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All Loop Scattering As A Sampling Problem

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Surfaceology Meets Borinsk-ology

All Loop Scattering as a Counting Problem

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Abstract: This is the first in a series of papers presenting a new understanding of scattering amplitudes based on fundamentally combinatorial ideas in the kinematic space of the scattering data. We study the simplest theory of colored scalar particles with cubic interactions, at all loop orders and to all orders in the topological 't Hooft expansion. We find a novel formula for loop-integrated amplitudes, with no trace of the conventional sum over Feynman diagrams, but instead determined by a beautifully simple counting problem attached to any order of the topological expansion. These results represent a significant step forward in the decade-long quest to formulate the fundamental physics of the real world in a radically new language, where the rules of spacetime and quantum mechanics, as reflected in the principles of locality and unitarity, are seen to emerge from deeper mathematical structures.

Tropical Monte Carlo quadrature for Feynman integrals

Michael Borinsky*

We introduce a new method to evaluate algebraic integrals over the simplex numerically. This new approach employs techniques from tropical geometry and exceeds the capabilities of existing numerical methods by an order of magnitude. The method can be improved further by exploiting the geometric structure of the underlying integrand. As an illustration of this, we give a specialized integration algorithm for a class of integrands that exhibit the form of a generalized permutahedron. This class includes integrands for scattering amplitudes and parametric Feynman integrals with tame kinematics. A proof-of-concept implementation is provided with which Feynman integrals up to loop order 17 can be evaluated.

A NUMERICAL strategy to evaluate AMPLITUDES at high (=10) loop orders

... which Amplitudes?

Which Amplitudes?

Today I will focus only on a very RESTRICTED setting:

Massive, $\mathrm{Tr}(\phi^3)$ theory, in D=2 space-time dimensions

Why? These amplitudes are FINITE: immediately amenable to numerical integration.

To extend the ideas presented here to more relevant theories: deal with **divergences & numerators**

I am optimistic this can be done and I will describe concrete steps to do so at the end of this talk

Outline

1. Surfaceology 101
2. The Dual Sampling Algorithm
3. Towards The Real World

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1. **Surfaceology 101**
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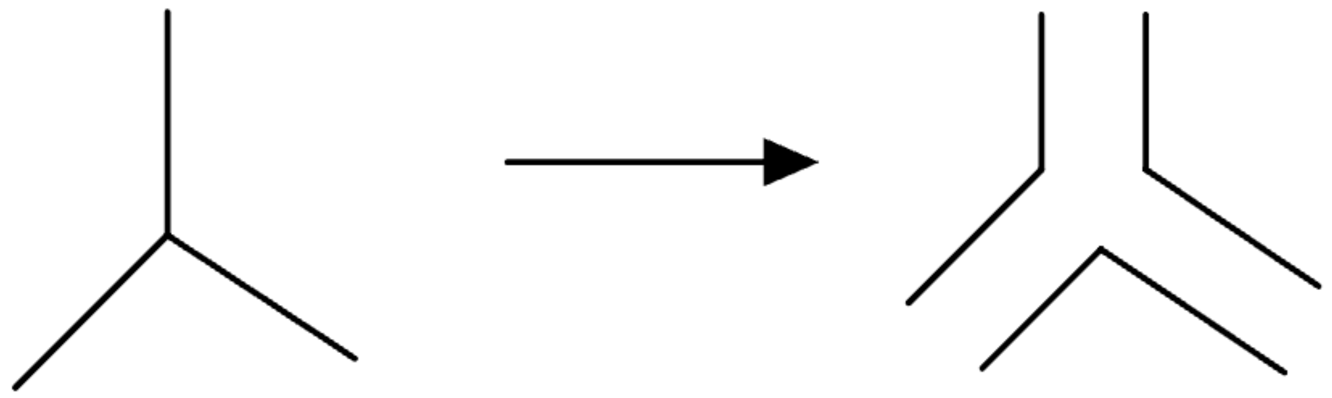
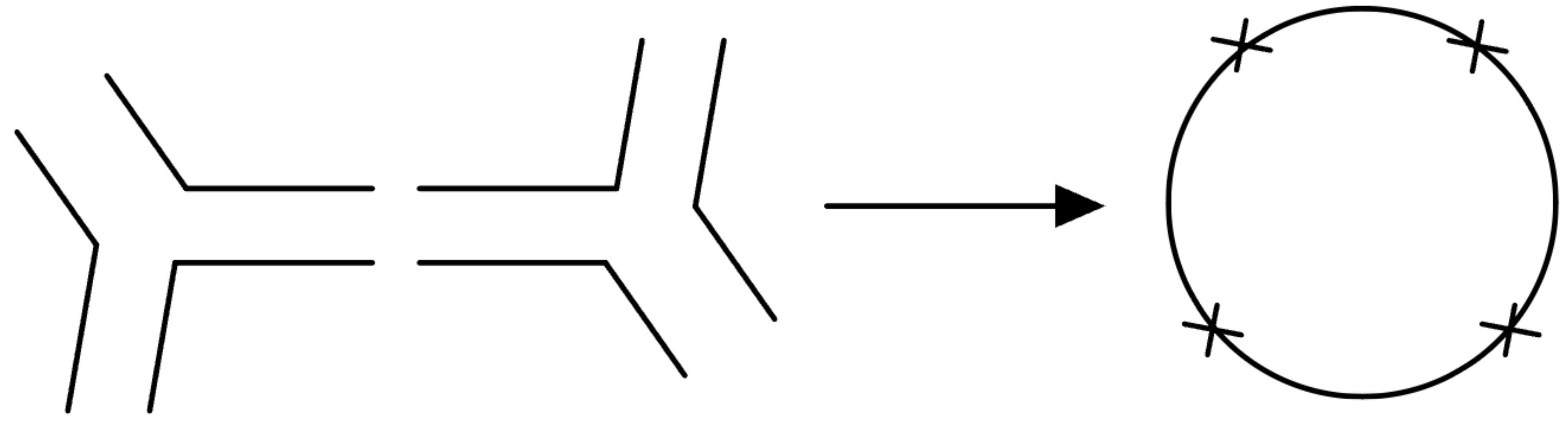
Tr ϕ^3 theory

$$\phi = \phi_{a,b} \quad a, b = 1, \ldots, N \rightarrow \mathscr{L} = \text{Tr } \phi^3$$

It is convenient to draw diagrams of the theory as *fatgraphs* Γ, which are associated to surfaces S





Γ $\qquad$ $S(\Gamma)$

Topological expansion

The amplitudes are organized in a topological expansion that separates color and kinematics

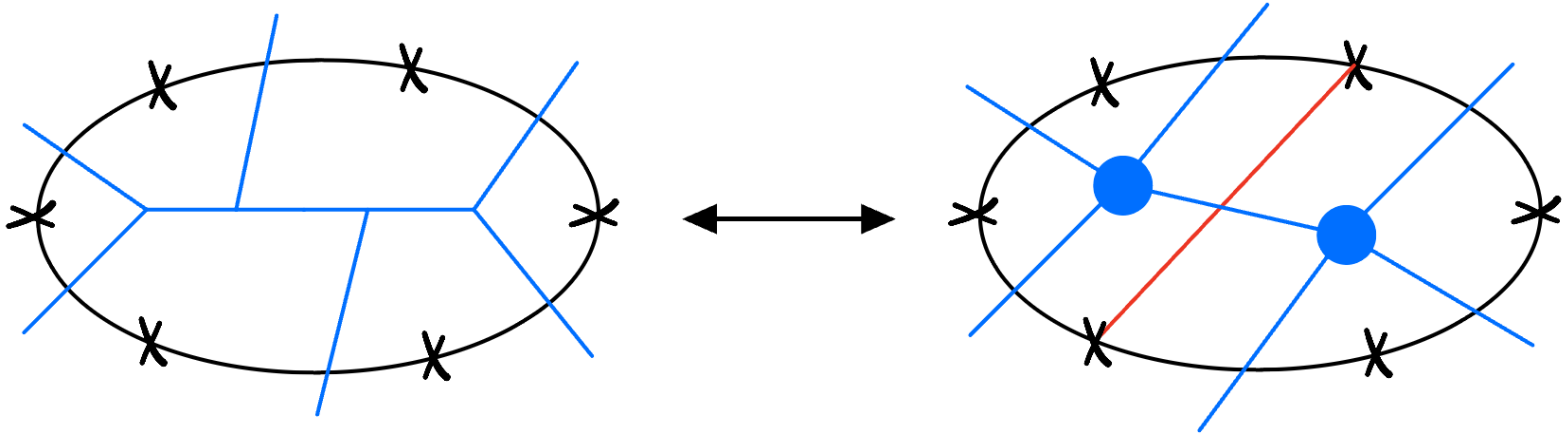
$$A_n(p_i, (a,b)_i) = \sum_S C_S A_S(p_i \cdot p_j)$$

The sum runs over all orientable surfaces S, the *color ordered* amplitude A_S is given by the corresponding fatgraphs

$$A_S = \sum_{\Gamma | S(\Gamma)=S} \mathrm{Val}(\Gamma)$$

Curves On Surfaces

The departure point of Surfaceology is to move from graphs to **curves**

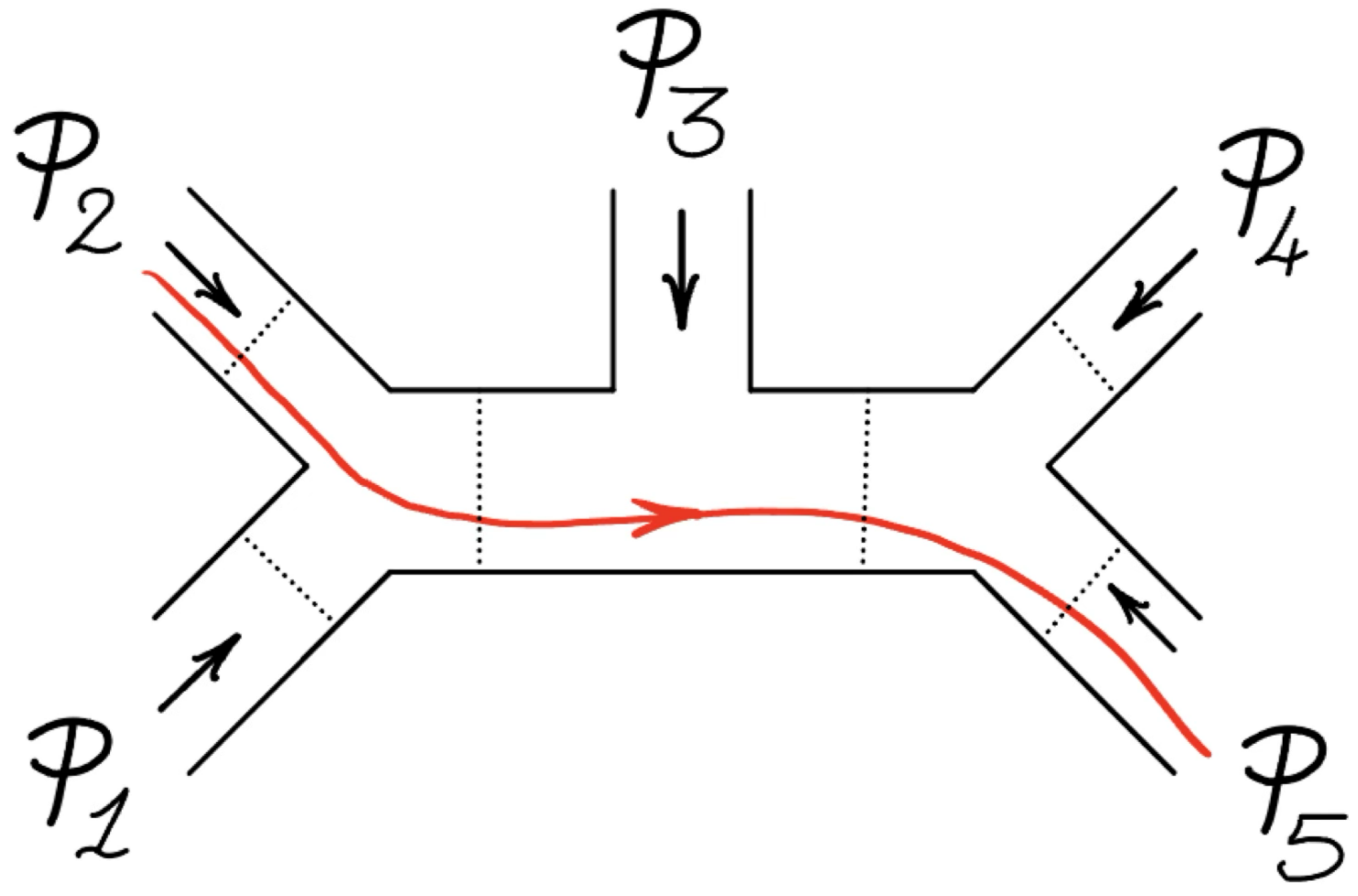


Maximal collections of non-crossing curves are dual to fatgraphs

There are many more graphs than there are curves!

Curves & Kinematics

A rule assigns every curve C to a momentum P_C



$$P_C^\mu = p_{\text{start}}^\mu + \sum_{\text{right turns}} p_{\text{from left}}^\mu$$

To the curve C we assign a "propagator" $X_C = P_c^2 + m^2$

At loop level this rule gives a **global momentum routing** for all diagrams

Headlight Functions

To a curve C we associate a matrix by the replacements

$$L \to \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad R \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad y_e \to \begin{bmatrix} 1 & 0 \\ 0 & y_e \end{bmatrix} \longrightarrow M_C = \begin{bmatrix} a_C & b_C \\ c_C & d_C \end{bmatrix}$$

We use this to construct the **Headlight Function**

$$u_C = \frac{a_C d_C}{b_C c_C} \longrightarrow \alpha_C = \text{Trop } u_C := u_C \Big|_{\substack{x+y \to \max(x,y) \\ x \times y \to x+y \\ x/y \to x-y}}$$

Curve Integrals

In Surfaceology, amplitudes are presented as a **global integral**

$$A_S = \int_{\mathbb{R}^{\mathbf{E}}} \frac{d\mathbf{t}}{\mathrm{MCG}} \int d^D\ell \, \exp\left(- \sum_{C \in \mathrm{Curves}(S)} X_C \, \alpha_C(t) \right)$$

Let us not worry about the MCG for now

Examples

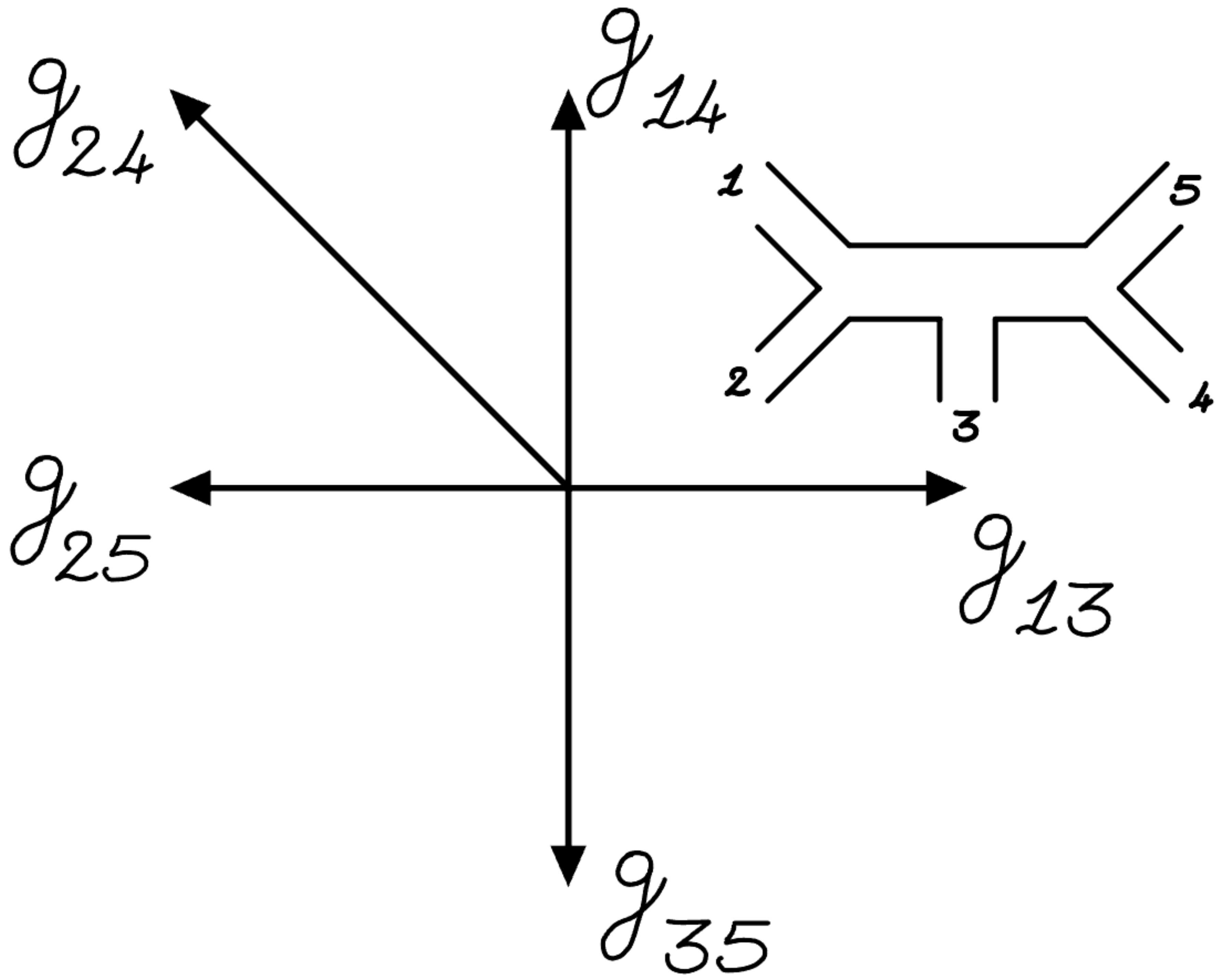
$$A_n^{\text{tree}} = \int_{S^{n-4}} \frac{\langle t\, d^{n-4} t \rangle}{\left[\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} 2 p_{i-1} \cdot p_{j-1} f_{ij} + \sum_{j=3}^{n-1} t_j (X_{1j} + m^2) \right]^{n-3}}$$

$$A_n^{\text{1-loop}} = \int_{\substack{S^{n-1}, \\ \sum_i t_i \geq 0}} \frac{\langle t\, d^{n-1} t \rangle}{\left(\sum_i \alpha_i\right)^{D/2} \left[\sum_{i,j} \alpha_i \alpha_j X_{ij} + \sum_k \alpha_k \left(m^2 \sum_i \alpha_i + \sum f_{ij} 2 p_i \cdot p_j + \sum_i T f_{i,i+n} + B f_{i+1,i+n} \right) \right]^{n-D/2}}$$

Where $X_{ij} = (p_i + \cdots + p_{j-1})^2$; $f_{ij} = \max(0, t_j, t_j + t_{j-1}, \ldots, t_j + \cdots + t_{i+1})$; $\alpha_i = f_{i,i+n} - f_{i,i+n+1}$

The Feynman Decomposition

The common domain of linearity of the Headlight functions are in 1-1 correspondence with **Feynman Diagrams**



In each domain the curve integrand collapses to

the Schwinger Parametrization of that diagram

Decomposing the integral in these regions

reproduces the usual sum-over-graphs

However, **different decompositions** are also possible!

Parametric Curve Integrals

The integrals over ℓ are Gaussian and can be performed exactly

$$A_S = \int_{\mathbb{R}^{\mathbf{E}}} \frac{d\mathbf{t}}{\mathrm{MCG}} \int d^D\ell \, \exp\left(-\sum_C X_C \, \alpha_C(t) \right)$$

Define

$$\sum_C \alpha_C X_C = \ell_i \Lambda_{ij} \ell_j + J_i \ell_i + Z \qquad \mathcal{U} = \det \Lambda \qquad \mathcal{F} = J^T \Lambda^{-1} J + Z\mathcal{U}$$

Then

$$A_S(X_C) = \Gamma(\mathbf{E} - LD/2) \int_{\mathbb{P}^{(\mathbf{E}-1)}_{\geq 0}} \frac{d\mathbf{t}}{\mathrm{GL}(1) \times \mathrm{MCG}} \mathcal{U}^{\mathbf{E}-(L+1)D/2} \mathcal{F}^{-(\mathbf{E}-LD/2)}$$

Outline

Monte Carlo Integration

$$A = \int I dx = \int \frac{I}{J}(J dx)$$

To estimate A by MC, generate N samples x_i distributed according to $d\mu = J(x)dx$, and use

$$A^{est} = \frac{1}{N}\sum_i \left(\frac{I(x_i)}{J(x_i)} \right)$$

The precision of the estimator is given by

$$\mathrm{Var}(A^{est}) = \frac{\mathrm{Var(I)_J}}{N}$$

Either we increase N or we decrease the variance changing the **sampling distribution** J (importance sampling)

Tropical Sampling

Borinsky's idea is to use as sampling distribution the **tropical integrand**

$$\mathcal{I}^{tr} = \mathcal{U}^{\mathbf{E}-(L+1)D/2}\mathcal{F}^{-(\mathbf{E}-LD/2)}\Big|\; x + y \to \max(x, y)$$

1. Compute the curve integral $H_S = \int \mathcal{I}^{tr}$ (a.ka. Hepp bound, Panzer)

2. Find a way to sample according to $\mathrm{J} = \dfrac{\mathcal{I}^{tr}}{H_S}$

Sector Decomposition

Consider the "sector" $\sigma = (C_1, C_2, \ldots, C_E)$ where $\alpha_{C_1} > \alpha_{C_2} > \ldots$

After a simple change of variable the tropical integrand simplifies to a monomial

$$\begin{aligned} \alpha_1 &= t_1 \\ \alpha_2 &= t_1 t_2 \\ \alpha_3 &= t_1 t_2 t_3 \\ &\ldots \end{aligned} \quad \longrightarrow \quad \mathcal{I}^{tr} = \prod_{i=2}^{E} t^{d_{S\setminus C_1 \ldots C_{i-1}} - 1}$$

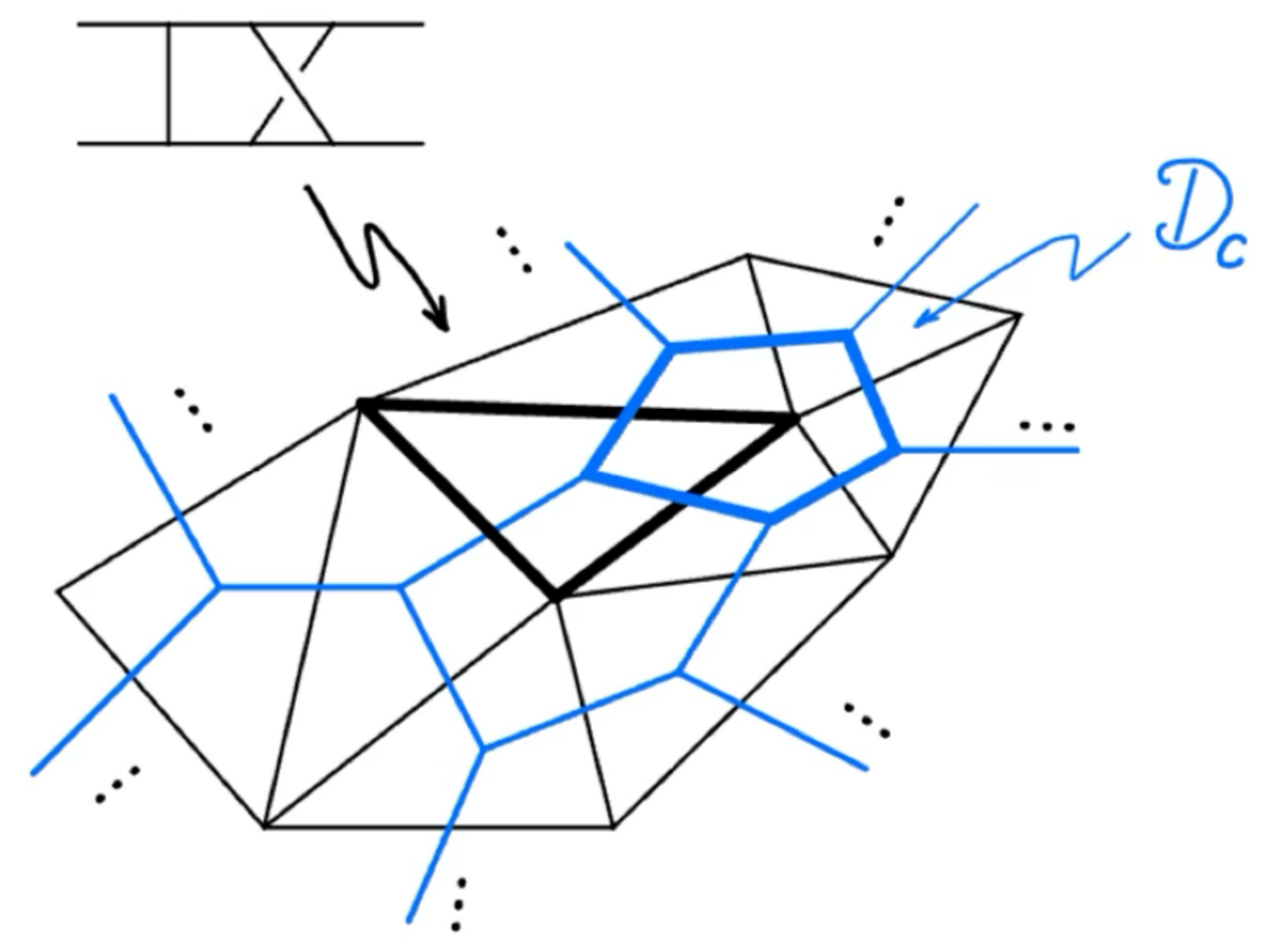
Within each sector the tropical integrand is essentially uniform. Each sector contributes with

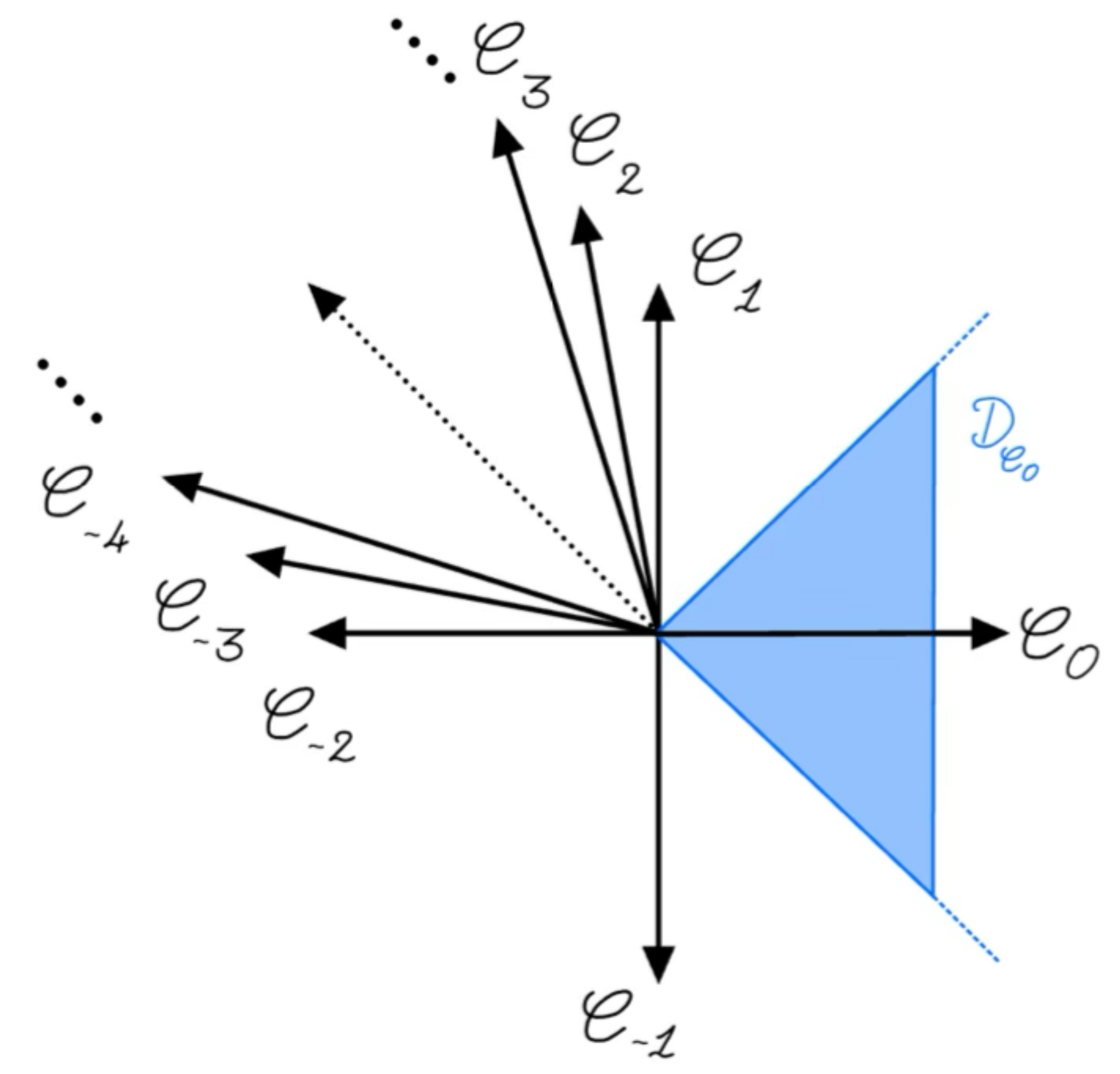
$$H_\sigma = \prod \frac{1}{d_{S\setminus C_1 \ldots C_i}} \quad \longrightarrow \quad H_S = \sum_\sigma H_\sigma$$

How to sample sectors according to their contribution? Too many to list them all! ($E! \times |\text{Graphs}|$)

The dual barycentric decomposition

As C runs over Curves(S)/MCG the cells $D_C = \{\alpha_C \geq \alpha_{C'}\}$ form a fundamental domain for MCG





Integrating over the dual cells gives a **recursive formula**

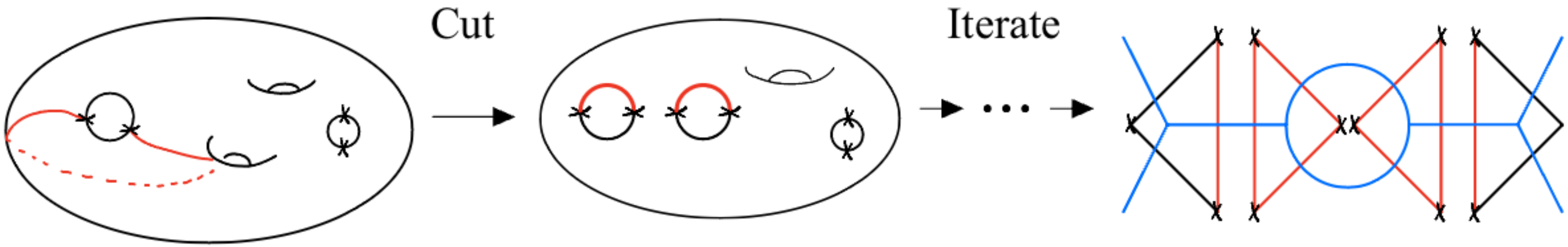
$$H_S = \sum_{C \in \text{Curves}(S)/\text{MCG}} \frac{H_{S \setminus C}}{d_{S \setminus C}}$$

Dual Sampling

Instead of sampling sectors or diagrams, sample **curves**

$$H_S = \sum_{C \in \mathrm{Curves}(S)/\mathrm{MCG}} \frac{H_{S\backslash C}}{d_{S\backslash C}} \to \text{ Pick a curve with probability } p(C) = \frac{H_{S\backslash C}}{d_{S\backslash C}} / H_S$$

Cut the surface along the curve, and re-iterate…stop when sufficiently many curves are drawn



Dual Sampling

Algorithm 1 Dual Sampling Algorithm

input: A surface S **output:** A sector of S

Set $i \leftarrow 1$

Set $S_{(i)} \leftarrow S$

while $i < E_S + 1$ **do**

  Sample a connected component S^{conn} of S with probability $p(S^{\text{conn}}|S) = \frac{d_{S^{\text{conn}}}}{d_S}$

  Sample a curve C_i on S^{conn} with probability $p(S' = S^{\text{conn}} \setminus C|S^{\text{conn}}) = \frac{\hat{H}_{S'}}{H_{S^{\text{conn}}}}$

  Set $S_{(i+1)}$ to be $S_{(i)}$ with S^{conn} replaced by S'

  Set $i \leftarrow i + 1$

end while

return $\mathcal{S} = (C_1, \ldots, C_{E_S})$

Tests

S	E_S	L_S	$\lvert\text{FatGraph}(S)\rvert$	n_{samples}	t_{sample}	A_S	δ/A_S	Memory
$\{1,\{1\}\}$	4	2	1	10^2	$\mathcal{O}(10^{-4}\text{ s})$	7×10^{-1}	7%	3 KB
$\{2,\{1\}\}$	10	4	105	10^4	$\mathcal{O}(10^{-3}\text{ s})$	4×10^{1}	3%	43 KB
$\{3,\{1\}\}$	16	6	50 050	10^4	$\mathcal{O}(10^{-3}\text{ s})$	6×10^{2}	5%	354 KB
$\{4,\{1\}\}$	22	8	56 581 525	10^6	$\mathcal{O}(10^{-2}\text{ s})$	7×10^{4}	9%	2.2 MB
$\{5,\{1\}\}$	28	10	117 123 756 750	10^7	$\mathcal{O}(10^{-2}\text{ s})$	1×10^{7}	5%	11 MB

Table 2: Results for surfaces of increasing genus: number of edges, loop order, number of fatgraphs, number of samples required to achieve target accuracy ($\leq 10\%$), average time per sample (including the generation of cut rules and Hepp bounds), numerical result for the amplitude (2.12), percentage relative accuracy, size of the final decision tree. All amplitudes listed here involve a single external particle with zero momentum.

NOTE: the number of sample points grow significantly less than the number of diagrams

Using **feyntrop** one observe that each diagram require roughly the same number of samples

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3. **Towards the Real World**

Towards the Real World

In order to apply to extend the dual sampling algorithm we need a **parametric curve integrand**

We need to sample many points: we need **fast numerical evaluation**

I have implicitly used this: determinants are quick to evaluate numerically!

$$A_S(X_C) = \Gamma(\mathbf{E} - LD/2) \int_{\mathbb{P}_{\geq 0}^{(\mathbf{E}-1)}} \frac{d\mathbf{t}}{\mathrm{GL}(1) \times \mathrm{MCG}} \mathcal{U}^{\mathbf{E}-(L+1)D/2} \mathcal{F}^{-(\mathbf{E}-LD/2)}$$

$$\sum_C \alpha_C X_C = \ell_i \Lambda_{ij} \ell_j + J_i \ell_i + Z \qquad \mathcal{U} = \det \Lambda \qquad \mathcal{F} = J^T \Lambda^{-1} J + Z\mathcal{U}$$

UV/IR divergences

In surfaceology we can make use of dim-reg: $A_S = \sum_i \epsilon^i A_S^{(i)}$

Can we find a **curve integrand** for each ϵ-order?

We can try with a **local subtraction** formula, like zero-momentum subtraction (Brown, Kreimer)

$$\mathcal{I}^{\text{ren}} = \sum_{\text{forests}} (-1)^F (\tau_{\gamma_1}\tau_{\gamma_2}\ldots)\mathcal{I}$$

$\tau \sim$ Taylor expansion

Exponentially many terms...

A better solution is **sector decomposition** because the subtraction commute and the formula factorize!

$$\mathcal{I}^{\text{ren}} = \prod_i (1 - \tau_i)\mathcal{I}$$

Faster to evaluate numerically!

Beyond Tr ϕ^3

Colorless theories can also be treated in surfaceology (see my talk at QCD meets gravity 2024)

Spinning particles involve numerators in loop representation

These can be converted in parametric space by Wick´s theorem

Fast numerical evaluation? How to enumerate quickly Wick´s contractions?

A good idea could be the derivative trick described in book by Mizera+Hannesdottir

Conclusions

1. Surfaceology+Tropical Sampling gives a powerful numerical strategy to evaluate amplitudes
2. Extension beyond toy theories plausible, work in progress…
3. Phase-space integrations? See new work by Borinsky 2504.09613

Thanks for the attention!

Backup slides

Why Dual?

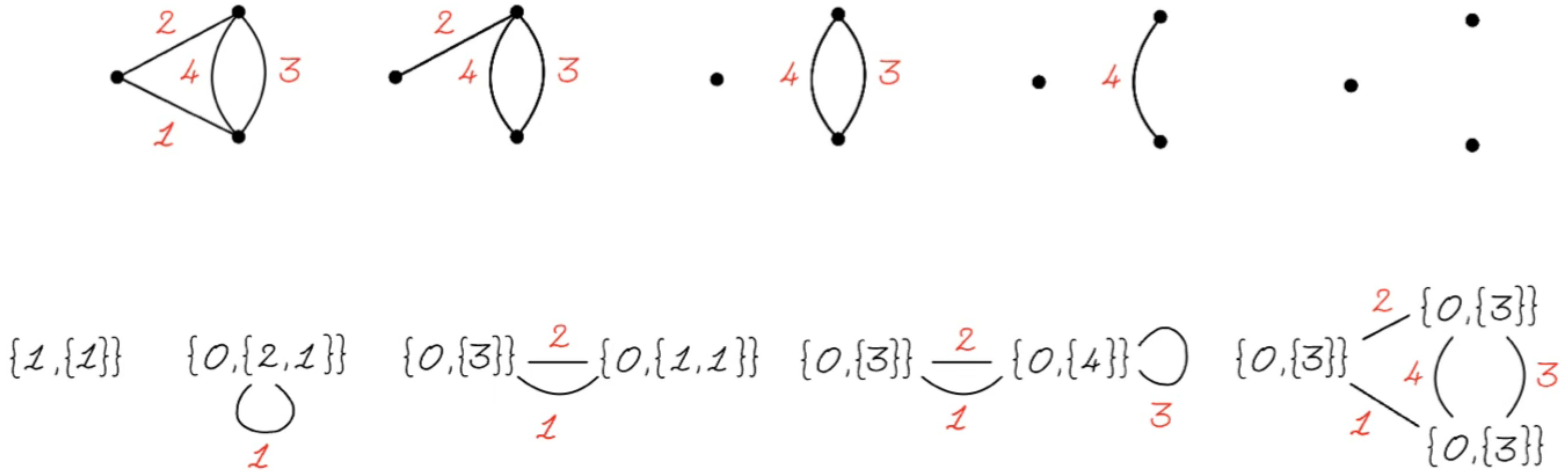

***Figure* 14**: *Dual processes.* The stochastic process of [13] starts from a graph and *removes* its edges (Top). The dual process starts from a vertex labelled by a surface and *adds* edges (Bottom).