Symbol Alphabets in QCD and Flag Cluster Algebras

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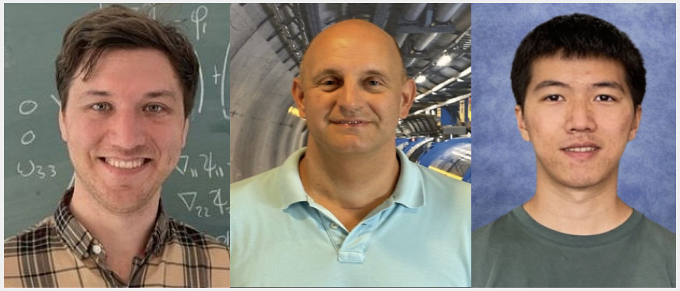
Pokraka, Spradlin, Weng

2506.11895



Amplitudes 2025, June 2025

Outline

0. Not talk about 2503.23579, 2504.11253
1. Symbol alphabet in N=4 Yang-Mills & Grassmannian cluster algebra
2. Symbol alphabet in massless QCD & Flag cluster algebra 2506.11895
3. Conclusion

Hidden Zeros of Cosmological Wavefunction

De, Paranjape, Pokraka, Spradlin, AV

2503.23579

Abstract: Motivated by the recent discovery of hidden zeros in particle and string amplitudes, we characterize zeros of individual graph contributions to the cosmological wavefunction of a scalar field theory. We demonstrate that these contributions split near these zeros for all tree graphs and provide evidence that this extends to loop graphs as well. We explicitly construct polytopal realizations of the relevant graph associahedra and show that the cosmological zeros have natural geometric and physical interpretations. As a byproduct, we establish an equivalence between the wavefunction coefficients of chain graphs and flat-space $\mathrm{Tr}(\phi^3)$ amplitudes, enabling us to leverage the cosmological zeros to uncover the recently discovered hidden zeros of colored amplitudes.

Large Deformations of Trφ^3 and the World at Infinity

Paranjape, Skowronek, Spradlin, AV

2504.11253

ABSTRACT: The amplitudes of the non-linear sigma model can be obtained from those of $\mathrm{Tr}(\Phi^3)$ theory by sending the kinematic (Mandelstam) variables to infinity in a certain direction. In this paper we characterize the behavior of $\mathrm{Tr}(\Phi^3)$ amplitudes under a general class of large kinematic shifts called g-vector shifts. The objects that live in this world at infinity retain certain key amplitude-like properties, most notably factorization, and admit descriptions in terms of polytopes, but they are not generally amplitudes of any cognizable theory. We identify particular g-vector shifts that lead at infinity to mixed amplitudes involving two pions and any number of scalars, allowing us to provide polytopal descriptions of these amplitudes.

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Planar N=4 Yang-Mills

n-point symbol alphabet is described by Grassmannian cluster algebra Gr(4,n).

Golden, Goncharov, Spradlin, Vergu, AV 1305.1617

2-loop 6/7-point in N=4 Yang-Mills

Symbol alphabet for 2-loop 6/7-point amplitudes in N=4 Yang-Mills has been worked out in

Goncharov, Spradlin, Vergu, AV 1006.5703

Caron-Huot 1105.5606.

$$\{\langle \mathbf{a\,a{+}1\,b\,c}\rangle, \langle \mathbf{1(23)(45)(67)}\rangle, \langle \mathbf{1(27)(34)(56)}\rangle\}$$

15/49 letters for 6/7-point

Refresher 1: Momentum Twistors

Hodges 0905.1473

$$\mathbf{p_i = \lambda_i \widetilde{\lambda}_i}$$

$$\mathbf{\lambda_i^a = \mathit{Z}_i^a}$$

$$\mathbf{\widetilde{\lambda}_i^a = \frac{\langle i{-}1\, i\rangle Z_{i+1}^{a+2} + \langle i{+}1\, i{-}1\rangle Z_i^{a+2} + \langle i\, i{+}1\rangle Z_{i-1}^{a+2}}{\langle i{-}1\, i\rangle \langle i\, i{+}1\rangle}}$$

$$\mathbf{\langle abcd\rangle = det(Z_a\, Z_b\, Z_c\, Z_d)}$$

$$\mathbf{\langle a(bc)(de)(fg)\rangle = \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle}$$

Refresher 2: Symbol Alphabet

Goncharov, Spradlin, Vergu, AV 1006.5703

$$dF^{(m)} = \sum_i F_i^{(m-1)} \, d\log x_i$$

$$S(F^{(m)}) := \sum_i S(F_i^{(m-1)}) \otimes x_i$$

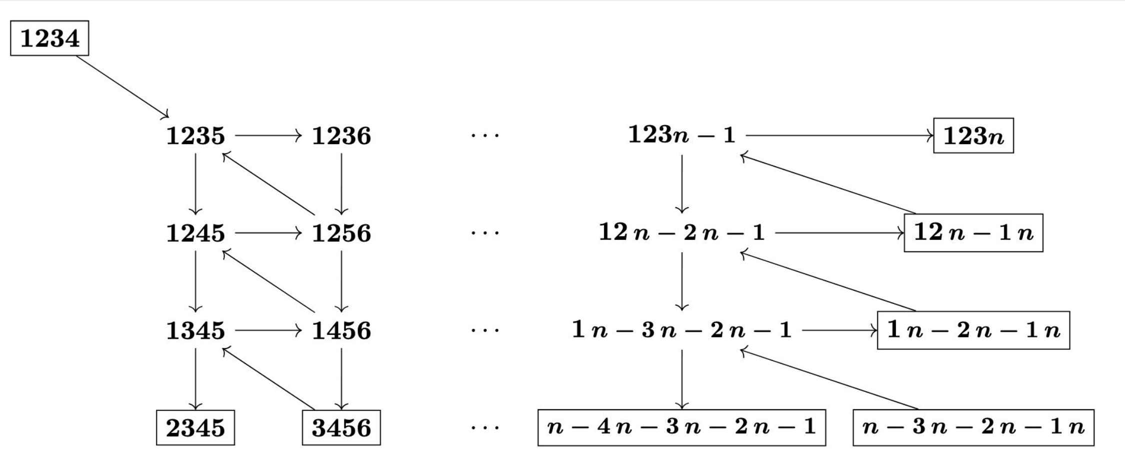
$$S(\log x) = x$$

$$dLi_2(x) = -\log(1-x)\, d\log(x)$$

$$S[Li_2(x)] = -(1-x) \otimes x$$

Refresher 3: Grassmannian Cluster Algebra Gr(4,n)

Initial Seed (variables, quiver)



Mutation Rule

$$a_k \to a'_k = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

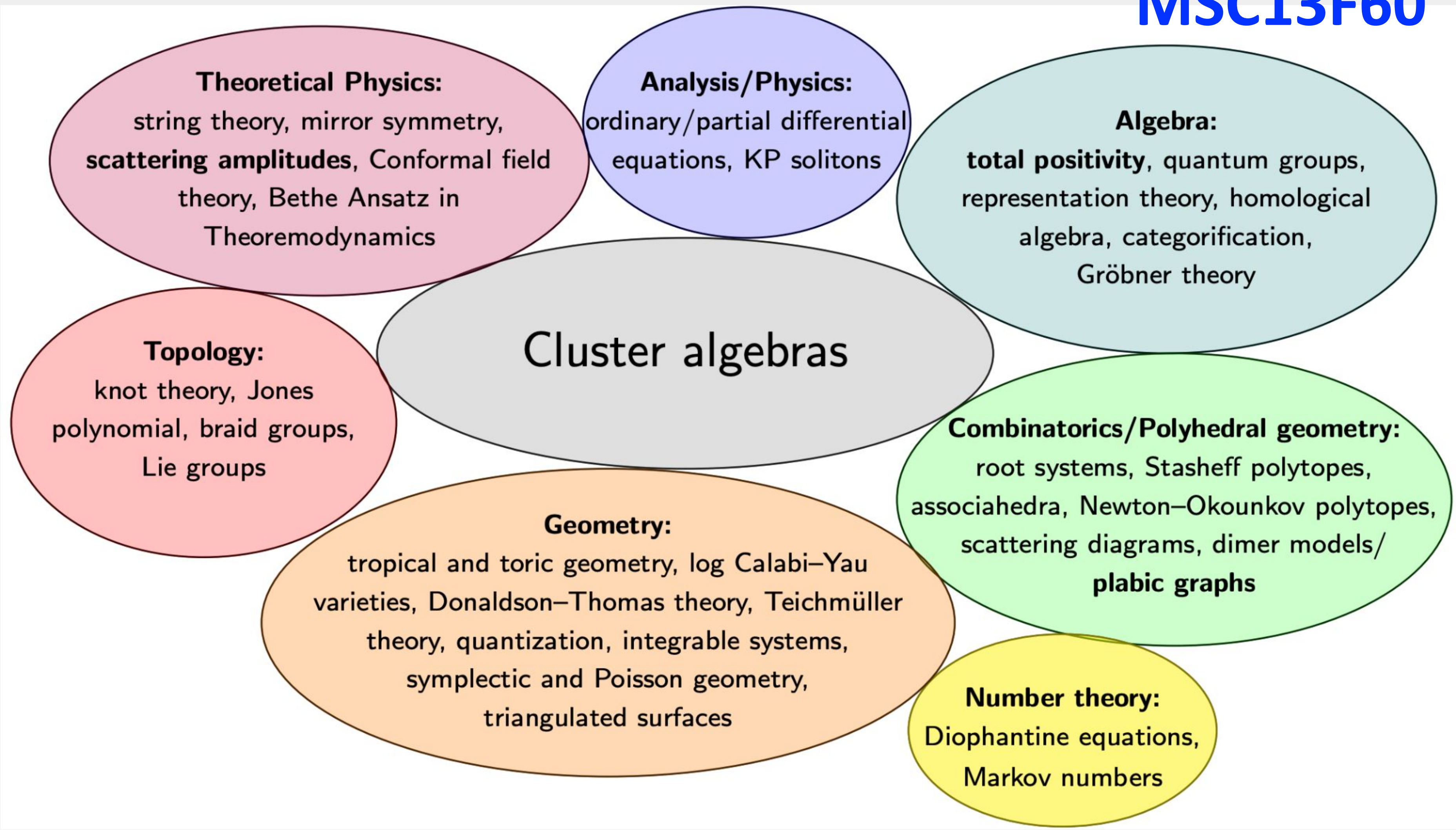
Cluster Variables

$$\{a_k\}$$

Fomin, Zelevinsky 0104151, Scott 0311148, Gekhtman, Shapiro, Vainshtein 0208033
Fomin, Williams, Zelevinsky "Introduction to Cluster Algebras"

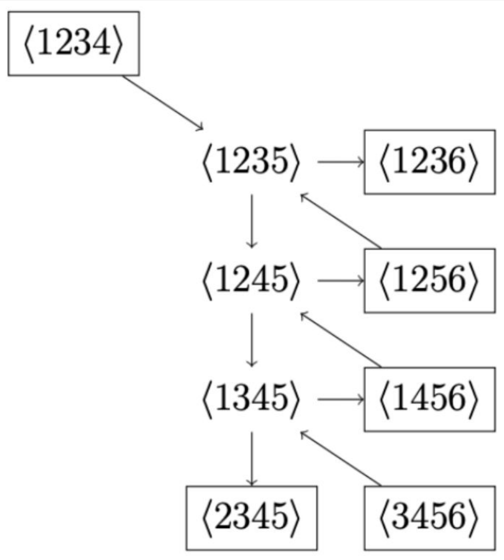
Cluster Algebras in Mathematics

MSC13F60



Slide from Bossinger, Harvard CMSA Summer School (2025)

Gr(4,6) Cluster Algebra & 2-loop 6-point Symbol Alphabet



Golden, Goncharov
Spradlin, Vergu, AV
1305.1617

Start with the initial seed

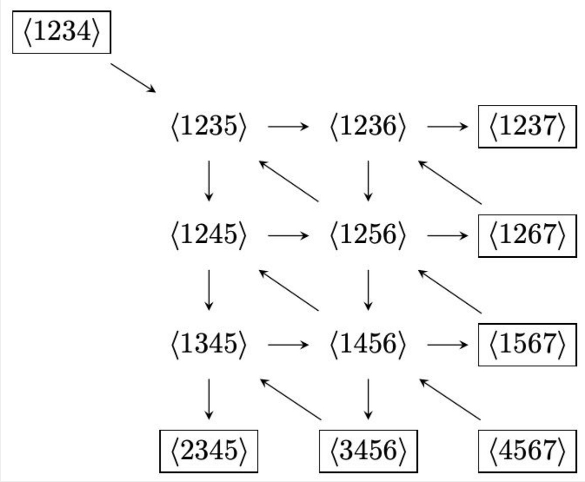
Perform all mutations

Obtain 15 cluster variables

$$\{\langle a\, a{+}1\, b\, c\rangle\}$$

Match 2-loop 6-point symbol alphabet!

Gr(4,7) Cluster Algebra & 2-loop 7-point Symbol Alphabet



Golden, Goncharov
Spradlin, Vergu, AV
1305.1617

Start with the initial seed

Perform all mutations

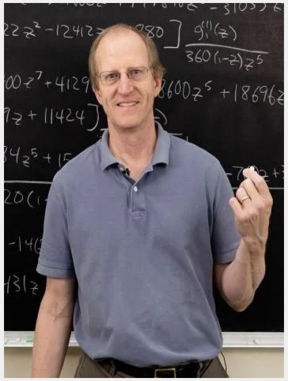
Obtain 35+7+7 cluster variables

$$\{\langle \mathbf{a}\,\mathbf{a}{+}\mathbf{1}\,\mathbf{b}\,\mathbf{c}\rangle, \langle \mathbf{1}(\mathbf{23})(\mathbf{45})(\mathbf{67})\rangle, \langle \mathbf{1}(\mathbf{27})(\mathbf{34})(\mathbf{56})\rangle\}$$

Match 2-loop 7-point symbol alphabet!

Symbol Bootstrap

Dixon and collaborators have computed
6- and 7-point amplitudes in N=4 Yang-Mills
to 8 and 4-loops by using a bootstrap approach
based on mathematical and physical inputs



- assuming only cluster variables appear
- assuming they satisfy adjacency
- discrete symmetries
- first-entry condition
- input from integrability

Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
Dixon, Liu 2308.08199

New Features at n>7

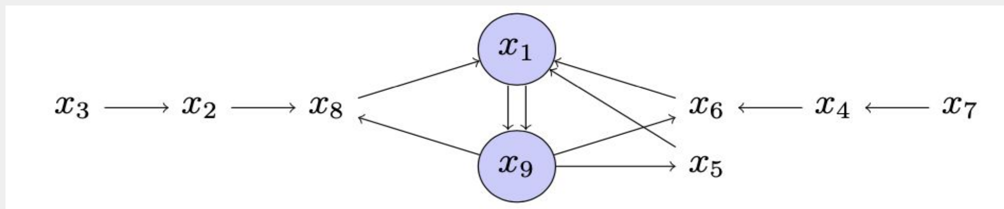
- Gr(4,n) cluster algebra is infinite for n>7.
- Some symbol letters are not cluster coordinates, but involve square roots when written in terms of Pluckers.

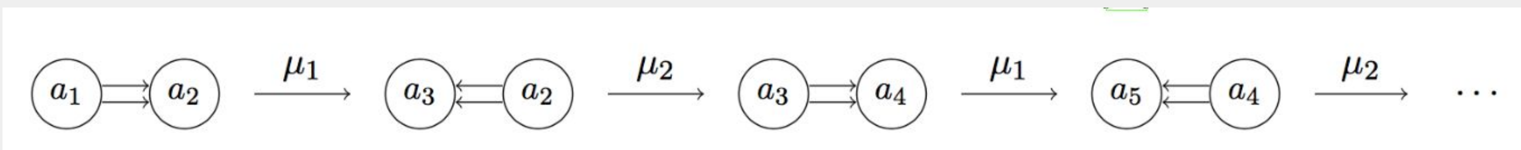
Li, Zhang 2110.00350

He, Li, Zhang 1911.01290
2009.11471

Algebraic Letters from Infinite Paths

The infinite cluster variables of Gr(4,8) arise from quivers that have a double arrow like





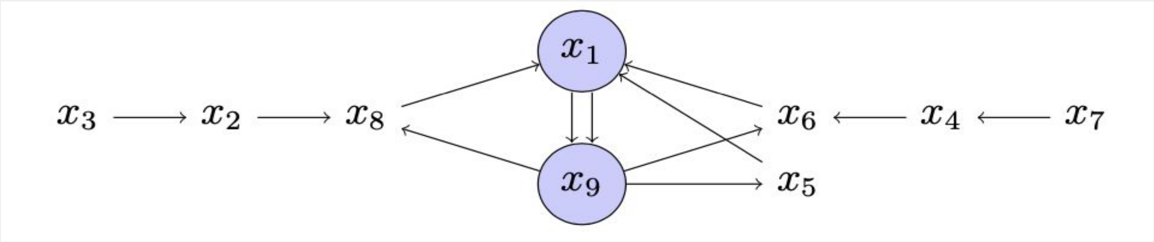
Square roots appear along infinite mutation sequence

$$\lim_{i\to\infty} \frac{a_i}{a_{i-1}} = \frac{a_2}{2a_1}\left(1 + x_1 + x_1x_2 + \sqrt{(1 + x_1 + x_1x_2)^2 - 4x_1x_2}\right)$$

$x_1 = 1/a_2^2,\ x_1 = a_1^2.$ Canakci, Schiffler 1608.06568

Algebraic Letters from Infinite Paths

All algebraic symbol letters known to appear in 8 and 9-point amplitudes are known to be associated to such infinite sequences inside Gr(4,8) and Gr(4,9).



Drummond, Foster, Gürdoğan, Kalousios 1912.08217

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   Grassmannian cluster algebra
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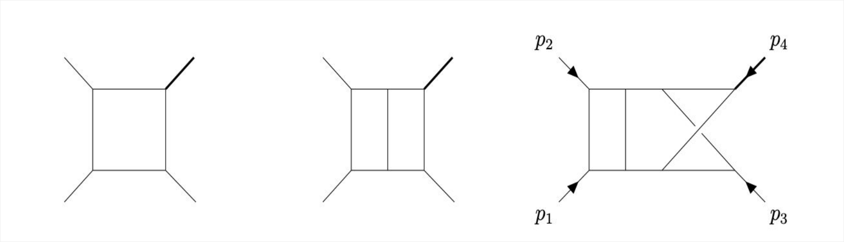
Cluster Algebras for Feynman Integrals

Chicherin, Henn, Papathanasiou 2012.12285,
Aliaj, Papathanasiou 2408.14544 found cluster algebras structure in certain Feynman integrals relevant to QCD.



A_2 C_2 G_2

Symbol alphabet for these integrals has been worked out to three-loops by Henn, Lin, Torres Bobadilla 2302.12776.

Cluster Algebras for Feynman Integrals

Start with the initial seed

$$\{\mathbf{a_1}, \mathbf{a_2}\}$$

Perform all mutations

$$\mathbf{a_{m+1}} = \begin{cases} \frac{\mathbf{1+a_m}}{\mathbf{a_{m-1}}} & m \ \mathbf{odd} \\ \frac{\mathbf{1+a_m^L}}{\mathbf{a_{m-1}}} & m \ \mathbf{even} \end{cases}$$

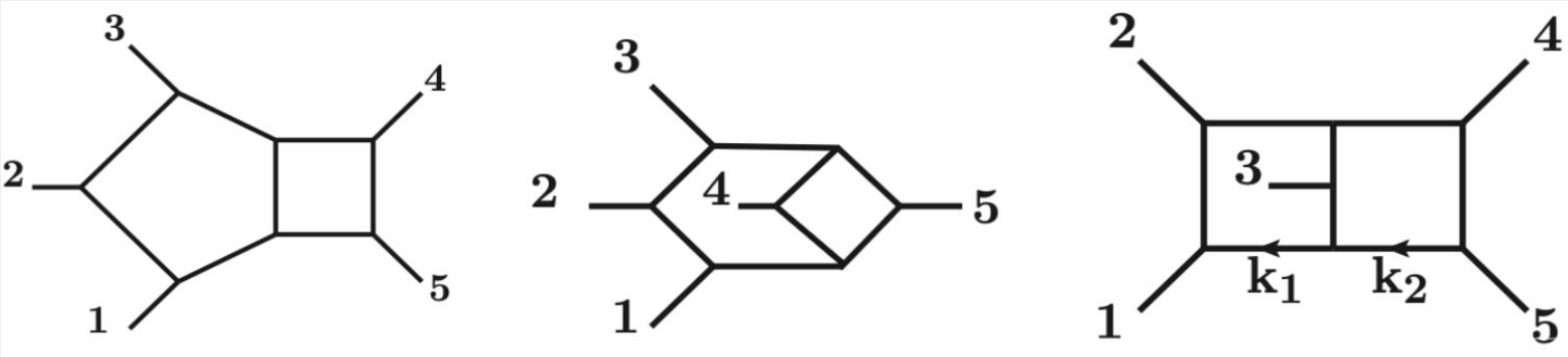
L=1,2,3
for
A_2, C_2, G_2

Obtain 5,6,8 cluster variables

Match 1,2,3-loop symbol alphabet

$$\mathbf{a_1} = \frac{\mathbf{s - p_4^2}}{\mathbf{t}} \qquad \mathbf{a_2} = \frac{\mathbf{s}}{\mathbf{t}} \frac{\mathbf{p_4^2 - s - t}}{\mathbf{p_4^2}}$$

2-loop 5-point Massless QCD



The full symbol alphabet for planar & non-planar 2-loop 5-point integrals has been worked out by Germann, Henn, Lo Presti 1511.05409, Chicherin, Henn, Mitev 1712.09610.

$$\left\{ s_{12}, s_{12}+s_{23}, s_{12}-s_{45}, s_{45}-s_{12}-s_{23}, s_{34}+s_{45}-s_{12}-s_{23}, \frac{a-\sqrt{\Delta}}{a+\sqrt{\Delta}} \right\}$$

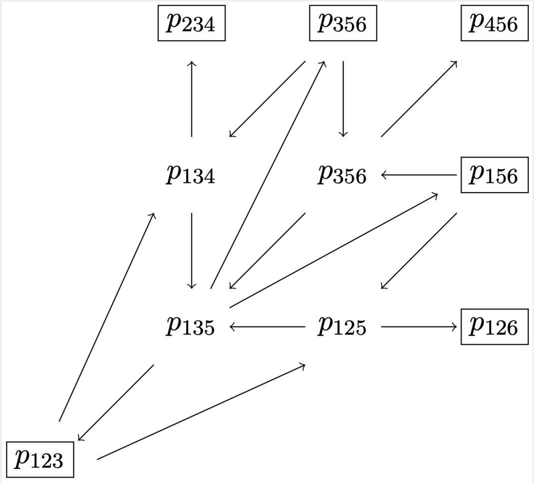
$$a = s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} - s_{51}s_{12}$$

$$\sqrt{\Delta} = \langle 12\rangle\langle 45\rangle[24][15] - \langle 24\rangle\langle 15\rangle[12][45]$$

+ 5 cyclic

Planar: 25+1 letters & Non-planar: +5 letters

These letters were shown to coincide with those from a construction based on Gr(3,6) = D_4 cluster algebra.



Bossinger, Drummond, Glew 2212.08931

2-loop Integrals & Flag Cluster Algebra

I will construct these letters in a different but equivalent way making use of 2408.14956. Bossinger and Li presented an embedding of partial flag varieties $F(2,n-2;n)$ into Grassmannians $Gr(n-2,2n-4)$ that respects the cluster algebra structure.

n=5: $F(2,3;5) \rightarrow Gr(3,6)$

n=6: $F(2,4;6) \rightarrow Gr(4,8)$

Flag Cluster Algebra

The (partial) flag variety

$$\mathbf{F}(\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{n}) = \{\mathbf{V_{d_1}} \subset \mathbf{V_{d_2}} \subset \cdots \subset \mathbb{C}^{\mathbf{n}}\}$$

generalizes the Grassmannian

$$\mathbf{Gr}(\mathbf{k}, \mathbf{n}) = \{\mathbf{V_k} \subset \mathbb{C}^{\mathbf{n}}\}$$

where V_d denotes vector space dimension d.

There is a cluster algebra structure associated to a flag manifold.

Geiss, Leclerc, Schroer 0609138

Flag Variety and Spinor Helicity

There is an isomorphism between
the spinor helicity variables for n-particles &
a point in F(2,n-2;n) represented by (n-2)x(n) matrix.

$$\underbrace{\begin{pmatrix} & Q & \end{pmatrix}}_{n} \Big\}\, n-2 \quad = \quad \underbrace{\begin{pmatrix} P \\ \hdashline \widetilde{P} \end{pmatrix}}_{n} \begin{matrix} \}\,2 \\ \Big\}\, n-4 \end{matrix}$$

Bossinger Li
2408.14956

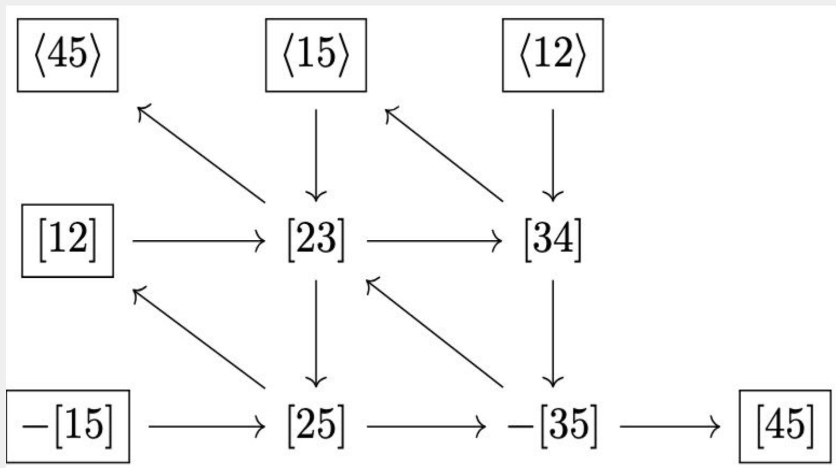
$$\langle \mathbf{i\,j} \rangle = \mathbf{det}(\mathbf{P_i P_j})$$

$$[\mathbf{i\,j}] = (-1)^{\mathbf{i+j+1}} \mathbf{det}(\mathbf{Q_i Q_2} \cdots \not{\mathbf{Q_i}} \cdots \not{\mathbf{Q_j}} \cdots \mathbf{Q_n})$$

Under this isomorphism:
the initial quiver for F(2,n-2;n) is that of Gr(n-2,2n-4)
(via mutations, freezings, deletions).

F(2,3;5) Cluster Algebra

The initial quiver of F(2,3;5) is identical to that of the Gr(3,6)=D_4 cluster algebra.



$$\{\langle \mathbf{i\,j}\rangle, [\mathbf{i\,j}], \langle \mathbf{2\,3}\rangle[\mathbf{2\,3}] - \langle \mathbf{4\,5}\rangle[\mathbf{4\,5}], \langle \mathbf{1\,2}\rangle[\mathbf{1\,2}\rangle - \langle \mathbf{3\,4}\rangle[\mathbf{3\,4}]\}$$

10+10+1+1 cluster variables

F(2,3;5) Cluster Algebra & 2-loop 5-point Symbol Alphabet

This set is not closed under the natural cyclic group $i \longrightarrow i+1$ of the 5-particles

but if we take the union under cyclic permutation (or under all permutations)

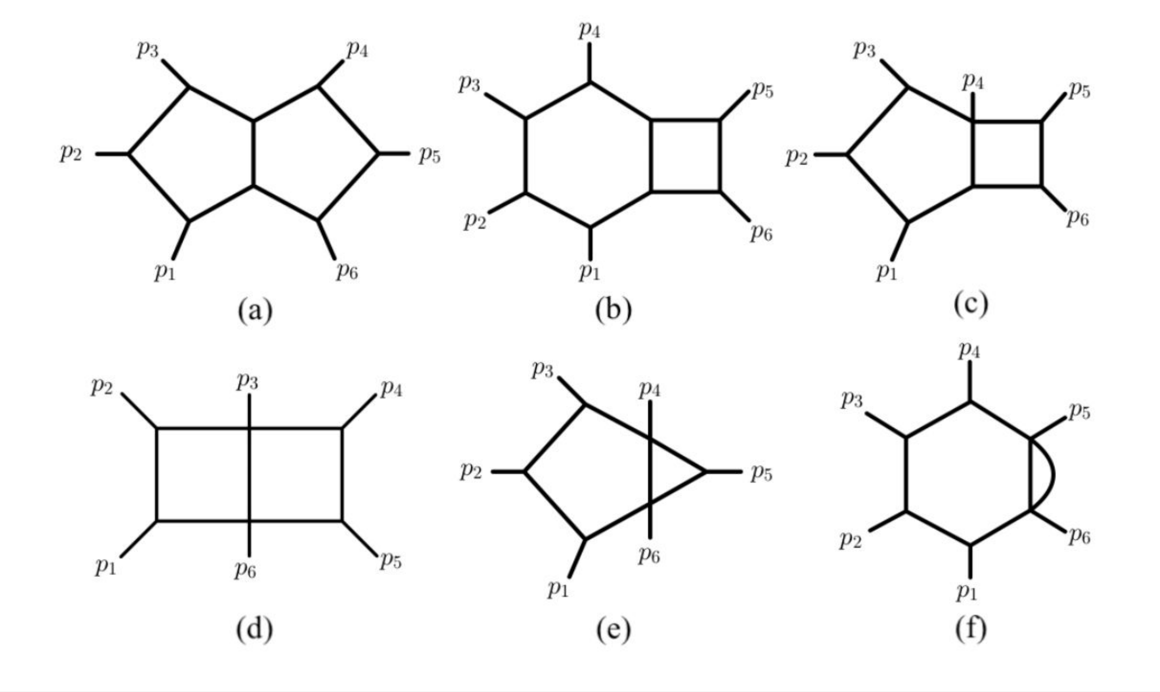
it produces exactly

25 planar (and 5 non-planar) letters!

(except for one letter $\sqrt{\Delta}$ that drops out of all known amplitudes in four dimensions).

Pokraka, Spradlin, AV, Weng 2506.11895

2-loop 6-point Planar Massless QCD

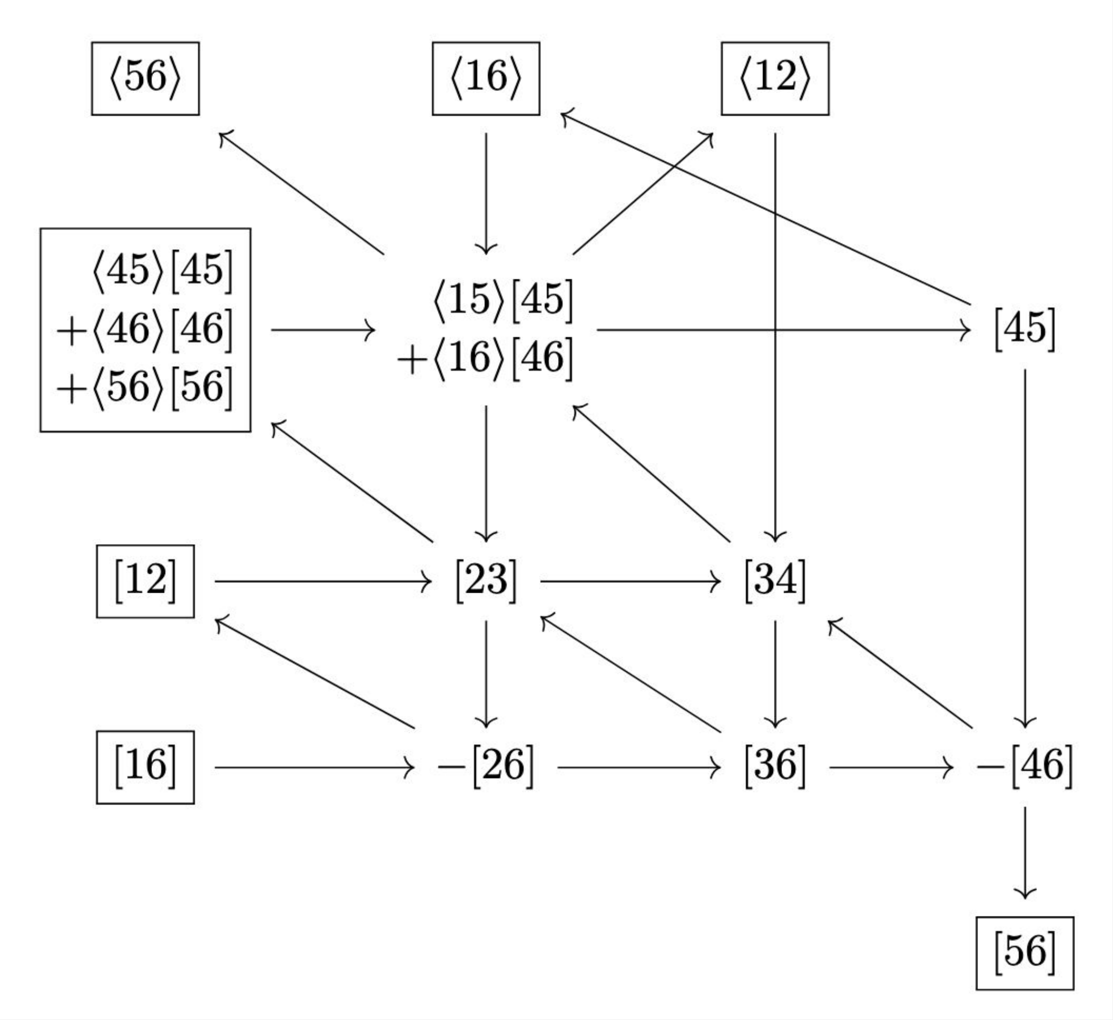


The full symbol alphabet for planar 2-loop 6-point integrals has been worked out by

Abreu, Monni, Page, Usovitsch 2412.19884;

Henn, Matijasic, Miczajka, Peraro, Xu, Zhang 2501.01847.

F(2,4;6) Cluster Algebra



F(2,4;6) Cluster Algerba & 2-loop 6-point Symbol Alphabet

245 symbol letters through all orders in dim reg

- 7 are analogs of $\sqrt{\Delta}$ in the 5-point case
- 135 are F(2,4;6) cluster variables (including permutations)
- 40 are algebraic and we reproduce all of them from infinite mutation sequences
- 27 of them are rational, are NOT cluster variables, but only appear at O(epsilon) so are not relevant in four dimensions
- 36 remain somewhat mysterious…

(24 of 36 appear in Carrolo, Chicherin, Henn, Yang, Zhang 2505.01245)

Pokraka, Spradlin, AV, Weng 2506.11895

Conclusion

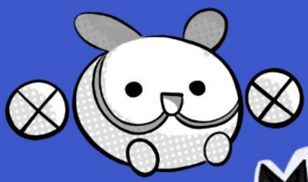
ABSTRACT: The full 245-letter symbol alphabet for all planar massless two-loop six-point Feynman integrals was recently determined in arXiv:2412.19884 and arXiv:2501.01847. In a parallel mathematical development, it was shown in arXiv:2408.14956 that there is cluster algebra preserving embedding of the partial flag variety $\mathcal{F}\ell_{2,n-2;n}$, which describes the kinematics of n massless particles, into the Grassmannian $\mathrm{Gr}(n-2, 2n-4)$. In this paper we connect these developments by showing that most of the rational symbol letters can be expressed in terms of flag cluster variables, and that all of the algebraic symbol letters arise from infinite mutation sequences.

- **Missing Letters (C_2?)**
- **Cluster Adjacency**
- **Cluster Functions**
- **Amplituhedron Tilings**
- **More data**

Chicherin, Henn, Trnka, Zhang 2410.11456

Symbology@15

15-18 December 2025



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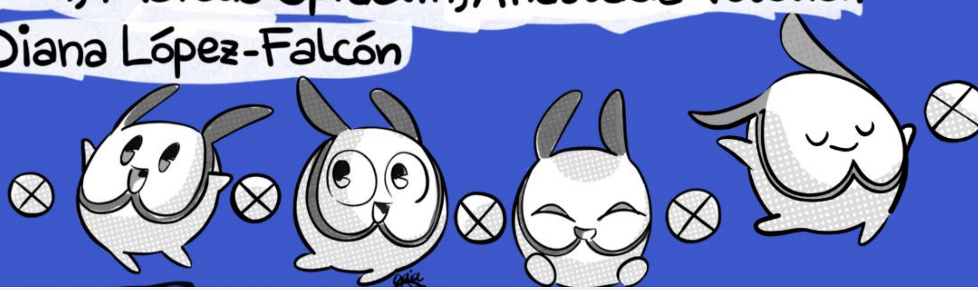

Thank you!