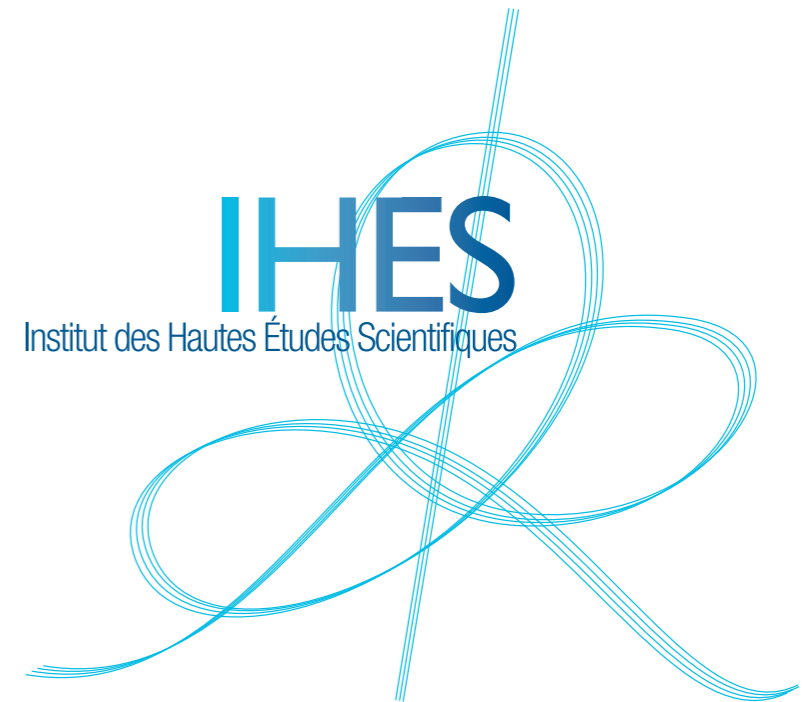


# Soft Theorems from Higher Symmetry

**Julio Parra-Martinez (IHES)**

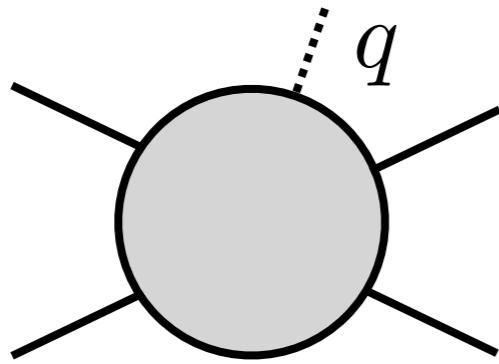
Based on 2505.03566

w/ Jonah Berean-Dutcher, Maria Derda



@ Amplitudes 2025 @ SNU, June 2025

The behavior of scattering amplitudes when the momentum of a particle is small is often universal



$$\lim_{q \rightarrow 0} \mathcal{A}_{n+1} = S \mathcal{A}_n$$

Earliest examples:

- Soft pion theorem  
[Adler]

$$\lim_{q_\pi \rightarrow 0} \mathcal{A}_{n+\pi} = 0$$

- Soft photon theorem  
[Low; Burnett, Kroll; Weinberg]

$$\lim_{q_\gamma \rightarrow 0} \mathcal{A}_{n+\gamma} = \sum_a Q_a \frac{\epsilon_\gamma \cdot P_a}{q_\gamma \cdot P_a} \mathcal{A}_n$$

# Many perspectives

*Symmetry:*    *See Sangmin's talk!*

“asymptotic (photons, gravitons), spontaneously broken (pions),...”

*Geometry*    *See Andrea's talk!*

“encode geometry of moduli space of vacua”

*Factorization (EFT)*

“soft theorems are just leading operators in EFT of soft modes”

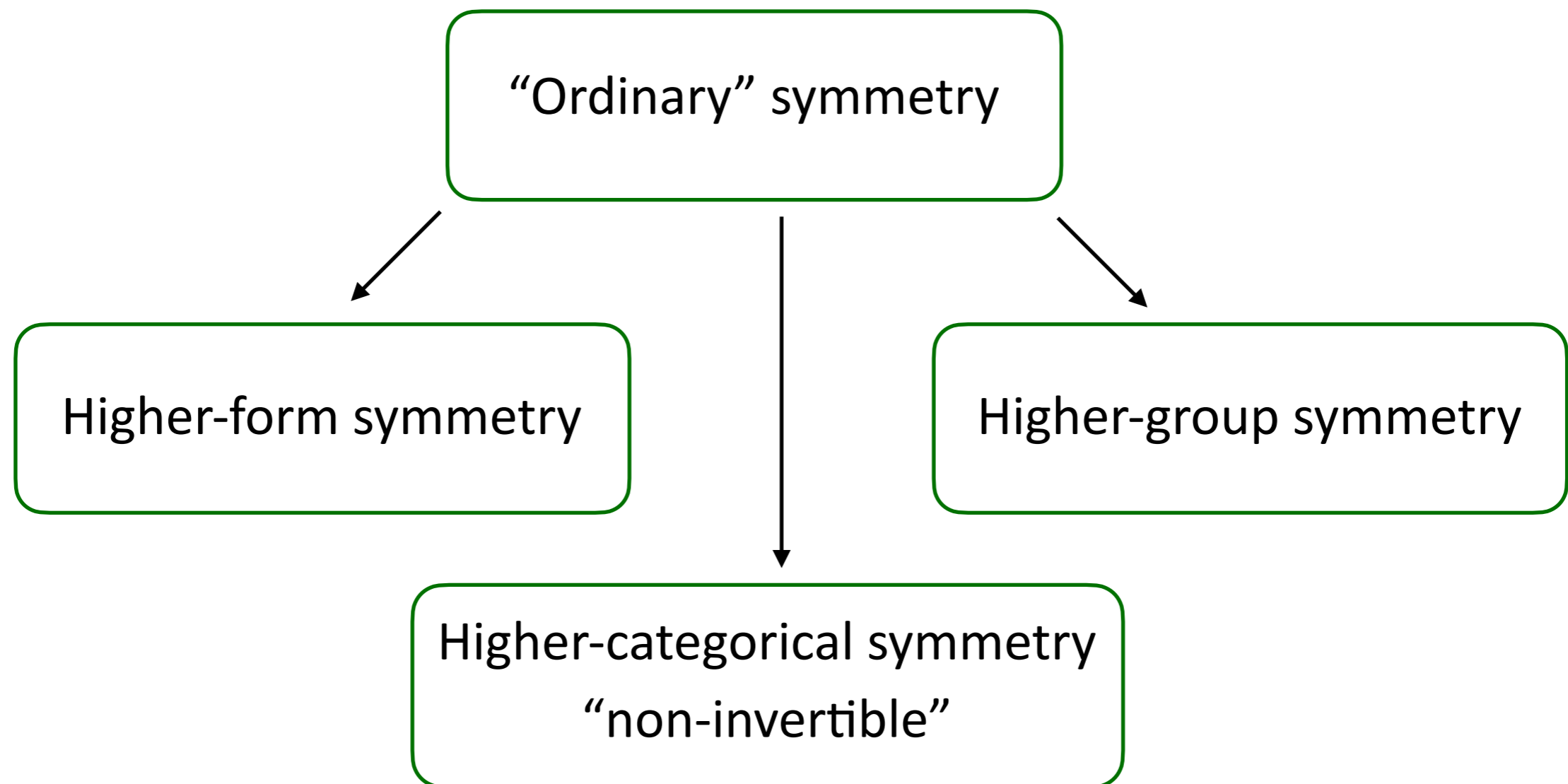
Connection between these?

It always bugged me that explaining soft scalar and soft photon theorems required invoking rather different ideas...

# “Higher” or “generalized” symmetry

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In the last decade there has been a great expansion in what we understand as symmetry



Thus far, these ideas have found surprisingly little application in scattering amplitudes, with some exceptions.

[van Beest, Boyle Smith, Delmastro, Komargodski, Tong;  
Copetti, Cordova, Komatsu]

*See Lucia's talk!*

Today I'll try to argue that they can provide a unifying perspective on soft theorems.

Furthermore, they suggest new structures!

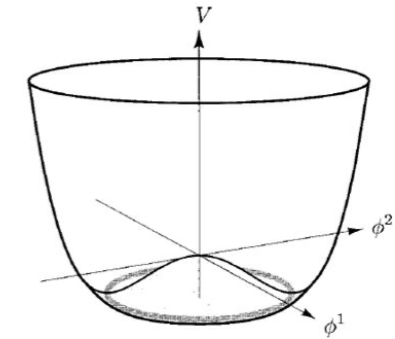
# Outline

- Review: soft pions
- Soft photons from 1-form symmetry
- 2-group soft theorem

**Review: soft pions**

# Spontaneously broken symmetry

Familiar situation  $\langle \mathcal{O}^a \rangle = v^a$  breaks  $G \rightarrow H$



$$[T^a, T^b] = i f^{abc} T^c$$

$$[T^a, X^b] = i f^{abc} X^c$$

$$[X^a, X^b] = i f^{abc} T^c$$

$T^a$  generate unbroken subgroup  $H$

$X^a$  broken generators in coset  $G/H$

Goldstone's theorem: SSB implies existence of NGB for each  $X^a$

$$\langle \partial^\mu j_\mu(x) O(0) \rangle \sim q \langle O(0) \rangle \delta^{(D)}(x) = qv \delta^{(D)}(x)$$

$$\langle j_\mu(x) O(0) \rangle \sim \partial_\mu \frac{qv}{x^2} \quad \text{power law} = \text{gapless modes}$$

“NGB parameterize fluctuations of VEV”  $O^a \sim v^a e^{i\pi^a X^a}$

# Pions as Nambu-Goldstone bosons

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More precisely:  $\langle 0 | j_\mu^a(x) | \pi^b(q) \rangle = f_\pi \delta^{ab} q_\mu e^{iq \cdot x}$

This implies current linear in the NBG and shift symmetry

$$j_\mu^a = \partial_\mu \pi^a + \mathcal{O}(\pi^3) \quad \pi^a \rightarrow \pi^a + c^a + \mathcal{O}(\pi^2)$$

E.g., QCD in the chiral limit ( $m_q = 0$ )

$$SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V \quad \langle \psi_L^a \psi_R^b \rangle = \Lambda_{QCD}^3 \delta^{ab}$$

$$N_f^2 - 1 \text{ pions at low energies} \quad \mathcal{L} = f_\pi^2 \text{Tr}(\partial U \partial U^\dagger) + \dots$$

# Soft pion: Adler Zero

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Considering current overlap with multi-particle state  $|\alpha\rangle$

$$\langle 0 | j^{a\mu}(q) | \alpha \rangle = -f_\pi \frac{q^\mu}{q^2} \mathcal{A}(\alpha + \pi^a) + \mathcal{N}^\mu$$

Ward identity implies relation

$$0 = \langle 0 | \partial_\mu j^{a\mu}(q) | \alpha \rangle = -f_\pi \mathcal{A}(\alpha + \pi^a) + q_\mu \mathcal{N}^\mu$$

Which gives Adler's zero [Adler]

$$\lim_{q \rightarrow 0} \mathcal{A}(\alpha + \pi^a) \sim \lim_{q \rightarrow 0} q_\mu \mathcal{N}^\mu \longrightarrow 0$$

# Double-soft pion

Current algebra also implies multi-soft pion theorems

$$\partial \cdot j^a(x) j^{b\mu}(y) \sim i f^{abc} \delta^{(4)}(x - y) j^{c\mu}(x)$$

By considering  $\langle 0 | j^{a\mu}(q_a) j^{b\nu}(q_b) | \alpha \rangle$  one finds

$$\lim_{q_{a,b} \rightarrow 0} \mathcal{A}_{n+2} = (S^{(0)} + S^{(1)}) \mathcal{A}_n$$

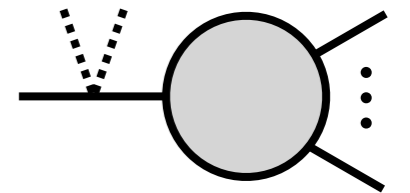
with

$$S^{(0)} = \sum_i \frac{p_i \cdot (q_a - q_b)}{p_i \cdot (q_a + q_b)} [X^a, X^b]$$

[Weinberg; Arkani-Hamed, Cachazo, Kaplan]

$$S^{(1)} = \sum_i \left( \frac{p_i \cdot (q_a - q_b)}{p_i \cdot (q_a + q_b)} \frac{q_a \cdot q_b}{p_i \cdot (q_a + q_b)} + \frac{q_a \cdot J_i \cdot q_b}{p_i \cdot (q_a + q_b)} \right) [X^a, X^b] + \frac{q_a \cdot q_b}{p_i \cdot (q_a + q_b)} \{X^a, X^b\}$$

[Cachazo; Low]



# Soft photons from 1-form symmetry

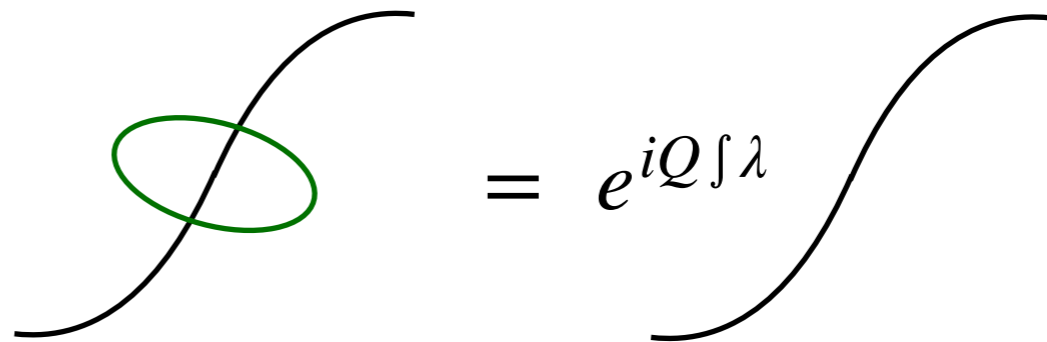
# 1-form symmetry

Conserved currents are rank two tensor  $J^{\mu\nu} = J^{[\mu\nu]}$

Charges supported on  
codimension-two surfaces

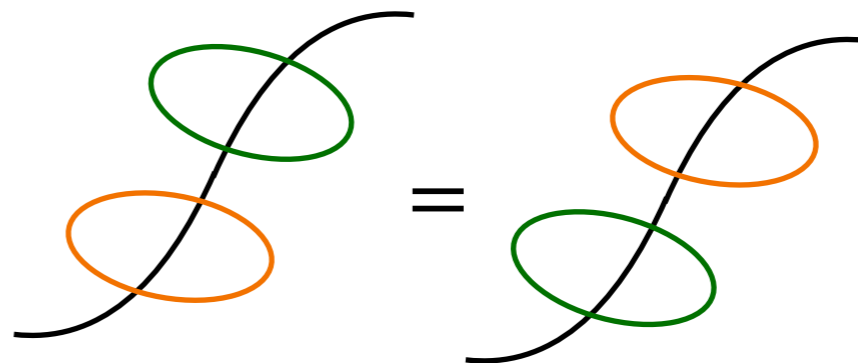
$$Q = \oint_{\Sigma} J_{\mu\nu} d\Sigma^{\mu\nu}$$

Charged operators are not  
local but line operators



The diagram shows a black line operator with a green loop around it. To the right is an equals sign followed by the expression  $e^{iQ \int \lambda}$ , and then another black line operator.

Necessarily abelian



The diagram shows two line operators. The left one has a green loop and an orange loop. The right one has an orange loop and a green loop. They are separated by an equals sign, indicating they are equivalent.

# 1-form Ward identity

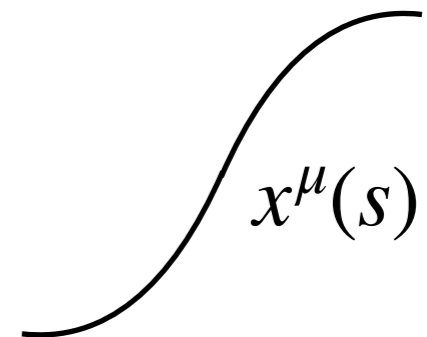
1-form symmetry implies the following operator eq.

$$\partial^\mu J_{\mu\nu} W_Q(\Gamma) \sim Q \delta_\nu^{(D-1)}(x_\Gamma) W_Q(\Gamma)$$

(c.f., 0-form WI  $\partial^\mu j_\mu(x) O(y) \sim q \delta^{(D)}(x - y) O(x)$ )

The codimension-1 delta function is defined as

$$\delta_\nu^{(D-1)}(x_\Gamma) = \int ds \dot{x}_\nu(s) \delta^{(D)}(x - x(s))$$

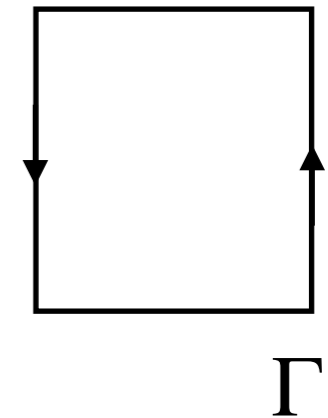


# 1-form symmetry breaking

Line operators, get a v.e.v

$$\langle W(\Gamma) \rangle = c e^{-L(\Gamma)} \sim 1$$

“perimeter law”



Version of Goldstone's theorem

$$\langle \partial^\mu J_{\mu\nu}(x) W \rangle \sim Q \delta_\nu^{(D-1)}(x_\Gamma)$$

$$\langle J_{\mu\nu}(x) W \rangle \sim v_{[\nu} \partial_{\mu]} \frac{Q}{x_\perp^2}$$

Photon parameterizes fluctuations of v.e.v

$$W(\Gamma) = e^{i\oint_\Gamma A}$$

# Photon as NG boson

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More precisely: 
$$\langle 0 | J_{\mu\nu}(x) | \gamma^h(q) \rangle = \frac{1}{g} (q_\mu \varepsilon_\nu^* - q_\nu \varepsilon_\mu^*) e^{iq \cdot x}$$

This implies current linear in the NGB and shift symmetry

$$J_{\mu\nu} \sim \partial_{[\mu} A_{\nu]} + \mathcal{O}(A^3) \quad A_\mu \rightarrow A_\mu + \lambda_\mu + \mathcal{O}(A^2)$$

with  $\partial_{[\mu} \lambda_{\nu]} = 0$

E.g., free Maxwell theory

$$U(1)_e^{(1)} \times U(1)_m^{(1)} \rightarrow \emptyset \quad J_{\mu\nu}^e = \frac{1}{g^2} F_{\mu\nu} \quad J_{\mu\nu}^m = *F_{\mu\nu}$$

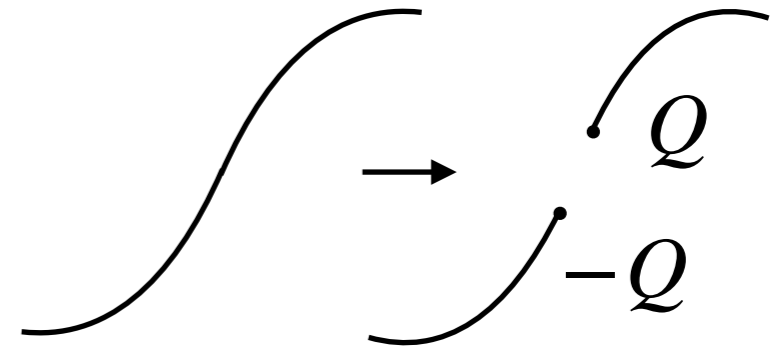
Photon is massless because it's a NGB

# Broken one-form symmetry

Electric one-form symmetry is broken by charged matter

$$\partial_\mu J_e^{\mu\nu} = j^\nu \neq 0$$

“screening”



This means it is a priori hard to interpret the soft photon theorem in terms of a 1-form symmetry.

However, 1-form symmetry is “robust”: No charged local operators can break it, so photon remains massless

The magnetic symmetry is unbroken, since it just follows from Bianchi

$$\partial^\mu J_{\mu\nu}^m = \partial^\mu *F_{\mu\nu} = 0$$

# Emergent one-form symmetry (boring)

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For energies  $E \ll m_Q$  we can integrate out charged matter

$$\mathcal{L} = -\frac{1}{4g^2}F^2 + \frac{c}{g^2m^4}F^4 + \dots \quad (\text{c.f., Euler-Heisenberg})$$

1-form symmetry emerges exactly  $J \sim \frac{1}{g^2}F + \frac{c}{g^2m^4}F^3 + \dots$

Only broken non-perturbatively  $\sim e^{-m|x|}$

It should be obvious that there is a photon soft theorem, but let's still go through the motions

# Photon Adler zero

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Considering current overlap with multi-particle state  $|\alpha\rangle$

$$\langle 0 | J^{\mu\nu}(q) | \alpha \rangle = \frac{1}{g} \frac{q^{[\mu} \varepsilon^{\nu]}}{q^2} \mathcal{A}(\alpha + \gamma) + \mathcal{N}^{\mu\nu}$$

Ward identity implies relation

$$0 = \langle 0 | \varepsilon_\nu^* \partial_\mu J^{\mu\nu}(q) | \alpha \rangle = \frac{i}{g} \mathcal{A}(\alpha + \gamma) + q_\mu \varepsilon_\nu^* \mathcal{N}^{\mu\nu}$$

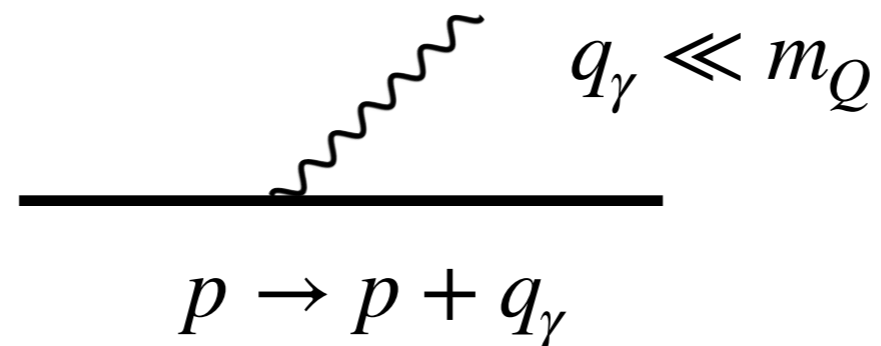
There is indeed an exact “Adler zero” following from WI

$$\lim_{q_\gamma \rightarrow 0} \mathcal{A}_{n+\gamma} = 0$$

A bit boring, because higher-form symmetries are Abelian, so there is no interesting double-soft theorem

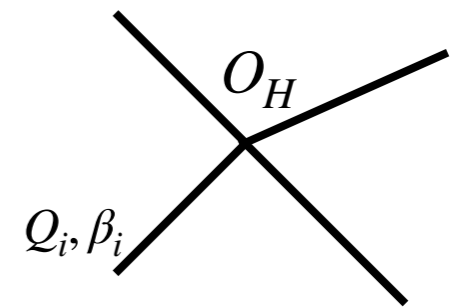
# Emergent one-form symmetry

If interested in scattering of matter by soft photons



In this limit amplitude is correlation function of Wilson lines

$$\mathcal{A} = \langle W_{Q_1}^{\beta_1} \dots W_{Q_n}^{\beta_n} O_H \rangle + \mathcal{O}((q/m)^\#)$$



along paths  $x_i^\mu(s) = \beta^\mu s$  with  $s \in [0, \infty]$

# Soft-photon theorem

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Consider the correlator  $\mathcal{C} = \langle \varepsilon^\mu \partial^\nu J_{\mu\nu} W_{Q_1}^{\beta_1} \dots W_{Q_n}^{\beta_n} O_H \rangle$

On the one hand creates a photon  $\langle \gamma(q) | W_{Q_1}^{\beta_1} \dots W_{Q_n}^{\beta_n} O_H \rangle + \varepsilon_\mu q_\nu \mathcal{R}^{\mu\nu}$

On the other use WI  $\sum_i Q_i \varepsilon^\mu \delta_\mu^{(D-1)}(x_{\Gamma_i}) \langle W_{Q_1}^{\beta_1} \dots W_{Q_n}^{\beta_n} O_H \rangle$

Contact term yields precisely soft factor

$$Q \text{ FT}[\delta_\nu^{(D-1)}(x_\Gamma)] = Q \int_0^\infty ds e^{-iq \cdot \beta s} \varepsilon \cdot \beta = Q \frac{\varepsilon \cdot \beta}{q \cdot \beta}$$

Combined gives soft theorem  $\lim_{q_\gamma \rightarrow 0} \mathcal{A}_{n+\gamma} = \sum_a Q_a \frac{\varepsilon_\gamma \cdot P_a}{q_\gamma \cdot P_a} \mathcal{A}_n$

# Heavy particle EFT (HEFT)

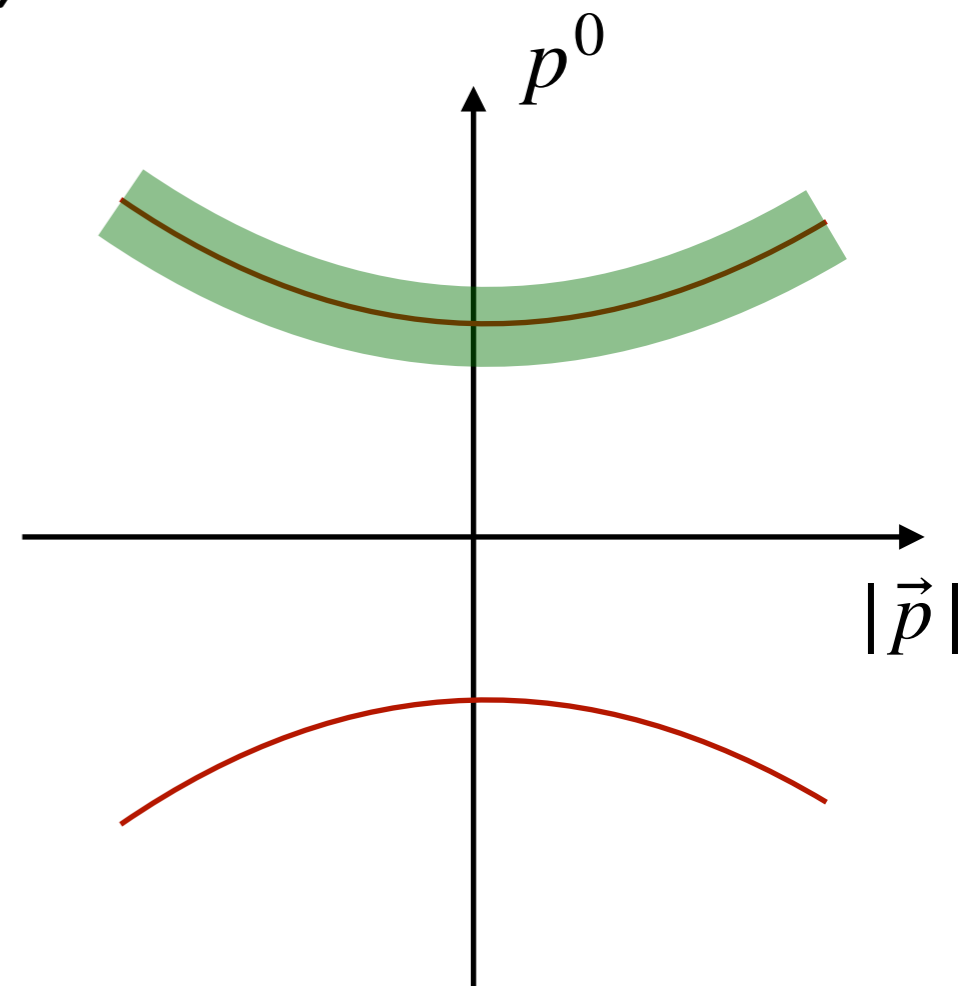
Split field into positive and negative frequency modes

$$\phi = e^{im\beta \cdot x} (P_+ \phi_\beta + P_- \phi_\beta^*)$$

Integrate out negative frequency modes, which have gap  $2m$

$$\mathcal{L} = \phi_\beta^* \beta \cdot D \phi_\beta + \phi_\beta^* \frac{D_\perp^2}{2m} \phi_\beta + \dots$$

This is a theory without antiparticles, no screening of Wilson lines!



# 1-form symmetry in HEFT

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BPS field redefinition  $\phi_\beta = \exp\left(iQ \int_0^\infty \beta^\mu A_\mu(s\beta)\right) \tilde{\phi}_\beta$

Makes claimed form of the amplitude manifest

Also makes effective action function of field strength

$$D_\mu \phi_\beta = W_Q \left( \partial_\mu + \frac{F_{\mu\nu} \beta^\nu}{\beta \cdot \partial} \right) \tilde{\phi}_\beta$$

Invariant under shift  $A \rightarrow A + \lambda$ , so 1-form symmetry is an exact symmetry of HEFT

So far no new soft theorem, only  
alternative perspective on old photon  
bringing it closer to pion

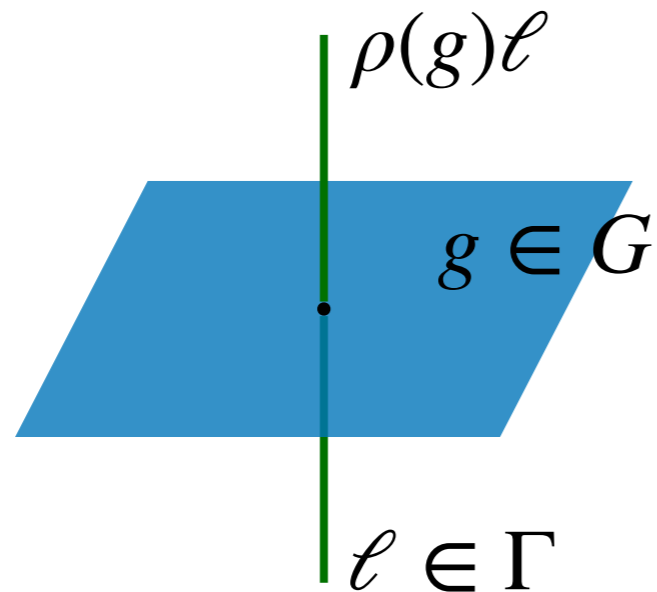
Is there some kind of symmetry that  
intertwines them?

# 2-group soft theorem

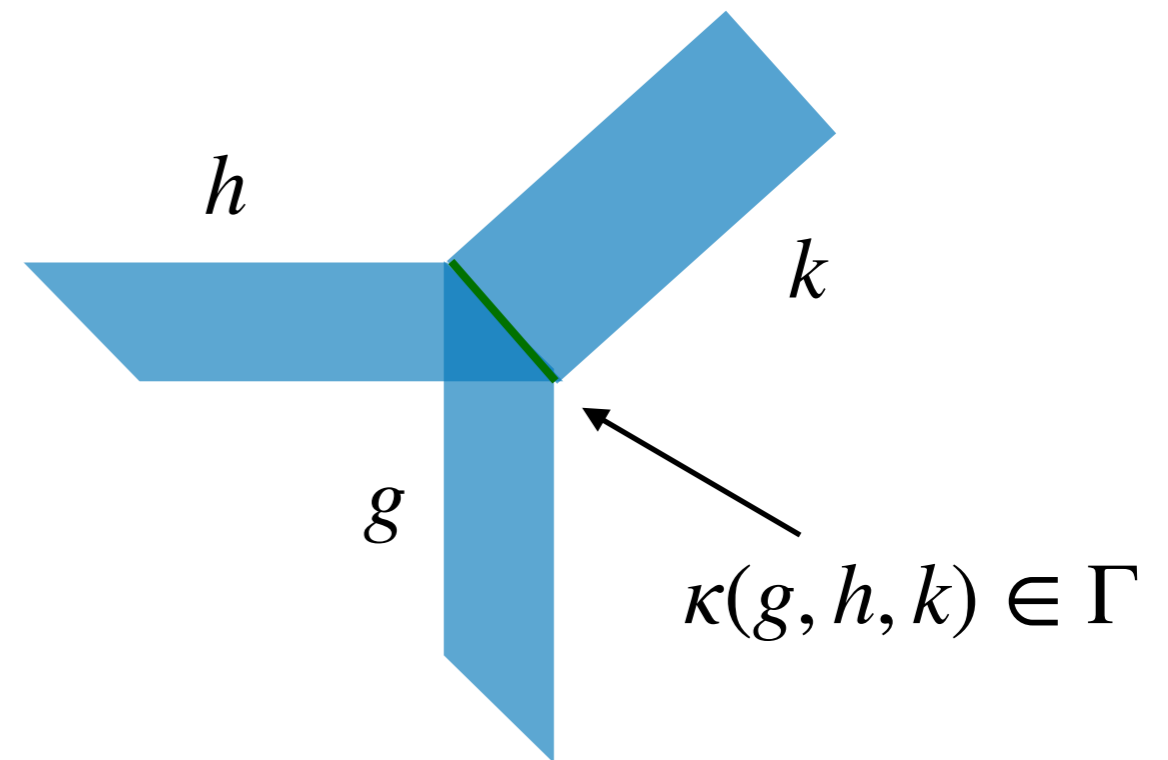
# 2-group symmetry

0-form,  $G$ , and 1-form,  $\Gamma$ , symmetries can be intertwined into a 2-group. Full description requires additional data:

- Action  $\rho : G \rightarrow \text{Aut}(\Gamma)$



- Class  $\kappa \in H^3(G, \Gamma)$



Roughly a non-trivial extension of 0-form by 1-form

# 2-group symmetry

For continuous 2-groups Ward identity modified:

$$\begin{aligned} \partial \cdot j^a(x) j^{b\mu}(y) &\sim i f^{abc} \delta^{(4)}(x-y) j^{c\mu}(x) + \frac{\kappa}{2\pi} \delta^{ab} \partial^\nu \delta^{(4)}(x-y) J_{\mu\nu} \\ &\sim f^{ab\gamma} \end{aligned}$$

We will denote this as

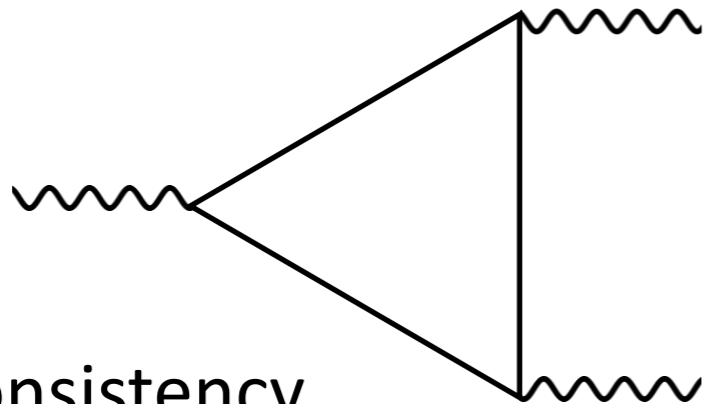
$$G \times_{\kappa} U(1)^{(1)}$$

Suggests there should be a soft theorem that mixes up photons and pions

# The fourth “triangle diagram”

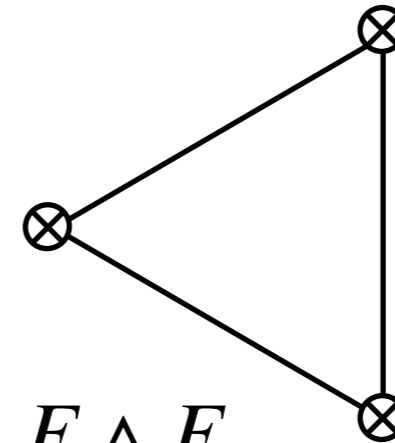
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Gauge anomaly



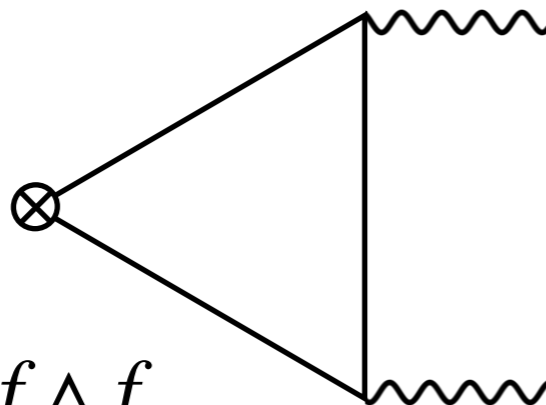
inconsistency

't Hooft anomaly



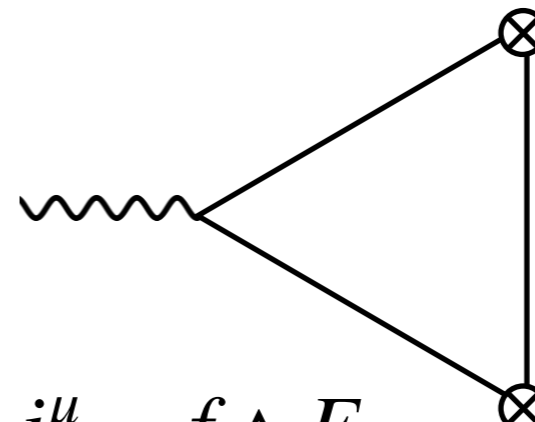
$$\partial_{\mu} j^{\mu} \sim F \wedge F$$

ABJ anomaly



$$\partial_{\mu} j^{\mu} \sim f \wedge f$$

2-group



$$\partial_{\mu} j^{\mu} \sim f \wedge F$$

# Aside: history of 2-group



First characterized in thesis by Hoàng Xuân Sính, student by correspondence of Grothendieck while he was at IHES and lectured near Hanoi when Vietnam war was raging on

# Example

Take QCD with gauged  $U(1)_V \sim$  Baryon number

$$\mathcal{L} = -f_\pi^2 \text{Tr}(\partial U \partial U^\dagger) - \frac{1}{4g^2} F^2 + A_\mu B^\mu + \dots$$

with 
$$B^\mu = i \frac{\kappa}{24\pi^2} \varepsilon^{\mu\nu\alpha\beta} \text{Tr} \left[ (iU^\dagger \partial_\nu U)(iU^\dagger \partial_\alpha U)(iU^\dagger \partial_\beta U) \right].$$

A mixed  $U(1)_V$ -Axial-Axial 't Hooft anomaly yields 2-group after gauging!

$$SU(N_f)_L \times SU(N_f)_R \times_\kappa U(1)_m^{(1)}$$

Magnetic 1-form symmetry: 
$$J_{\mu\nu} = \frac{1}{4\pi} \varepsilon_{\mu\nu\sigma\rho} F^{\sigma\rho}$$

# 2-group soft theorem

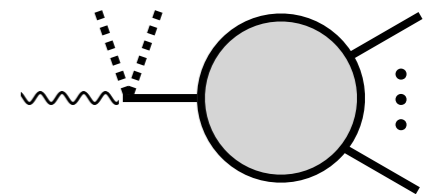
[Berean-Dutcher, Derda, JPM]

Double-soft pion theorem modified

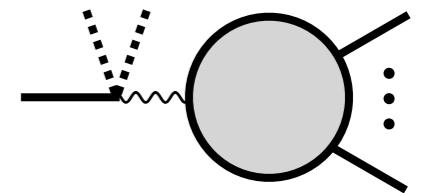
$$\lim_{q_{a,b}^\pi \rightarrow 0} \mathcal{A}_{(n_\pi+2, n_\gamma)} = (S^{(0)} + S^{(1)} + S_K^{(1)}) \mathcal{A}_{(n_\pi, n_\gamma)}$$

with

$$S_K^{(1)} \mathcal{A}_{(n_\pi, n_\gamma)} = \sum_{i \in \gamma} \frac{\varepsilon(q_a, q_b, p_i, \epsilon_i)}{p_i \cdot (q_a + q_b)} f^{abd} f^{dc\gamma} \mathcal{A}_{(n_\pi+1, n_\gamma-1)}^{\dots c \dots}$$



$$+ \sum_{i \in \pi} \sum_h \frac{\varepsilon(q_a, q_b, p_i, \epsilon_{(h)}^*)}{p_i \cdot (q_a + q_b)} f^{abd} f^{dc\gamma} \mathcal{A}_{(n_\pi-1, n_\gamma+1)}^{\dots \hat{c} \dots}$$



Where we suggestively wrote  $f^{ab\gamma} \sim \delta^{ab} \kappa$

# Comments

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- Proof follows by considering  $\langle 0 | j^{a\mu}(q_a) j^{b\nu}(q_b) | \alpha \rangle$  and using 2-group WI, which has new term.

$$\partial \cdot j^a(x) j^{b\mu}(y) \sim i f^{abc} \delta^{(4)}(x - y) j^{c\mu}(x) + \frac{\kappa}{2\pi} \delta^{ab} \partial^\nu \delta^{(4)}(x - y) J_{\mu\nu}$$

- Extra derivative on contact term consistent with sub-leading
- Not corrected by loops or other effects!
- We have checked it explicitly in all tree amplitudes up to 6-point.

# Example

[Berean-Dutcher, Derda, JPM]

2-photon-4-pion amplitude

$$\mathcal{A}_{(4,2)}^{a_1 a_2 a_3 a_4} = f^{a_1 a_2 c} f^{c a_3 a_4} K^2 \left[ \frac{\epsilon(p_1 p_2 p_5 e_5) \epsilon(P_{125} p_3 p_6 e_6)}{s_{12} + s_{15} + s_{25}} + \frac{\epsilon(p_1 p_2 p_6 e_6) \epsilon(P_{126} p_3 p_5 e_5)}{s_{12} + s_{16} + s_{26}} \right] \\ + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)$$

Taking  $p_1, p_2 \rightarrow 0$  we predict

$$S_K^{(1)} \mathcal{A}_{(2,2)} = -iK f^{a_1 a_2 c} \left[ \frac{\epsilon(q_1 q_2 p_5 e_5)}{s_{15} + s_{25}} \mathcal{A}_{(3,1)}^{c a_3 a_4}(p_5, p_3, p_4, p_6) + \frac{\epsilon(q_1 q_2 p_6 e_6)}{s_{16} + s_{26}} \mathcal{A}_{(3,1)}^{c a_3 a_4}(p_6, p_3, p_4, p_5) \right] \\ = K^2 f^{a_1 a_2 c} f^{c a_3 a_4} \left[ \frac{\epsilon(q_1 q_2 p_5 e_5)}{s_{15} + s_{25}} \epsilon(p_5 p_3 p_6 e_6) + \frac{\epsilon(q_1 q_2 p_6 e_6)}{s_{16} + s_{26}} \epsilon(p_6 p_3 p_5 e_5) \right]$$

# Summary and comments

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- New angle on soft theorems from 1-form symmetry, in direct analogy to pions
- Seems to also work for soft gluons & gravitons, from emergent 1-form symmetry at weak coupling
- Is there a connection to the story at  $\mathcal{I}$ ? Or to geometric perspective?
- Emergent higher symmetries in Lorentzian EFTs likely ubiquitous (e.g., HQET, SCET), connection to factorization?
- New soft theorem from 2-group structure. More to discover? Non-invertible?