Cluster Promotion Maps

Matteo Parisi

Harvard University, *Center of Mathematical Sciences and Applications*

AMPLITUDES 2025

Seoul National University, 20 June, 2025

Introduction

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[*Physics*] Singularities of scattering amplitudes described using cluster algebras [Golden, Goncharov, Spradlin et al. '13; Drummond, Foster, Gürdoğan '17].

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In this talk: new ways to generate cluster maps from the amplituhedron!

1. *BCFW tilings and cluster adjacency for the amplituhedron*, PNAS, 2024
   Cluster algebras and tilings for the $m = 4$ amplituhedron, arXiv:2310.17727, 2023
2. *Plabic Tangles and Cluster Promotion Maps*, in preparation, 2025.

with Even-Zohar, Sherman-Bennett, Tessler, Williams.

Outline

The positive Grassmannian

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$$\mathrm{Gr}_{k,n} := \{V : V \subseteq \mathbb{R}^n, \dim(V) = k\}$$

Represent V by a full-rank $k \times n$ matrix C (modulo row operations).

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[Postnikov '06] If $S_{\mathcal{M}}$ non-empty, it is homeomorphic to an open ball. So we have:

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Cells have very nice combinatorics (bijections with: decorated permutations, *plabic graphs*, ...)

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A *plabic graph* G is a planar graph embedded in a disk, such that:

- each vertex is either black or white (*bicolored*)
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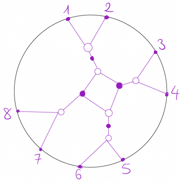

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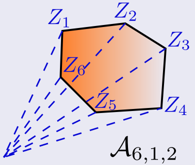

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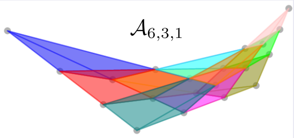

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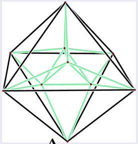

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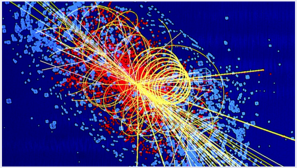

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$m = 4$: its 'volume' gives the tree-level *scattering amplitudes* in $N = 4$ super Yang-Mills theory for n particles in the $\mathrm{N}^k\mathrm{MHV}$ helicity sector.
To compute the volume we 'tile' the amplituhedron and sum over the volumes of each tile.

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Tiles and tilings have beautiful combinatorics - e.g. relation to tropical geometry, *magic number*, ... and cluster algebras!

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We can keep some variables frozen, i.e. we never mutate them.

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In practice: start with an *initial* cluster and generate all the others by mutations.

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$\mathcal{A}(Gr_{m,n})$ is much more complicated ✗
infinite # of clusters, cluster vars are polynomials of Plückers (of arbitrary high degree), no combinatorial rule for compatibility between cluster vars, etc.

Amplitudes, Amplituhedron and Clusters

Amplitudes in planar $\mathcal{N} = 4$ SYM admit an integral representation whose integrand $\mathcal{I}^{(L)}_{k;n}$ is a rational function of external kinematics and loop momenta.

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( ) : degree of $\tilde{Z}$ as a rational map on $Gr_{k;n}(\mathbb{C})$ VS $\tilde{Z}$ as a map on $Gr^{\,0}_{k;n}$.

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( ) : MPLogs not enough beyond $(L, n, k) = (2, 10, 3)$: elliptic MPlogs and LeS, Calabi-Yaus...
[Caron-Huot, Larsen '12][Bourjaily, Kalyanapuram, Langer, Patatoukos, Spradlin '20]

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Key for our proof: describing amplituhedron images of BCFW cells using cluster algebras

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BCFW cells are obtained recursively: if S_{G_L} and S_{G_R} are BCFW cells, then S_{G_L / G_R} is a new BCFW cell, where G_L / G_R is the plabic graph obtained by placing G_L (G_R) inside the left/blue (right/red) disk (blob). The plabic graph G_{core} in the center/purple is called core.

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The poles of a Yangian invariant in a BCFW expression are compatible cluster vars for $Gr_{4;n}$.

BCFW Recursions, Amplituhedron and Clusters

BCFW cells are obtained recursively: if S_{G_L} and S_{G_R} are BCFW cells, then S_{G_L / G_R} is a new BCFW cell, where G_L / G_R is the plabic graph obtained by placing G_L (G_R) inside the left/blue (right/red) disk (blob). The plabic graph G_{core} in the center/purple is called core.

( ) : we also need cyclic shift and re ection

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Q: How do we nd the collection of compatible cluster vars x_G in $Gr_{4;n}$ for a BCFW cell S_G?

Cluster Maps

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Definition (Cluster Map)

A map $\Psi : A' \to A$ between two cluster algebras A' and A is a *cluster map* if Ψ maps any collection of compatible cluster variables of A' into a collection of compatible cluster variables of A (up to Laurent monomials in the frozen variables).

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Let $G = G_L \bowtie G_R$, with S_G BCFW cell. Then $\mathbf{x}_G$ is obtained recursively as the image of a cluster map:

$$\Psi_{BCFW} : \mathcal{A}(\mathrm{Gr}_{4;N_L} \times \mathrm{Gr}_{4;N_R}) \to \mathcal{A}(\mathrm{Gr}_{4;n})$$
$$\mathbf{x}_G = \Psi_{BCFW}\left(\mathbf{x}_{G_L} \cup \mathbf{x}_{G_R}\right) \cup \mathbf{x}_{G_{core}}$$

$\mathbf{x}_{G_{core}} = 5$ Plückers on indices $\{a, b, c, d, n\}$ frozen.

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$\mathbf{x}_{G_L}$ and $\mathbf{x}_{G_R}$ are compatible cluster vars $\implies$ $\mathbf{x}_G$ are compatible cluster vars for $\mathrm{Gr}_{4;n}$.

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E.g. on $\mathrm{Gr}_{4;N_L}$: $\Psi_{BCFW}(\langle ij\ell b\rangle) = \dfrac{\langle ij\ell b\rangle\langle acdn\rangle \quad \langle ij\ell a\rangle\langle bcdn\rangle}{\langle acdn\rangle} =$

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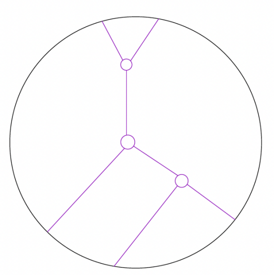
Theorem (BCFW Cluster Promotion Map) [ELPSBTW '23]

Ψ_{BCFW} is a cluster map, i.e. it maps collections of compatible cluster variables of $\mathrm{Gr}_{4;N_L} \times \mathrm{Gr}_{4;N_R}$ to collections of compatible cluster variables of $\mathrm{Gr}_{4;n}$ (up to frozens).

Promotion Maps via Plabic Graphs

Definition (Promotion Map via Plabic Graph) [EPSBTW '25]



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- solve each v_j in terms of $\{z_i\}$. Theorem: (1) , unique solution (up to scaling);

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We define a *promotion map* Ψ:

$$\Psi : A(\ \grave{}\,\mathrm{Gr}_{4;N\grave{}}) \to A(\mathrm{Gr}_{4;n})$$

on $\mathrm{Gr}_{4;N\grave{}}$: $u \mapsto {}_u v_{b_u}$

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Place s disks each in a face of G_{core} and with boundary $D_{\grave{}}; \grave{} \in [s]$. Connect in a planar way each vertex $u \in D_{\grave{}}$ to a black vertex b_u of G_{core}, corresponding to the vector v_{b_u}.

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We call $(G_{core}; \{D_\ell\}_{\ell=1}^{s})$ a *plabic tangle* .

( ): inspired by *planar tangles* in the work on planar algebras by F. R. Jones (1999).

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Intersection-number one plabic graphs (OR rational Yangian invariants OR amplituhedron tiles) generate cluster maps on the Grassmannian.

Amplitrees

Theorem (Characterization of intersection-number one trees) [EPSBTW '25]

A plabic tree G has intersection-number one if and only if G is an *amplitree*.

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Definition (Amplitree) [EPSBTW '25]

Let G be a bipartite plabic tree of type $(k, km+1)$. We say that G is a (k, m)-*amplitree* if for each edge e of G, if we cut $G \setminus \{e\} = G_1 \sqcup G_2$ (giving "half" the edge e to each G_i), then

$$m(k_i - 1) < n_i - 1 \leq mk_i, \quad \text{where } G_i \text{ is a plabic tree of type } (k_i, n_i).$$

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For many classes, we proved amplitrees give cluster maps (BCFW only the tip of the iceberg).

Beyond intersection-number one and clusters

Definition (4-Mass Box Promotion) [EPSBTW '25]

$$\Psi \; : \mathbb{C}(\mathrm{Gr}_{4;N'}) \to \mathbb{C}(\mathrm{Gr}_{4;n})[\sqrt{\Delta}]$$

$$2 \mapsto \frac{(21) \cap (56X\;)}{\langle 156X\;\rangle}; \quad 7 \mapsto z_7 + \quad z_8 =: X$$

$$:= \frac{B \quad \sqrt{\Delta}}{2A}; \quad \Delta := B^2 \quad 4AC$$

$$A = \langle 128|34|568\rangle; \; C = \langle 127|34|567\rangle$$

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The 4-mass box promotion map Ψ preserves positivity of cluster variables for $\mathrm{Gr}_{4;N'}$, i.e. if x is a cluster variable for $\mathrm{Gr}_{4;N'}$, then $\Psi \ (x)$ is a positive function on $\mathrm{Gr}_{4;n}^{>0}$.

Beyond intersection-number one and clusters

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Algebraic singularities that are images of Ψ appear in Yangian invariants for $n \quad 9$ and $k \quad 3$.

A Positive Outlook

Conjecture (Positivity Phenomenon) ['Feeling of the community' or Spradlin '24]

Scattering amplitudes in planar $N = 4$ SYM are regular on $\mathrm{Gr}_{4;n}^{>0}$. In other words, singularities are of the type $f(\mathbf{z}) = 0$, where $f(\mathbf{z}) > 0$ on $\mathrm{Gr}_{4;n}^{>0}$.

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Cluster variables > 0 on $\mathrm{Gr}_{4;n}^{>0}$ (from Laurent positivity of cluster algebras)) $f(\mathbf{z}) > 0$

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- When $f(z)$ is algebraic (algebraic letters and singularities of non-rational Yangian invariants, e.g. 4-mass box): no general mechanism that would explain positivity. Some results in the frameworks of limiting rays and cluster algebraic functions.
  [Drummond, Foster, Gürdoğan, Kalousios `19][Arkani-Hamed, Lam, Spradlin `19]

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Singularities of non-rational Yangian invariants are images of promotion maps (int-num $d > 1$)

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When $f(z)$ is rational (rational letters and poles of Yangian invariants): $f(z)$ conjectured to be a cluster variable for $Gr_{4;n}$.
Cluster variables >0 on $Gr^{>0}_{4;n}$ (from Laurent positivity of cluster algebras)) $f(z)>0$

Singularities of rational Yangian invariants are images of cluster promotion maps (int-num 1)

When $f(z)$ is algebraic (algebraic letters and singularities of non-rational Yangian invariants, e.g. 4-mass box): no general mechanism that would explain positivity. Some results in the frameworks of limiting rays and cluster algebraic functions. [Drummond, Foster, Gürdoğan, Kalousios `19][Arkani-Hamed, Lam, Spradlin `19]

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(Commutative Algebra): Cluster promotion maps can shed light on the cluster structure of $Gr_{m;n}$ itself, e.g. classifying families of cluster vars or finding compatibility criteria.

A Positive Outlook

[redacted]

Scattering amplitudes in planar $\mathcal{N} = 4$ SYM are regular on $Gr^{>0}_{4;n}$. In other words, singularities are of the type $f(z) = 0$, where $f(z) > 0$ on $Gr^{>0}_{4;n}$.

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(Commutative Algebra): Cluster promotion maps can shed light on the cluster structure of $Gr_{m;n}$ itself, e.g. classifying families of cluster vars or nding compatibility criteria.

(Bootstrap): Are symbol letters images of promotion maps?

Questions?

Thank you!