Instanton Scattering

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Introduction

- Perturbing around the self-dual sector of Yang-Mills is good for computing helicity amplitudes.
- It's possible to generalise to perturb around non-trivial self-dual backgrounds, e.g., self-dual plane waves, dyons, etc. **[Adamo et al. 20, 23, 25; Garner, Paquette 23, 24; . . .]**
- In this talk I will consider the most famous such backgrounds: instantons. **[BPST, 75; Ward, 77; ADHM, 78;. . .]**

Why is this worth doing?

- By understanding amplitudes in instanton backgrounds can hope to learn about non-perturbative gauge theory.

We will adopt the following strategy:

- First compute the one-loop all-plus amplitudes in $\mathrm{SU}(N_c)$ gauge theory on an instanton background for the $\mathrm{SU}(N_f)$ flavour symmetry.
- Then gauge the $\mathrm{SU}(N_f)$ flavour symmetry, i.e., integrate over instanton moduli space with one-loop determinant measure.

Yields instanton contributions to the one-loop amplitude in $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ gauge theory coupled to bifundamental Diracs, where $\mathrm{SU}(N_1)$ gluons scatter off an $\mathrm{SU}(N_2)$ instanton.

We will obtain explicit formulae and observe some odd behaviour.

Self-Dual Gauge Theory

Self-dual Yang-Mills has fields

$$A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})\,, \qquad B \in \Omega^2_-(\mathbb{R}^4, \mathfrak{g})\,,$$

a gauge connection and anti-self-dual 2-form Lagrange multiplier. The action is

$$\int_{\mathbb{R}^4} \mathrm{tr}\big(B \wedge F(A)\big)\,.$$

Deforming with $\mathrm{tr}(B^2)$ yields full Yang-Mills up to a θ-term

$$\int_{\mathbb{R}^4} \mathrm{tr}\big(B \wedge F(A)\big) - \frac{g^2}{8}\int_{\mathbb{R}^4} \mathrm{tr}(B \wedge B) \simeq \frac{2}{g^2}\int_{\mathbb{R}^4} \mathrm{tr}\big(F_-(A) \wedge F_-(A)\big)\,.$$

The amplitudes we compute do not involve the $\mathrm{tr}(B^2)$ vertex, so it is enough to work in the self-dual theory.

We couple self-dual Yang-Mills to N_f massless fundamental Diracs

$$\sum_{i=1}^{N_f} \int_{\mathbb{R}^4} \mathrm{d}^4x \, \langle \bar{\psi}_i, \not{D}\psi^i \rangle_{\mathrm{F}}$$

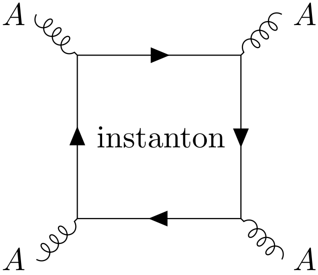
to get self-dual gauge theory.

To switch on a flavour background $a_f \in \Omega^1(\mathbb{R}^4, \mathfrak{su}(N_f))$ one deforms the action with

$$\sum_{i,j=1}^{N_f} \int_{\mathbb{R}^4} \mathrm{d}^4x \, \langle \bar{\psi}_i, (\not{a}_f)^i{}_j \psi^j \rangle_{\mathrm{F}} \, .$$

We will take a_f to be an instanton.

The one-loop amplitudes of self-dual gauge theory in the background a_f are generated by diagrams of the following form:



Notice that the trace-ordered amplitudes are independent of N_c, so we can compute these amplitudes with $N_f = N_c$.

Let

$$\mathcal{D}(a_f;x) = -\frac{1}{8\pi^2}\mathrm{tr}\big(F(a_f)^2\big)$$

be the instanton density, and $\widehat{\mathcal{D}}(a_f;p) = \int_{\mathbb{R}^4} \mathrm{d}^4x\, e^{\mathrm{i}p\cdot x}\mathcal{D}(a_f;x)$ be its Fourier transform. Then:

Theorem [RB, Kevin Costello]

The single-trace amplitude of self-dual gauge theory in the background a_f is

$$\mathcal{A}^{\text{single-trace}}(p_1,\dots,p_n;a_f) = -2\frac{\widehat{\mathcal{D}}(a_f;P)}{\langle 12\rangle\dots\langle n1\rangle}$$

for $P = p_1 + p_2 + \cdots + p_n$ the total momentum.

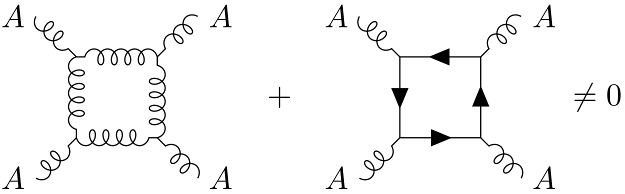
This is the one-loop all-plus amplitude of massless QCD in the same flavour background.

Green-Schwarz Mechanism for Amplitudes

To get this formula we exploited twistor methods. I will briefly explain how these work.

One-loop amplitudes in self-dual gauge theories are anomalies to integrability: the Coleman-Mandula theorem requires that any $4d$ integrable theory has trivial S-matrix. **[Bardeen, 96; Costello, 21]**

Self-dual gauge theory with $N_f = N_c$ suffers from such an anomaly.

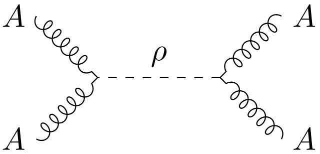


Remarkably, integrability can be restored using the Green-Schwarz mechanism. **[Costello, 21; Costello, Paquette, 22; Dixon, Morales, 25]**

More precisely, the anomaly can be cancelled by coupling self-dual gauge theory to a dimension zero scalar via an axion vertex.

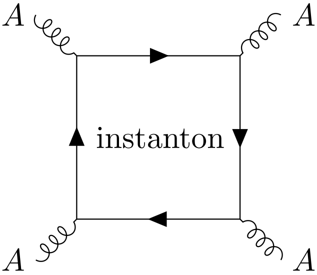
$$S_{\mathsf{GS}}[\rho; A] = \int_{\mathbb{R}^4} \mathrm{d}^4x \, (\triangle\rho)^2 + \frac{\mu}{2\pi} \int_{\mathbb{R}^4} \rho \, \mathrm{tr}\big(F(A)^2\big) \, .$$

By tuning μ the one-loop all-plus amplitudes cancel against tree-level exchange of ρ, e.g., the four-point amplitude cancels against:



Suppose we cancel the integrability anomaly in self-dual gauge theory with $N_f = N_c$ as above. Now consider switching on a flavour instanton.

We immediately run into a mixed t' Hooft anomaly between the integrable color and flavour symmetries.

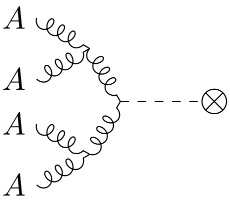


The amplitudes we compute are avatars of this t' Hooft anomaly.

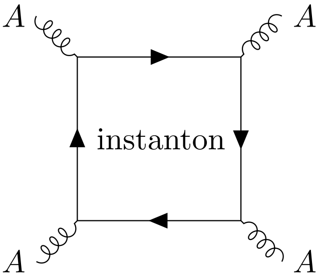
The one-loop amplitudes can be cancelled by modifying the previous Green-Schwarz mechanism. We simply add a source term for the dimension zero scalar coupling it to the instanton density.

$$-\frac{\mu}{2\pi}\int_{\mathbb{R}^4} \mathrm{d}^4x\,\rho\,\mathrm{tr}\big(F(a_f)^2\big)\,.$$

The one-loop diagrams we're interested in are equal to minus the following trees.



$\int_{\mathbb{R}^4}\rho(x)\mathcal{D}(a_f;x)$



The trees can be readily evaluated to deduce that

$$\mathcal{A}^{\textsf{single-trace}}(p_1,\ldots,p_n;a_f)=-\frac{2}{\langle 12\rangle\ldots\langle n1\rangle}\int_{\mathbb{R}^4}\mathcal{D}(a_f;x)e^{\mathrm{i}P\cdot x}$$

as claimed.

Soft Algebra

If a_f is an instanton of charge K, then

$$\lim_{\omega\to 0} \widehat{\mathcal{D}}(a_f; P) = K\,.$$

Let

$$\mathcal{A}^{\text{disc.}}\big((p_1, T_1), \ldots, (p_n, T_n); a_f\big)$$

be the full disconnected amplitude, where $T_i \in \mathfrak{su}(N_c)$ encode the color structure. We find

$$\lim_{\omega_i\to 0} \mathcal{A}^{\text{disc.}}\big((p_1, T_1), \ldots, (p_n, T_n); a_f\big) = \langle J_{T_1} \ldots J_{T_n}\rangle_{-2K}$$

where the right hand side denotes the chiral WZW correlator at level $-2K$.

The leading soft gluon theorem predicts

$$\begin{aligned}&\lim_{\omega_1\to 0}\mathcal{A}^{\text{disc.}}\big((p_1,T_1),\dots,(p_n,T_n);a_f\big)\\&=\sum_i\frac{1}{\langle 1i\rangle}\mathcal{A}^{\text{disc.}}\big((p_2,T_2),\dots,(p_i,[T_1,T_i]),\dots,(p_n,T_n);a_f\big)\,.\end{aligned}$$

But we also find a second order pole

$$-2\sum_i\widehat{\mathcal{D}}(a_f;p_i)\frac{\operatorname{tr}(T_1T_i)}{\langle 1i\rangle^2}\mathcal{A}^{\text{disc.}}\big((p_2,T_2),\dots,\widehat{(p_i,T_i)},\dots,(p_n,T_n);a_f\big)\,.$$

The leading soft gluon theorem leads to a Kac-Moody symmetry at level zero **[He, Mitra, Strominger, 16]**. In the presence of an instanton background the level shifts to $-2K$.

It's tempting to interpret the whole amplitude as a chiral algebra correlator, but this depends on the analytic properties of $\widehat{\mathcal{D}}(a_f;P)$.

In terms of ADHM data $X^{\dot{\alpha}\alpha}, I^{\beta}, J^{\gamma}$ the instanton density is **[Osborn, 81]**

$$\mathcal{D}(a_f;x) \propto \triangle^2 \mathrm{tr}\log(\langle IJ\rangle + (x-X)^2)\,.$$

Example

In the charge one case we can directly compute

$$\widehat{\mathcal{D}}(a_f;P) = \frac{1}{2}e^{\mathrm{i}P\cdot X}\rho^2 P^2 K_2(\rho|P|)$$

for $\rho^2 = \langle IJ\rangle$ and K_2 a modified Bessel function of the second kind.

In the holomorphic collinear limit the two-point amplitude has singularities

$$\mathcal{A}^{\text{single-trace}}(p_1, p_2; a_f) \sim -\frac{2}{\langle 12\rangle^2} + \frac{\rho^2}{2}\frac{[12]}{\langle 12\rangle} + \frac{\rho^4}{8}[12]^2 \log\langle 12\rangle + \text{regular}\,.$$

We notice that

- the first term is associated to the Kac-Moody central extension,
- the second is an extension considered in **[Melton, Narayanan, Strominger, 22]**,
- the third is not analytic in $\langle 12\rangle$, and so the amplitude is not a chiral algebra correlator.

IR & UV Divergences

Now suppose we gauge the flavour symmetry, i.e., integrate $\mathcal{A}^{\text{disc.}}$ over instanton moduli space. This can lead to IR & UV divergences coming from big & small instantons respectively.

We can decompose the space of charge K instantons as

$$\mathcal{M}^K_{\text{inst.}} = \widehat{\mathcal{M}}^K_{\text{inst.}} \times \mathbb{R}^4 \times \mathbb{R}_{>0}$$

where $\widehat{\mathcal{M}}^K_{\text{inst.}}$ has finite volume, $X \in \mathbb{R}^4$ is the centre of mass and $\rho \in \mathbb{R}_{>0}$ the scale. The one-loop measure is

$$\mathrm{vol}_{\widehat{\mathcal{M}}^K_{\text{inst.}}} \mathrm{d}^4 X \,\mathrm{d}\rho\, \rho^{b_0 - 5}$$

where $b_0 = \frac{11}{3} N_c - \frac{2}{3} N_f$. This has UV divergences for $b_0 \leqslant 4$.

Let's begin by considering the IR. In the charge one case as $\rho|P| \to \infty$

$$\widehat{\mathcal{D}}(a_f;P) \sim \rho^{3/2}|P|^{3/2}e^{-\rho|P|}$$

so big instantons are suppressed.

Note

Suppression only occurs for $P^2 > 0$, but in single-trace terms integrating over the instanton centre of mass sets $P = 0$.

Multi-trace terms are IR finite (for a suitable choice of contour).

For example, the double-trace amplitude on a charge one instanton is proportional to

$$\sum_{S\subset\{1,\dots,n\}} \frac{1}{|P_S|^{(b_0-4)}}\mathrm{PT}(S)\mathrm{PT}(S^c)\int_0^\infty \mathrm{d}\rho\,\rho^{b_0-1}K_2(\rho)^2\,.$$

Let's take $b_0 = 4$. The amplitude has a logarithmic UV divergence

$$\frac{\partial}{\partial \log \mu} \mathcal{A}^{\text{single-trace}}(p_1, \ldots, p_n; a_f) = -\frac{2}{\langle 12\rangle \ldots \langle n1\rangle} \delta^{(4)}(P)\,.$$

Note

This cannot be removed by a local counterterm.

Instead it indicates that the measure on instanton moduli space flows by a δ-function contribution at the singular points.

There are infinitely many ways of adding such δ-functions to the measure compatible with factorisation: suggests that self-dual gauge theory is non-perturbatively non-renormalizable for $b_0 \leqslant 4$.

Thank you for listening.