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Hexagonal bootstrap for Wilson-loops with Lagrangian insertion

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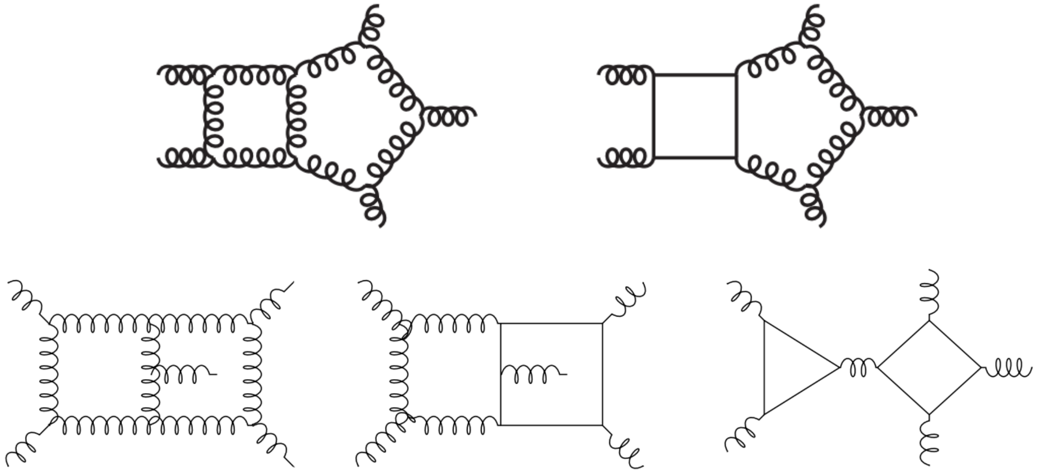
Based on 2505.01245

Sérgio Carrôlo, Dmitry Chicherin, Johannes Henn and Yang Zhang

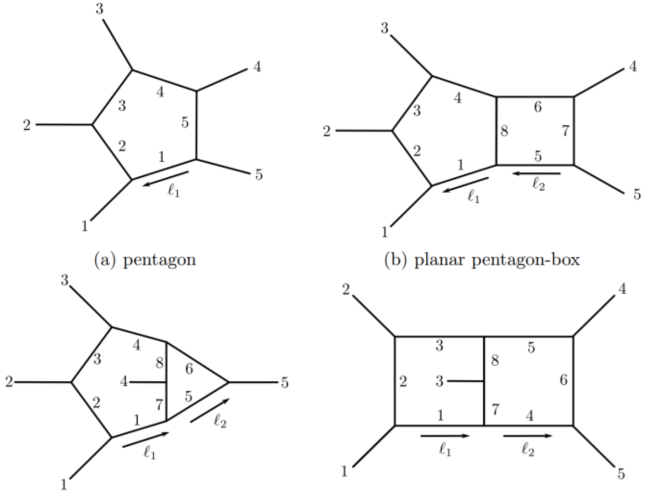
Towards Six-Parton scattering amplitudes (1/2)

- Integrand → (usually numerical) integration-by-part to **master integrals** → Reconstruct **kinematics prefactors** → Hard function from **divergence subtractions** [Matthias & Bayu's talks on Mon.]

Canonical differential equations



IBP reduction ⟶



(a) pentagon (b) planar pentagon-box (c) non-planar hexagon-box (d) non-planar double pentagon

$$\mathcal{A}_n^{(L)} = \sum_{ij} c_{ij} \boxed{R_{i,n}} \boxed{\mathcal{I}_{j,n}^{(L)}}.$$

- Five-Parton scattering amplitudes at NNLO from **pentagon functions** [Abreu, Agarwal, Buccioni, De Laurentis, Devoto, Dormans, Febres Cordero, Gambuti, Ita, Klinkert, Page, Sotnikov, Tancredi, von Manteuffel..; Gerhmann, Henn, LoPresti, Chicherin, Sotnikov]

Towards Six-Parton scattering amplitudes (2/2)

- Planar Hexagon functions at two loops [Henn, Matijašić, Miczajka, Peraro, Xu, Zhang; Abreu, Monni, Page, Usovitsch] [See Yang's poster] ; Six-point amplitudes at NNLO?

$$\mathcal{A}_n^{(L)} = \sum_{ij} c_{ij} \, R_{i,n} \, \mathcal{I}_{j,n}^{(L)}.$$

Coefficients (kinematics independent)

Kinematics prefactors

Master integral basis

- Q1. IBP reduction for the full integrand is computationally intensive
- Q2. Understanding of kinematics prefactors in QCD amplitudes remains limited

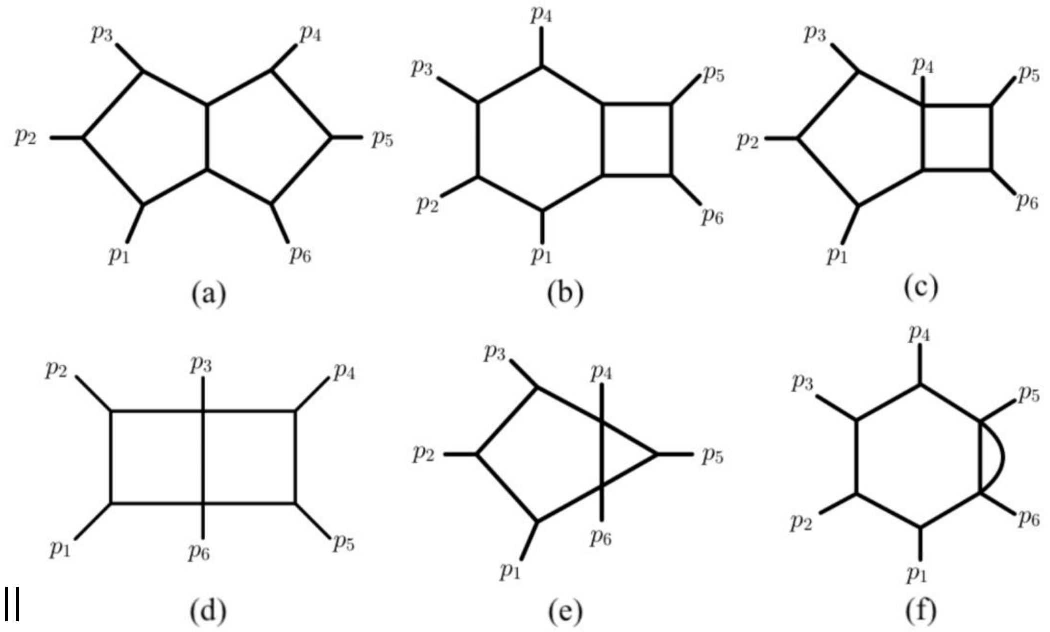
Any advanced understanding for Q1 and Q2?

Hexagon functions at two loops [Henn, Matijašić, Miczajka, Peraro, Xu, Zhang; Abreu, Monni, Page, Usovitsch]

- **Two-loop massless scattering processes with six particles,** State-of-art for Feynman integral calculation
- Up to weight four, **hexagon function space** is constructed (at symbol level)

Transcendental weight	1	2	3	4
# All symbols	9	62	319	945
# Two-loop six-point symbols	9	62	266	639
# Two-loop five-point one-mass symbols	9	59	263	594
# One-loop squared symbols	9	59	221	428
# Genuine two-loop six-point symbols	0	0	3	45

Basis for all planar six-point two-loop observables (symbol level)



Bootstrap

- **(Steinman/cluster) bootstrap for scattering amplitudes/form factors** : **[Golden, Goncharov, Spradlin, Vergu, Volovich; Basso, Caron-Huot, Dixon, Drummond, Duhr, Dulat, Foster, Gurdogan, Harrington, Henn, von Hippel, Liu, McLeod, Papathanasiou, Wilhelm⋯]** **[Ruth's talk on Mon.]**
- Extremely powerful in calculating six-point (up to **eight loops**) and seven-point (up **four loops**) amplitudes in N=4 SYM theory

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Table 1: The dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

weight n	0	1	2	3	4	5	6	7
First entry	1	7	42	237	1288	6763	?	?
Steinmann	1	7	28	97	322	1030	3192	9570
Ext. Stein.	1	7	28	97	308	911	2555	6826

Table 2: The dimensions of the heptagon, Steinmann heptagon, and extended Steinmann heptagon spaces at symbol level.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathscr{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

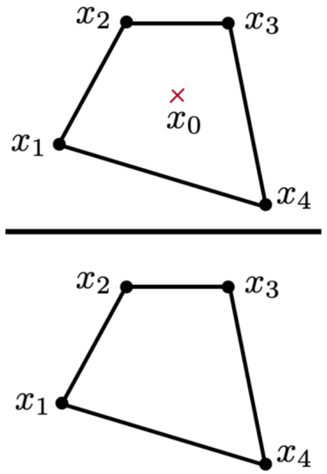
$$\mathcal{A}_n^{(L)} = \sum_{ij} c_{ij} \; R_{i,n} \; \mathcal{I}_{j,n}^{(L)} .$$

- Any possible QCD amplitudes we can compute based on this idea?
- **Feynman integral basis from CDE as the ansatz, but what about prefactors?**

Review: Wilson-loops with Lagrangian insertion

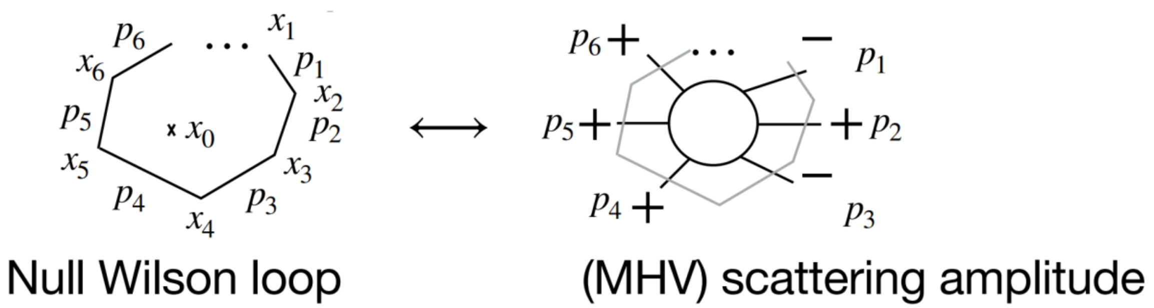
[Alday, Buchbinder, Chicherin, Engelund, Henn, Heslop, Korchemsky, Mistlberger , Roiban, Sikorowski, Tseytlin···]

Wilson-loops with Lagrangian insertion (1/2)



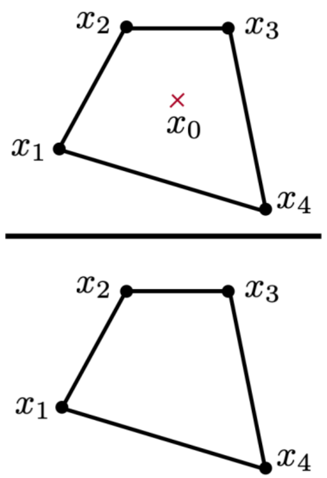
$$F_n(x_1,\ldots,x_n;x_0) := \pi^2 \frac{\langle 0|W_n[x_1,\ldots,x_n]\mathcal{L}(x_0)|0\rangle}{\langle 0|W_n[x_1,\ldots,x_n]|0\rangle},$$

- **(Dual) Conformal invariant** observables in planar N=4 SYM theory



- Integrand: logarithm of scattering amplitudes by **WL/amp duality** **[Alday,Maldacena; Drummond, Korchemsky, Henn, Sokatchev; Brandhuber, Heslop, Travaglini]**
- Integrated: **IR/UV finite, like QCD hard function;** uniform weight functions

Wilson-loops with Lagrangian insertion (2/2)



$$F_n(x_1, \ldots, x_n; x_0) := \pi^2 \frac{\langle 0|W_n[x_1, \ldots, x_n]\mathcal{L}(x_0)|0\rangle}{\langle 0|W_n[x_1, \ldots, x_n]|0\rangle},$$

- Kinematics is the same as that for n-parton scattering processes, depending on 3n-11 ratios of Mandelstam variables
- **Dual to the maximal transcendental weight part of (L+1)-loop all-plus pure Yang-Mills amplitudes** [Chicherin, Henn]

$$f_n(p_1, \ldots, p_n) = \lim_{x_0 \to \infty} (x_0^2)^4 F_n(p_1, \ldots, p_n; x_0)$$

- Previous results: four-point observables up to three loops, five-point observables up to two loops.
- **Today's focus: The six-point two-loop observable**

Negative geometries and leading singularity classification (1/2)

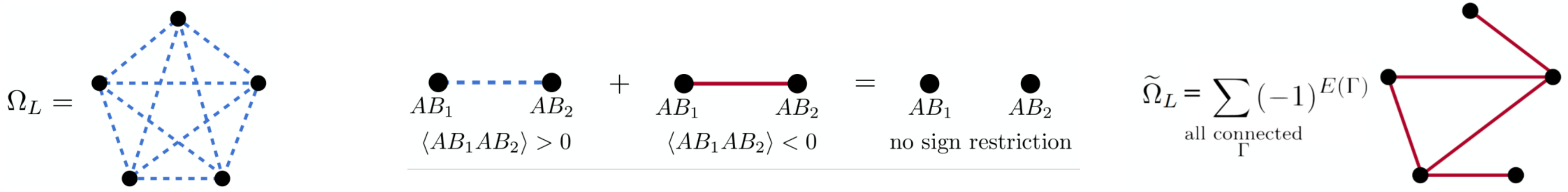
- In N=4 SYM theory, its integrand-level loop amplitudes can be computed from **Amplituhedron (Canonical functions from positive geometry)**

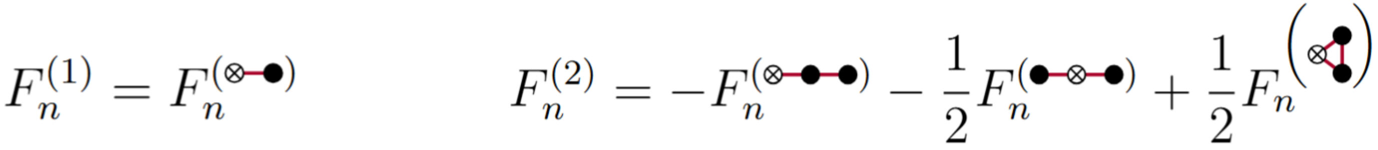
[Arkani-Hamed, Trnka]

"Simplest QFT"
hidden symmetries

- Geometrical expansion for the integrand: **Negative geometries**

[Arkani-Hamed, Henn, Trnka] [See Jaroslav's talk on Fri.]





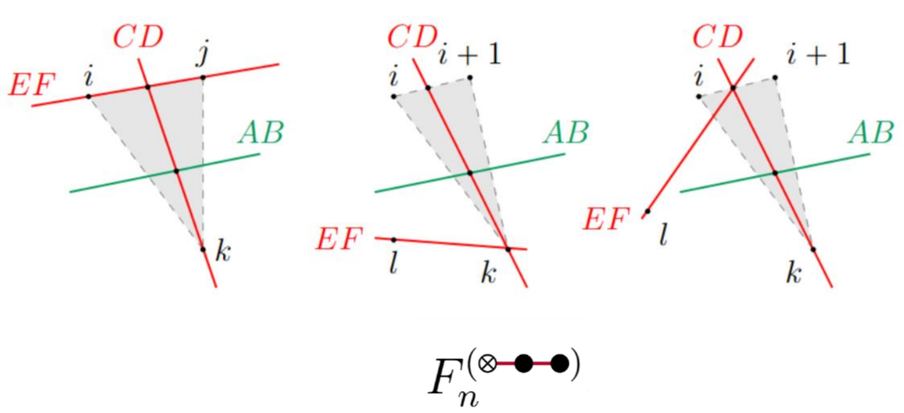
- **A (generally non-planar) basis totally different from Feynman integral expansion**

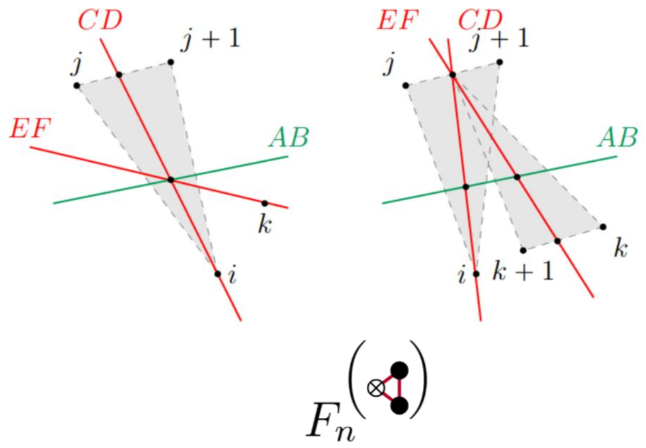
Negative geometries and leading singularity classification (2/2)

[Brown, Henn, Mazzucchelli, Trnka, 2503.17185]

- Idea: prefactors/leading singularities are from maximal residues of the integrand
- LS: conformal invariants from **Kermit basis** (proved from **amplituhedra viewpoint**)

Maximal boundaries from negative geometry basis at two loops





Kermit basis for all LS

$$B_{ijklm} := \frac{\langle AB(mij) \cap (jkl)\rangle^2}{\langle ABjm\rangle\langle ABij\rangle\langle ABjk\rangle\langle ABlj\rangle\langle ABmi\rangle\langle ABkl\rangle},$$

$$B_{ijkl} := \frac{\langle ijkl\rangle^2}{\langle ABij\rangle\langle ABjk\rangle\langle ABkl\rangle\langle ABli\rangle}.$$

- Leading singularities for the maximal transcendental part of all plus YM amplitudes (also conformal invariants)

Hexagonal bootstrap for the two-loop observable

[based on work 2505.01245]

$$F_6^{(2)} = \sum_{ij} c_{ij} \, R_i \, \mathcal{I}_j^{(2)} .$$

Coefficients (to be fixed)

Independent prefactors (from geometric input)

Master integral basis (from CDE input)

Ansatz

	n	5	6	7	8	9	10	11	12
$L=1$	$\frac{n(n-3)}{2}$	5	9	14	20	27	35	44	54
$L\geqslant 2$	$\frac{(n-1)(n-2)^2(n-3)}{12}$	6	20	50	105	196	336	540	825

- At six points, there are **20 linear-independent leading singularities** from Kermit basis and **945 hexagon functions (at symbol level)** from CDE

$$F_6^{(2)} = \sum_{i=1}^{20}\sum_{j=1}^{945} c_{i,j}\, R_i\, \mathcal{I}_j^{(2)}$$

- **1718** genuine unknown coefficients by dihedral symmetry

Physical conditions and constraints

- We impose each physical condition individually

and count number of constraints from it.

All unknowns in the ansatz are fully fixed by imposing all the physical conditions simultaneously

weight	0	1	2	3	4
unknowns in dihedral ansatz	5	22	139	644	1892
genuine **unknowns**	4	20	125	585	1718
constraints:					
soft	3	20	116	515	1439
collinear	3	20	121	551	1539
spurious $s_{24} = 0$	1	12	76	360	1044
spurious $s_{25} = 0$	1	6	36	165	483
scaling dimension	0	4	20	125	585
triple collinear	1	5	31	134	353
total **constraints**	4	20	125	585	1718
unfixed **unknowns**	0	0	0	0	0

Physical limits (soft/colinear limit) from six points to five points are very restrictive

Comparing: Five-point two-loop observable **CANNOT** be fixed by these conditions from pentagonal ansatz!

→six-point bootstrap is simpler than five-point case!

arXiv > hep-th > arXiv:1412.3763

High Energy Physics - Theory

[Submitted on 11 Dec 2014 (v1), last revised 10 Nov 2015 (this version, v3)]

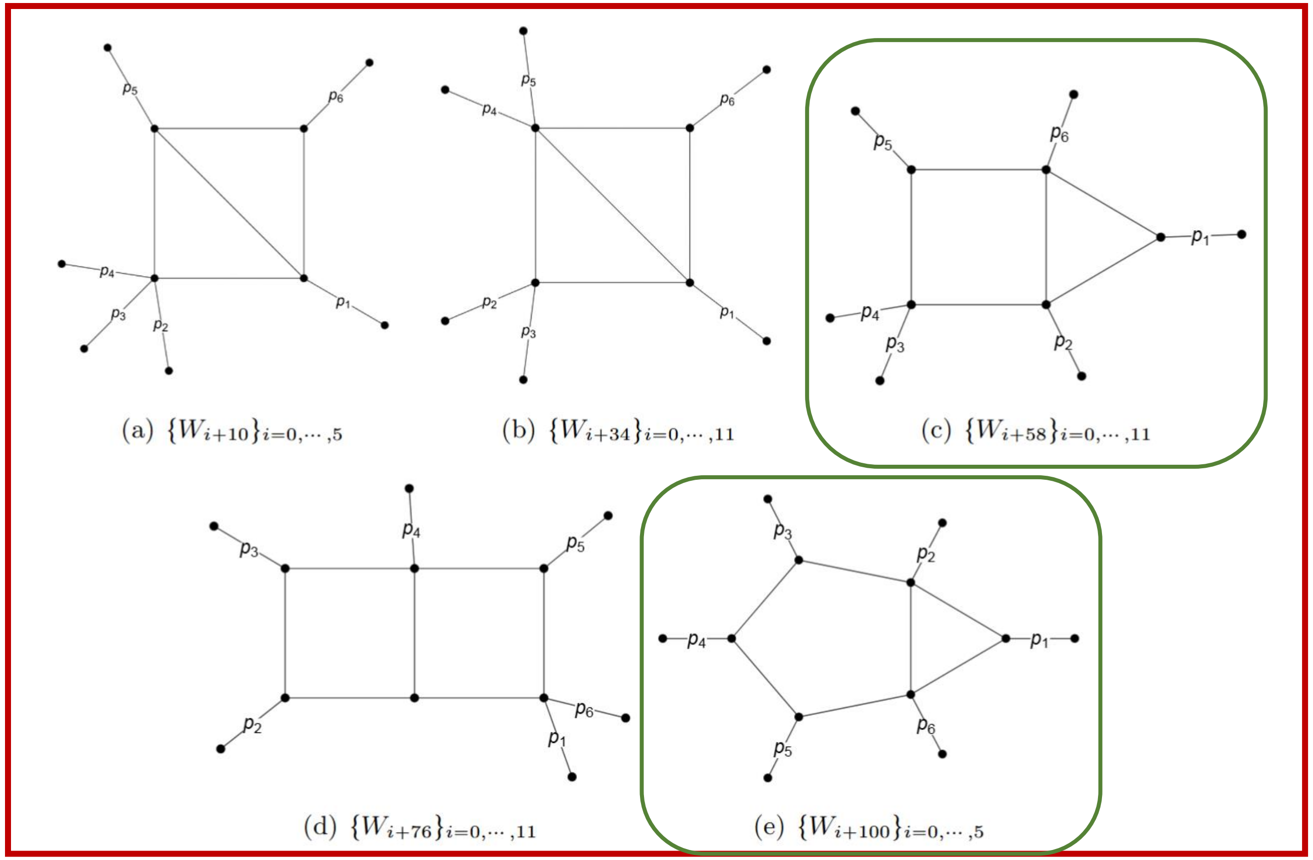
A Symbol of Uniqueness: The Cluster Bootstrap for the 3-Loop MHV Heptagon

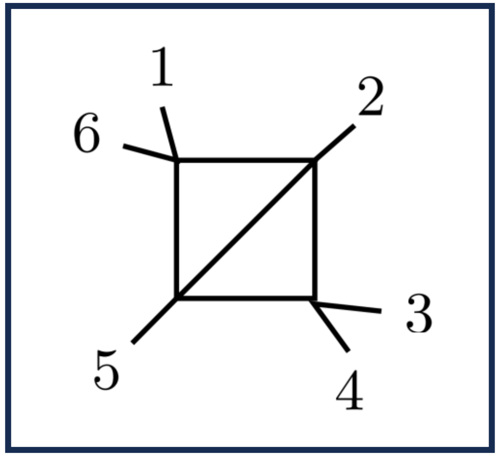
James M. Drummond, Georgios Papathanasiou, Marcus Spradlin

Singularities and Alphabet of bootstrapped result (1/2)

- Among all **245** two-loop planar symbol letters, only **137** of them remain in the final result
- Among 137 physical letters, **48** rational letters are contributed by two-loop topologies and their symbol letters [Anastasia's talk today]

Sectors yielding contribution





One of the sectors that do not contribute

Singularities and Alphabet of bootstrapped result (2/2)

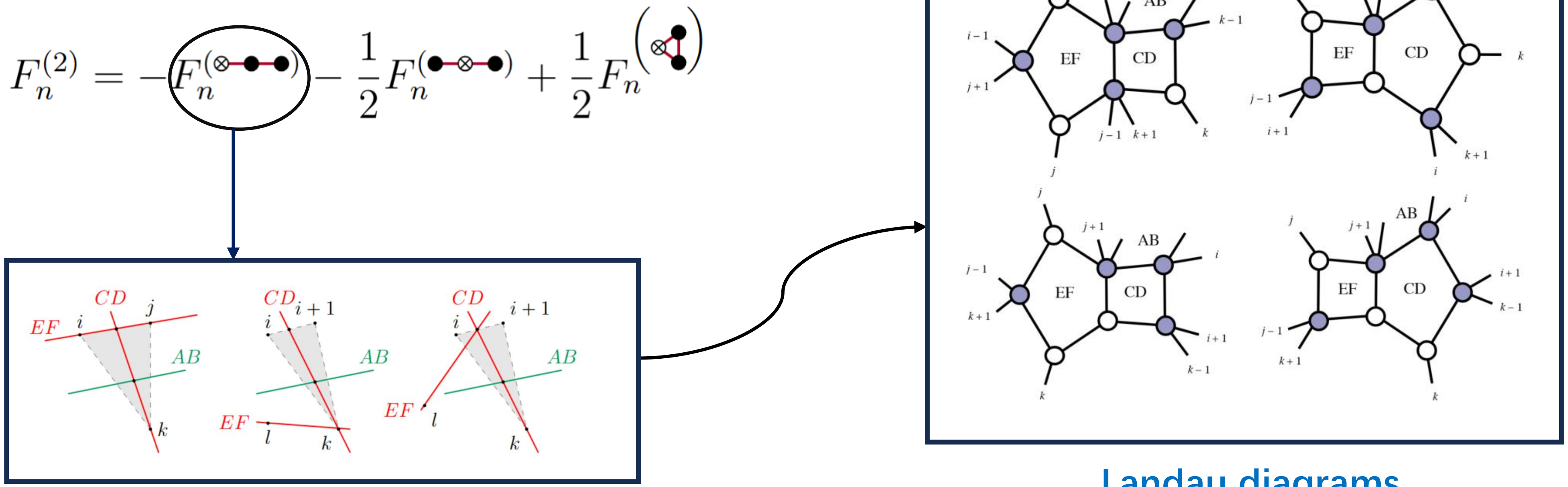
- **Why only those five diagrams and only 137 letters?** Any geometrical understanding about the singularities/symbol letters from amplituhedron and positive geometry picture ? **[Prlina, Spradlin, Stankowicz, Stanojevic]**

- **Exploring individual negative geometries!**
- **(Generally non-planar objects/bootstrap from integrable symbols and symbol letters)**

$$F_n^{(2)} = -F_n^{(\otimes - \bullet - \bullet)} - \frac{1}{2} F_n^{(\bullet - \otimes - \bullet)} + \frac{1}{2} F_n^{(\otimes \triangleleft)}$$

Geometric Landau bootstrap for negative geometries

[Based on work 2506.XXXXX, with Dmitry Chicherin, Johannes Henn, Elia Mazzucchelli, Jaroslav Trnka and Shun-Qing Zhang] [See Elia's poster]



Landau diagrams

Duality to all-plus Yang-Mills amplitude

- The observables are conjectured to cover the **maximal transcendental weight part of the all-plus pure YM amplitudes**

$$\langle 0|W_n \mathcal{L}(x_0)|0\rangle_{x_0\to\infty} \sim \frac{\mathcal{A}_n^{\rm YM}(1^+,\ldots,n^+)}{\mathcal{A}_n^{{\rm YM},(1)}(1^+,\ldots,n^+)},$$

$$F_n \mathcal{H}^{\rm MHV} \sim \mathcal{H}^{\rm YM}$$

- A strong evidence: **Steinman relation**

$$F_6 \times \mathcal{H}_6 = g^2 F_6^{(0)} + g^4 \left(F_6^{(1)} + F_6^{(0)} \mathcal{H}_6^{(1)}\right) + g^6 \left(F_6^{(2)} + F_6^{(1)}\mathcal{H}_6^{(1)} + F_6^{(0)}\mathcal{H}_6^{(2)}\right) + O(g^8),$$

$$\underset{s_{i,i+1,i+2}=0}{\rm Disc}\ \underset{s_{i-1,i,i+1}=0}{\rm Disc} \left(F_6^{(2)} + F_6^{(1)}\mathcal{H}_6^{(1)} + F_6^{(0)}\mathcal{H}_6^{(2)}\right) = 0\,.$$

Maximal transcendental weight part of three-loop all-plus pure YM amplitude

Not Steinman-satisfied by itself!
(it contains products of amplitudes)

Summary

- Our result provides a first amplitude-type observable that uses the novel two-loop planar hexagon function space

- Two bottlenecks of calculating QCD amplitudes are solved in our case by

1. Symbol bootstrap based on hexagon function basis instead of IBP

   (Six-point bootstrap is simpler than five-point case due to various strong physical limits)

2. Leading singularities classified by geometrical framework

   (Nice properties for prefactors are expected at maximal transcendental weight part **[Henn, Torres Bobadilla]**)

- Generalization to function level, which allows us to explore various properties (positivity, complete monotonicity **[Henn, Raman]** ..)
- Any further observables within this functions space? with more symbol letters? Cluster algebraic structures?

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Thanks!

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