

## Writing Systems of Linear PDEs in Matrix Form via Gröbner Bases

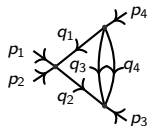
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# Differential equations behind holonomic functions

$f(x_1, \dots, x_n)$  a holonomic function  $x = (x_1, \dots, x_n)$

E.g., various special functions, Feynman integrals, ...



**Linear PDEs fulfilled by  $f$  occur in two shapes:**

polynomial coefficients

- 1 holonomic system of linear PDEs with polynomial coefficients

$$\sum_{\substack{\alpha, \beta \in \mathbb{N}^n, \\ \text{finite}}} c_{\alpha\beta} x^\alpha f^{(\beta_1, \dots, \beta_n)}(x) = 0$$

- 2 in matrix form  $dF = A \cdot F$

$A$  a matrix of differential one-forms

$F$  a vector of functions

e.g., master integrals

How to compute the **connection matrix  $A$**  from a system of PDEs?

**Aim:** systematic computation of connection matrices

**Tools:** algebraic analysis, Gröbner basis theory, computer algebra software  
symbolic computations

[1] P. Görlach, J. Koefler, A.-L.S., M. Sayrafi, H. Schroeder, N. Weiss, and F. Zaffalon. Connection Matrices in *Macaulay2*. Preprint [arXiv:2504.01362](https://arxiv.org/abs/2504.01362). [ConnectionMatrices.m2](#)

[2] M. Saito, B. Sturmfels, and N. Takayama. *Gröbner deformations of hypergeometric differential equations*. Volume 6 of *Algorithms and Computation in Mathematics*. Springer, 2000.

[3] D. R. Grayson and M. E. Stillman. *Macaulay2*, a software system for research in algebraic geometry. Available at <https://macaulay2.com/>. *open source!*

# Non-commutative algebras of differential operators

$f$  a function of  $x = (x_1, \dots, x_n)$

From homogeneous, linear differential equations to differential operators:

The PDE  $\sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} x^\alpha f^{(\beta_1, \dots, \beta_n)}(x) = 0$  is encoded by  $P = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} x^\alpha \partial^\beta$ .

**Example:**  $f''(x) - x \cdot f(x) = 0$  is encoded by  $P_{\text{Airy}} = \partial^2 - x$ .  
ODE ( $n = 1$ )

$$P \bullet f = 0, \partial_i \bullet f = \frac{\partial f}{\partial x_i}$$

**Weyl algebra:**  $D_n = \mathbb{C}[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle$

All generators commute, except  $\partial_i, x_i$ . They obey the Leibniz rule:  $\overbrace{\partial_i x_i - x_i \partial_i} = [\partial_i, x_i] = 1$ .

**⚠ Left  $D$ -ideals encode crucial properties of their solutions.**  $I = \langle P_1, \dots, P_k \rangle \subset D_n$   
such as singularities, growth behavior, number of data needed to encode the function, ...

## Examples:

- ◇ The **annihilator** of  $f$  is  $\text{Ann}_D(f) := \{P \in D_n \mid P \bullet f = 0\}$ . a left  $D_n$ -ideal  
E.g.,  $\text{Ann}_D(x^n) = \langle x\partial - n, \partial^{n+1} \rangle \subset D_1$ .
- ◇ If  $x_1 \partial_1 + \dots + x_n \partial_n - k \in \text{Ann}_D(f)$ , then  $f$  is **homogeneous of degree  $k$** .  
I.e.,  $f(\lambda x_1, \dots, \lambda x_k) = \lambda^k \cdot f(x_1, \dots, x_n)$  for all  $\lambda \neq 0$  and all  $x$ .
- ◇ **Invariance** under the conformal group is encoded as being a solution to a  $D$ -ideal.  
 $\rightsquigarrow$  Ward identities

$\mathbb{C}(x) = \left\{ \frac{p}{q} \mid p, q \in \mathbb{C}[x], q \neq 0 \right\}$  the field of rational functions in  $x = (x_1, \dots, x_n)$

**Rational Weyl algebra:**  $R_n = \mathbb{C}(x_1, \dots, x_n) \langle \partial_1, \dots, \partial_n \rangle$ .

⚠ *The differences between  $I \subset D_n$  and  $R_n I \subset R_n$  are subtle.*

The **holonomic rank** of a  $D_n$ -ideal  $I$  is the dimension of the  $\mathbb{C}(x_1, \dots, x_n)$ -vector space underlying  $R_n/R_n I$ :

$$\text{rank}(I) = \dim_{\mathbb{C}(x_1, \dots, x_n)} (R_n/R_n I) \quad P \sim Q \Leftrightarrow P - Q \in R_n I$$

- ◇ **generically:**  $\text{rank}(I) =$  dimension of space of analytic solutions = size of connection matrix
- ◇ **for a single ODE** ( $n = 1, I = D_1 P$ ):  $\text{rank}(I) = \text{ord}(P)$

**Example:**  $I = \langle x_1 \partial_2^2 - x_2 \partial_1^2 + \partial_1 - \partial_2, x_1 \partial_1 + x_2 \partial_2 + 1 \rangle \subset D_2$

- ◇  $\{1, \partial_1\}$  is a  $\mathbb{C}(x_1, x_2)$ -basis of  $R_2/R_2 I$
- ◇  $\text{rank}(I) = 2$
- ◇  $\text{Sing}(I) = V(x_1^3 - x_2^3) \subset \mathbb{C}^2$  singular locus of  $I$

## Writing $D$ -ideals in matrix form

Let  $I \subset D_n$  with  $\text{rank}(I) = m$  and  $\{s_1 = 1, s_2, \dots, s_m\}$  a  $\mathbb{C}(x_1, \dots, x_n)$ -basis of  $R_n/R_n I$ . For  $f \in \text{Sol}(I)$  a solution to  $I$ , denote  $F = (f, s_2 \bullet f, \dots, s_m \bullet f)^\top$ . Then there exist unique matrices  $A_1, \dots, A_n \in \text{Mat}_{m \times m}(\mathbb{C}(x_1, \dots, x_n))$  such that

$$\partial_i \bullet F = A_i \cdot F$$

for any  $f \in \text{Sol}(I)$ . The **connection matrices**  $A_i$  fulfill the **integrability conditions**:

$$\partial_i \bullet A_j - \partial_j \bullet A_i = [A_i, A_j] \quad \text{for all } i, j = 1, \dots, n.$$

**Gauge transformation:** For a change of basis  $\tilde{F} = BF$  with  $B \in \text{GL}_m(\mathbb{C}(x_1, \dots, x_n))$ ,

the transformed system is  $\partial_i \bullet \tilde{F} = \tilde{A}_i \tilde{F}$  with  $\tilde{A}_i = BA_i B^{-1} + (\partial_i \bullet B) \cdot B^{-1}$ .

**Facts about connection matrices:**

- ◇ geometrically, they arise from a **vector bundle with an integrable connection**
- ◇ systematic computation requires **Gröbner bases in  $R_n$**
- ◇ dually, the **connection matrix** is  $A = A_1 dx_1 + \dots + A_n dx_n$

**Particularly popular in dimensional regularization:**  $\varepsilon$ -factorized form  $dF = \varepsilon \tilde{A} F$

Then:  $D_n(\varepsilon) = \mathbb{C}(\varepsilon)[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle$

$\tilde{A}$  independent of  $\varepsilon$

## A friendly example

$$I = \langle x_1 \partial_2^2 - x_2 \partial_1^2 + \partial_1 - \partial_2, x_1 \partial_1 + x_2 \partial_2 + 1 \rangle \subset D_2 = \mathbb{C}[x_1, x_2] \langle \partial_1, \partial_2 \rangle$$

- ◇  $\{1, \partial_1\}$  is a  $\mathbb{C}(x_1, x_2)$ -basis of  $R_2/R_2 I$      $\text{rank}(I) = 2$
- ◇  $\text{Sol}(I) = \mathbb{C} \cdot \left\{ \frac{1}{x_1 - x_2}, \frac{1}{x_1 - x_2} \log(x_1/x_2) \right\}$
- ◇  $\text{Sing}(I) = V(x_1^3 - x_2^3) = V((x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2)) \subset \mathbb{C}^2$

Denoting  $F = (f, \partial_1 \bullet f)^\top = \begin{pmatrix} f \\ \frac{\partial f}{\partial x_1} \end{pmatrix}$  for  $f \in \text{Sol}(I)$ ,

$$\partial_1 \bullet F = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{x_1(x_1 - x_2)} & \frac{-3x_1 + x_2}{x_1(x_1 - x_2)} \end{pmatrix}}_{= A_1} \cdot F \quad \text{and} \quad \partial_2 \bullet F = \underbrace{\begin{pmatrix} -\frac{1}{x_2} & -\frac{x_1}{x_2} \\ \frac{1}{(x_1 - x_2)x_2} & \frac{x_1 + x_2}{(x_1 - x_2)x_2} \end{pmatrix}}_{= A_2} \cdot F.$$

Dually:

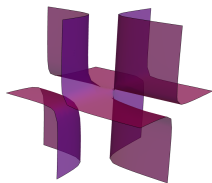
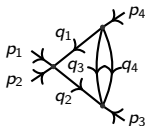
$$dF = \underbrace{\begin{pmatrix} -\frac{1}{x_1(x_1 - x_2)} dx_1 + \frac{1}{(x_1 - x_2)x_2} dx_2 & -\frac{3x_1 - x_2}{x_1(x_1 - x_2)} dx_1 + \frac{x_1 + x_2}{(x_1 - x_2)x_2} dx_2 \\ -\frac{1}{x_2} dx_2 & \frac{dx_1 - \frac{x_1}{x_2} dx_2}{x_2} \end{pmatrix}}_{= A} \cdot F.$$

↪ Systematic computation with the help of **Gröbner bases** in Weyl algebras!

## Commutative setup

- ◇ ideal  $I \subset \mathbb{C}[x_1, \dots, x_n]$  in a polynomial ring
- ◇ **algebraic variety**  $V(I) \subset \mathbb{C}^n$ : common vanishing set of all  $f \in I$

**Example:** graph polynomial of a Feynman diagram  $G$ ,  $\mathcal{G}_G = \mathcal{F}_G + \mathcal{U}_G \in \mathbb{C}\{\{\alpha_i\}\}$  in Schwinger parameters  $\alpha_i$ .



The set of real zeros of  $\mathcal{G}_G = \mathcal{U}_G + \mathcal{F}_G$  for  $G$  the massless parachute diagram.

*Ideals can be represented in various different ways!*

$$I = \langle p_1, \dots, p_k \rangle = \langle q_1, \dots, q_\ell \rangle = \dots$$

*In the Weyl algebra  $D_4$ :*  $\langle \partial_1, \partial_2, \partial_3, \partial_4 \rangle =$   
 $= \langle \partial_1, x_1^4 x_4 \partial_4 + x_1^2 x_3 \partial_3 + x_1^3 \partial_4 + x_1 \partial_3 + \partial_2 \rangle$

## Gröbner bases of ideals

- ◇ particular generating set, depending on choice of a term order on  $\mathbb{C}[x_1, \dots, x_n]$   
 e.g., lexicographic order built on  $x_1 \succ \dots \succ x_n$
- ◇ comparability and properties of ideals
- ◇ **computations:** solving polynomial systems, geometric operations, ...

Every  $P \in D_n$  can be written as

$$P = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha, \beta} x^\alpha \partial^\beta. \quad \text{finitely many } c_{\alpha, \beta} \neq 0$$

A **term order** on  $D_n$  is a **total order**  $\prec$  on the set of monomials  $\{x^\alpha \partial^\beta\}$  in  $D_n$ .

+ technical assumptions adapting to non-commutativity

**Example:**  $\prec_{(0, \nu)}$  with  $\nu \in \mathbb{R}_{>0}^n$  and  $\prec$  the lexicographic order *first compare  $(0, \nu)$ -weight,*  
 built upon  $\partial_1 \succ \dots \succ \partial_n \succ x_1 \succ \dots \succ x_n$ . *lex as a tie breaker*

This defines an **elimination order** on  $D_n$ , i.e.,  $\partial^\beta \prec \partial^\gamma$  implies  $x^\alpha \partial^\beta \prec \partial^\gamma \forall \alpha \in \mathbb{N}^n$ .

## Gröbner bases of $D_n$ -ideals

- ♦ The **initial monomial** of  $P \in D_n$ , denoted  $\text{in}_\prec(P)$ , is the monomial  $x^\alpha \partial^\beta$  in  $\mathbb{C}[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$  for which  $x^\alpha \partial^\beta$  is the largest monomial of  $P$ .
- ♦ The **initial ideal** of  $I \subset D_n$ ,  $\text{in}_\prec(I)$ , is the ideal in  $\mathbb{C}[x, \partial]$  generated by  $\{\text{in}_\prec(P) \mid P \in D_n\}$ .
- ♦ A finite set  $G = \{G_1, \dots, G_\ell\} \subset D_n$  is a **Gröbner basis** of  $I$  w.r.t.  $\prec$  if  $I = D_n G$  and  $\text{in}_\prec(I)$  is generated by  $\{\text{in}_\prec(G_i) \mid G_i \in G\}$ .
- ♦ The **standard monomials** of  $I$  with respect to  $\prec$  are the monomials  $x^\alpha \partial^\beta \notin \text{in}_\prec(I)$ .

**Example:**  $I = \langle x_1 \partial_1^2 - \underline{x_2 \partial_2^2} + \partial_1 - \partial_2, x_1 \partial_1 + \underline{x_2 \partial_2} + 1 \rangle \subset D_2$

A Gröbner basis of  $I$  with respect to  $\prec_{(0, \nu)}$  for  $\nu = (1, 2)$  is

$$\left\{ \underline{x_2 \partial_2} + x_1 \partial_1 + 1, \underline{x_1^2 \partial_1^2} - x_1 x_2 \partial_1^2 + 3x_1 \partial_1 - x_2 \partial_1 + 1, \underline{x_1 \partial_1 \partial_2} + x_1 \partial_1^2 + \partial_2 + \partial_1 \right\}.$$

## Computation of connection matrices

$R_n = \mathbb{C}(x_1, \dots, x_n) \langle \partial_1, \dots, \partial_n \rangle \quad \mathbb{C}(x)$  now considered as coefficients

Restricting a term order  $\prec$  on  $D_n$  to monomials in the  $\partial_i$ 's yields a term order  $\prec'$  on  $R_n$ .

**Nota bene:** The standard monomials of a Gröbner basis of  $R_n I$  are a  $\mathbb{C}(x_1, \dots, x_n)$ -basis of  $R_n/R_n I$ . *counterpart of master integrals*

### Entries of the connection matrices

Let  $G$  be a Gröbner basis of  $R_n I$  with respect to a term order  $\prec'$  on  $R_n$  with standard monomials  $\{s_1 = 1, s_2, \dots, s_m\}$ . Therefore, one can write

$$\partial_i s_j = \sum_{k=1}^m a_{jk}^{(i)} s_k + Q_j^{(i)} \quad \text{with } Q_j^{(i)} \in R_n I. \quad F = (f, s_2 \bullet f, \dots, s_m \bullet f)^\top$$
$$\partial_i \bullet F = A_i \cdot F \text{ for any } f \in \text{Sol}(I)$$

Hence  $a_{jk}^{(i)}$  is the  $(j, k)$ -th entry of the matrix  $A_i$ . *reduction in  $R_n$*

**Main step:** normal form algorithm in  $R_n \rightsquigarrow \text{normalForm}(\partial_i s_j, G)$  yields  $A_i$

⚠ The rational Weyl algebra  $R_n$  is not implemented in *Macaulay2*.

**Workaround:** If  $G$  is a Gröbner basis of a  $D_n$ -ideal  $I$  w.r.t. an **elimination order**  $\prec$  on  $D_n$ , then  $G$  is also a Gröbner basis of  $R_n I \subset R_n$  with respect to the order  $\prec'$  on  $R_n$ .

## Example code in *Macaulay2*

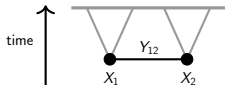
Package & documentation available at: <https://mathrepo.mis.mpg.de/ConnectionMatrices>

*Try it out on your favorite system of PDEs!*

```
i1 : needsPackage "ConnectionMatrices";
i2 : D = makeWeylAlgebra(QQ[x,y],{1,2});
i3 : I = ideal(x*dx^2-y*dy^2+dx-dy,x*dx+y*dy+1);
i4 : SM = standardMonomials I
o4 = {1, dx}
i5 : A = connectionMatrices(I,SM)
o5 = { | 0          1          | , | (-1)/y   (-x)/y   | }
      | (-1)/(x2-xy) (-3x+y)/(x2-xy) | | 1/(xy-y2) (x+y)/(xy-y2) |
i6 : connectionMatrix I
o6 = | (-1)/ydy          (-x)/ydy+dx          |
      | 1/(xy-y2)dy+(-1)/(x2-xy)dx (x+y)/(xy-y2)dy+(-3x+y)/(x2-xy)dx |
```

More interesting examples provided in the documentation:

- ◇ Gauss' hypergeometric function  ${}_2F_1$
- ◇ massless one-loop Feynman triangle integral
- ◇ cosmological correlator for 2-site chain + *gauged transformation to  $\varepsilon$ -factorized form*



[5] J. M. Henn, E. Pratt, A.-L.S., and S. Zoia. *D-Module Techniques for Solving Differential Equations in the Context of Feynman Integrals*. *Letters in Mathematical Physics*, 114(87), 2024.

[6] C. Fevola, G. L. Pimentel, A.-L.S., and T. Westerdijk. Algebraic Approaches to Cosmological Integrals. Special volume on Positive Geometry, *Le Matematiche*, 80(1):303–324, 2025.

*Algebraic analysis provides tools for the systematic investigation and manipulation of systems of PDEs behind holonomic functions in the sciences.*

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