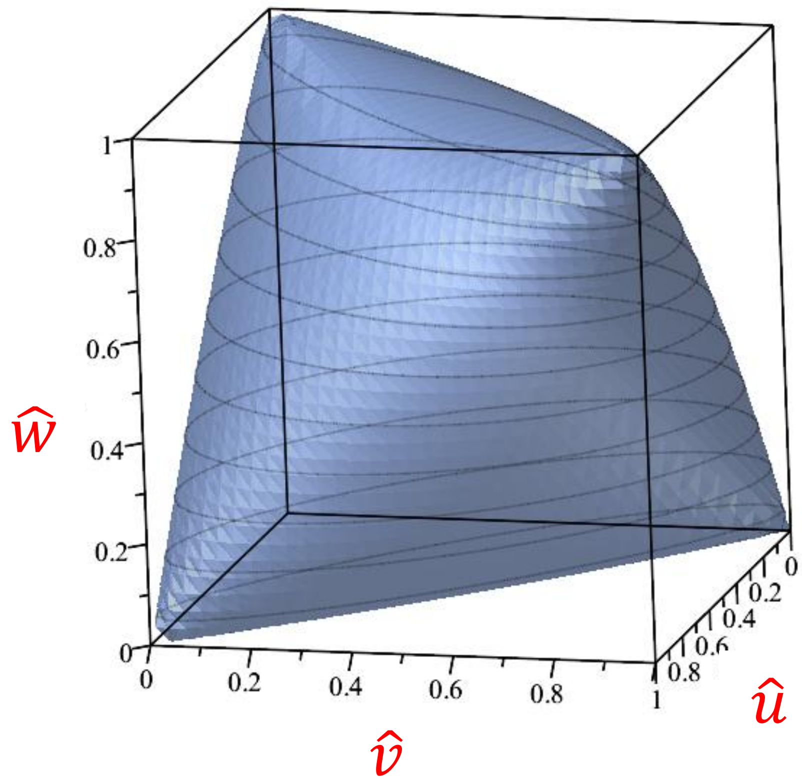
AD explains many patterns in F_3

- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- Empirically, 12 pairs of adjacent letters are forbidden:

$$\ldots a \otimes d \ldots, \quad \ldots b \otimes e \ldots, \quad \ldots c \otimes f \ldots$$
$$\ldots d \otimes a \ldots, \quad \ldots e \otimes b \ldots, \quad \ldots f \otimes c \ldots$$
$$\ldots d \otimes e \ldots, \quad \ldots e \otimes f \ldots, \quad \ldots f \otimes d \ldots$$
$$\ldots e \otimes d \ldots, \quad \ldots f \otimes e \ldots, \quad \ldots d \otimes f \ldots$$

- Resemble constraints from causality:
  Steinmann relations Steinmann, Helv. Phys. Acta (1960)
- But not really, which mystified us for a while…
- However, the relations are antipodally dual to the (extended) Steinmann relations for A_6 !!

Parity-preserving surface



$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

$\Delta = 0$ means that kinematics lies in a 3d subspace of 4d spacetime → parity invariant

FFOPE kinematical variables for F_4

$$u_i = \frac{s_{i,i+1}}{s_{1234}}, \qquad v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$$

$$-u_1 + u_3 + v_4 + v_1 = 1$$
$$-u_2 + u_4 + v_1 + v_2 = 1$$
$$-u_3 + u_1 + v_2 + v_3 = 1$$

$$u_1 = \frac{T^2 T_2^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_2 = \{1 + T^2 + \frac{S^2[(1+F_2^2)S_2T_2 + F_2(1+S_2^2+T^2+T_2^2)]}{F_2 S_2^2}\}^{-1}$$

$$u_3 = \frac{S^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_4 = \frac{S^2T^2}{S_2^2}\, u_2$$

$$v_1 = \frac{T_2^2+1}{S^2+T^2+T_2^2+1}$$

- OPE limit takes $T, T_2 \to 0$, interpolates between 2-collinear limit $T_2 \to 0$ and 3-collinear limit $T \to 0$

ASD beyond 2 loops

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410v2;
LD, Z. Li, to appear

- Bootstrapped symbol of F_4 at 3 loops, using same 113 letter (2-loop) alphabet.
- **Unique result**, which obeys all the FFOPE predictions we could check.
- 2 loop symbol uses only 34 of the letters [3,784 terms]
- 3 loop symbol uses only 88 of the letters [3,621,202 terms]
- 4 loop symbol uses only 88 of the letters [???? terms]
- No OPE checks yet at 4 loops, but few constraints needed to fix it.
- **ASD holds in 3D at 4 loops!**

Palindromic criterion not sufficient for ladders

- First entries: $\{\, z\bar{z},\ (1-z)(1-\bar{z})\,\}$
- Last entries (for $p > 1$): $\{\, z,\ \bar{z}\,\}$
- First pair of entries: 2 of 3 are products
- Last pair of entries (for $p > 2$): All 3 of 3 are products:

  $$\{\, z \otimes z, \qquad z \otimes \bar{z} + \bar{z} \otimes z, \qquad \bar{z} \otimes \bar{z} \,\}$$

- Since dimensions of first-pair space and last-pair space differ after projecting out products, there is no kinematic map for which the f_p ladders are antipodally self-dual ($p > 2$)