

Collinear scattering in self-dual gauge theories

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Based on work in collaboration with



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Building off earlier work in collaboration with K. Costello
(See also Amplitudes 2024 review talk by K. Costello)

Context for today's talk

- Study families of quantum integrable 4d QFTs, which are not-necessarily supersymmetric
- They have nontrivial collinear scattering to arbitrary loop order $\mathcal{O}(\hbar^\#)$. No branch cuts, poles only in holomorphic collinear limits.
- You can obtain analytic expressions for their scattering! We did with 2d CFT techniques
- The theories: Self-dual YM coupled to special choices of matter (twistor space origin)
- SDYM captures certain helicity amplitudes/form factors in YM, so there exists a small subsector of massless QCD w/ certain matter that is exactly solvable
- Are these formulas useful for non-rational theories? (L. Dixon, A. Morales, K. Costello, NMP,...)

Plan of today's talk

1. Introduce the 4d theories
2. 2d chiral algebras govern scattering
3. Obtaining collinear scattering to all-loop order (w/ Fernandez)
4. Conclusions

Part 1: Solvable 4d QFTs

$$B \in \Omega_-^2(\mathfrak{g})$$

$$B_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$$

$$F_{\mu\nu,-} = \frac{1}{2}(F_{\mu\nu} - *F_{\mu\nu})$$

$$S = \int \text{tr}(B \wedge F(A)_-)$$

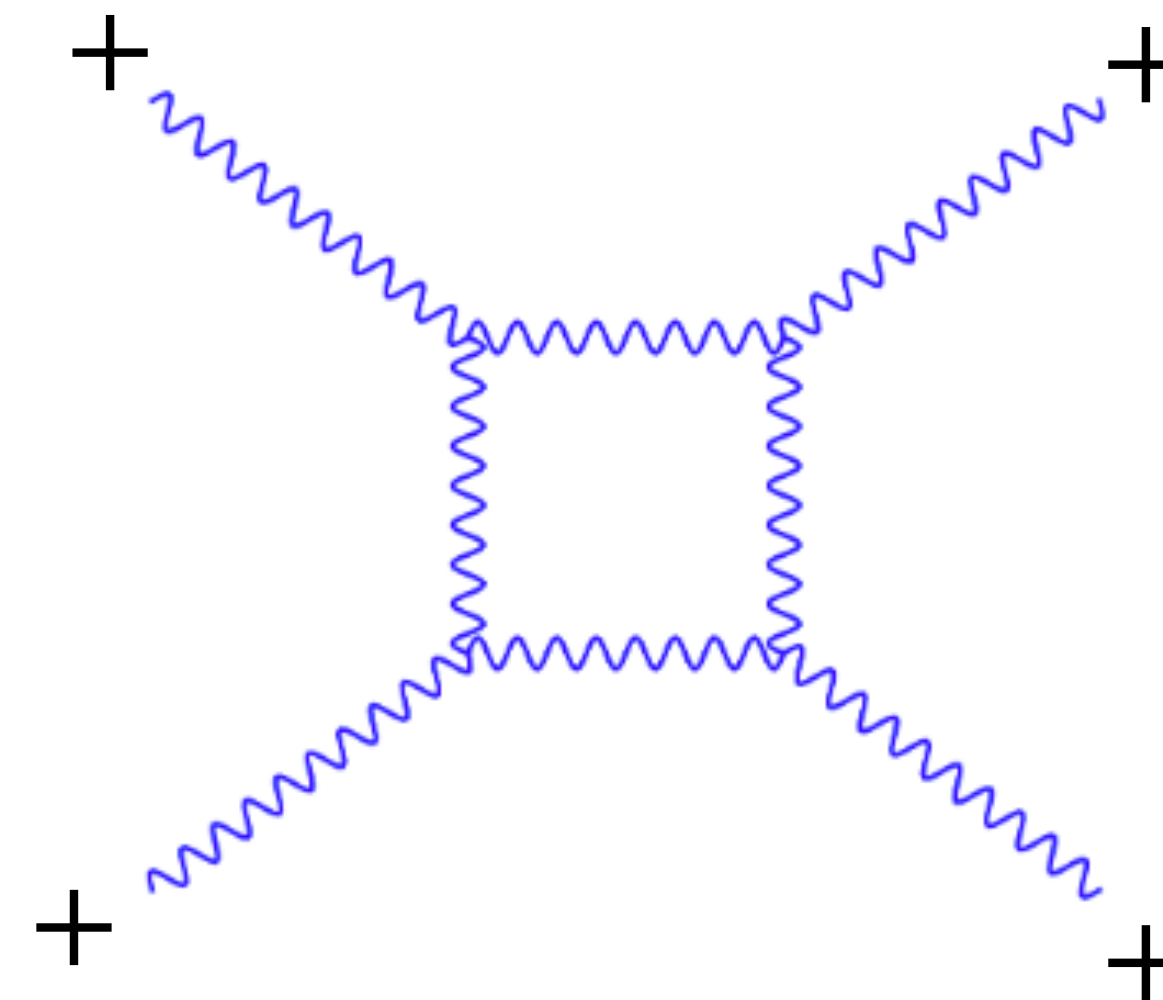
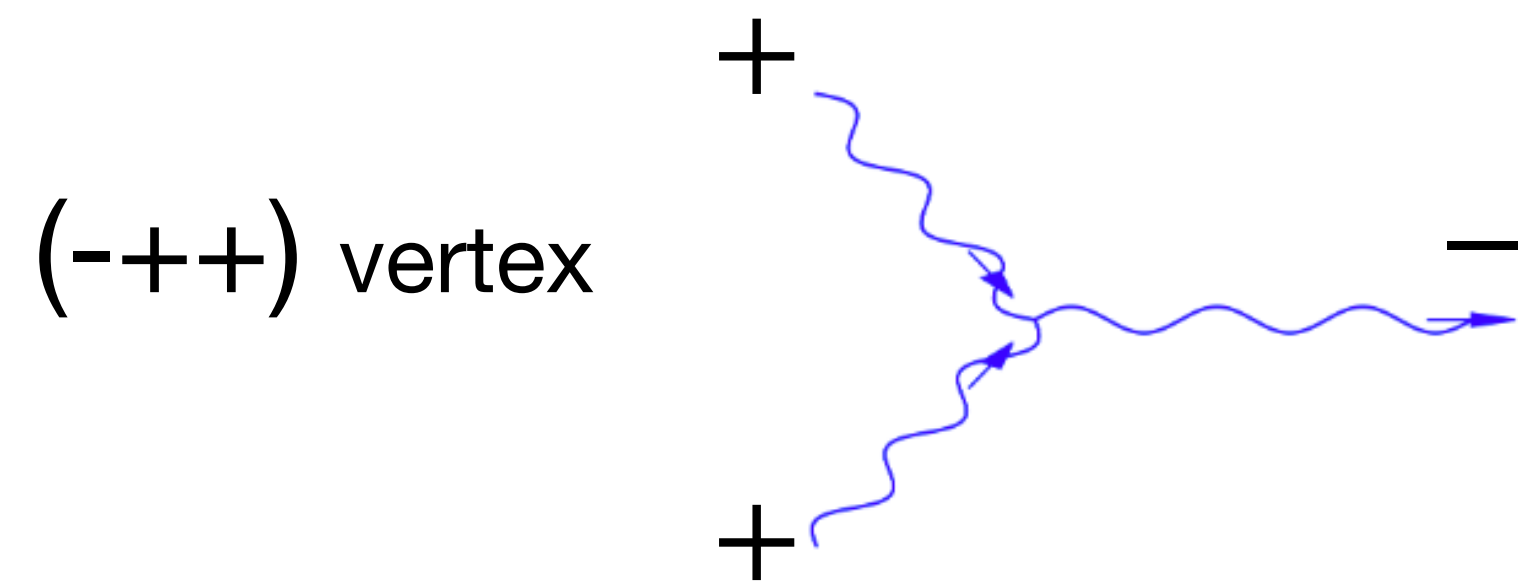
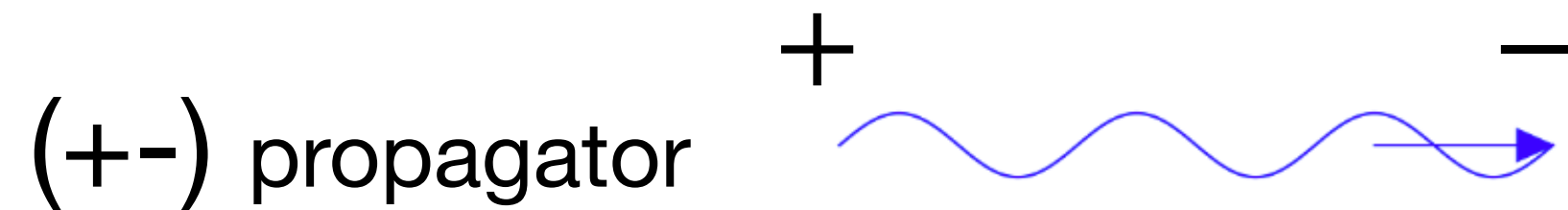
Self-dual Yang-Mills review

[Chalmers-Siegel,...]

B: negative helicity gluons

A: positive helicity gluons

one-loop exact scattering



[Bern-Dixon-Kosower] collinear splitting

$$+\frac{g_{YM}^2}{2} \int \text{tr}(B^2)$$

add term to action: perturbative YM in first order formalism

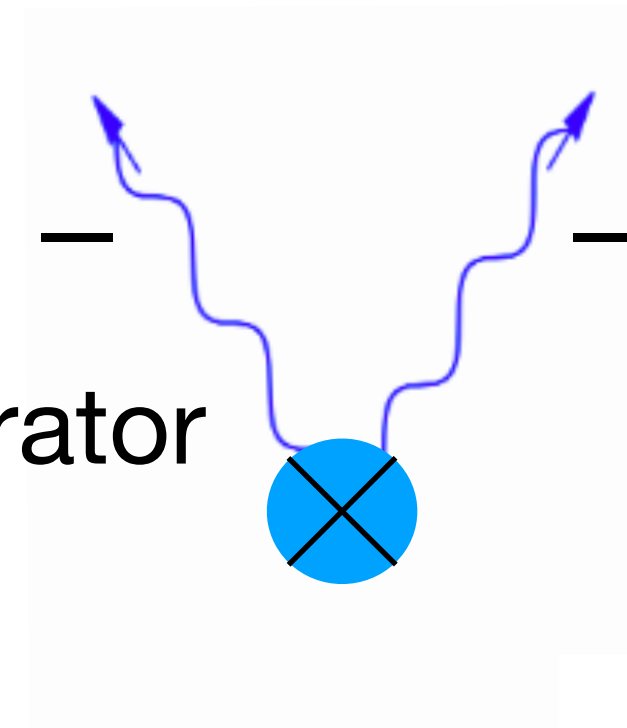
$$e^{S_{BF}} \rightarrow e^{S_{BF} + \frac{1}{2} g_{YM}^2 \int \text{tr}(B \wedge B)}$$

$$\simeq e^{S_{BF}} \left(1 + \frac{1}{2} g_{YM}^2 \int \text{tr} B^2 \right)$$

Form factors in SDYM

Perturbative expansion around self-dual sector

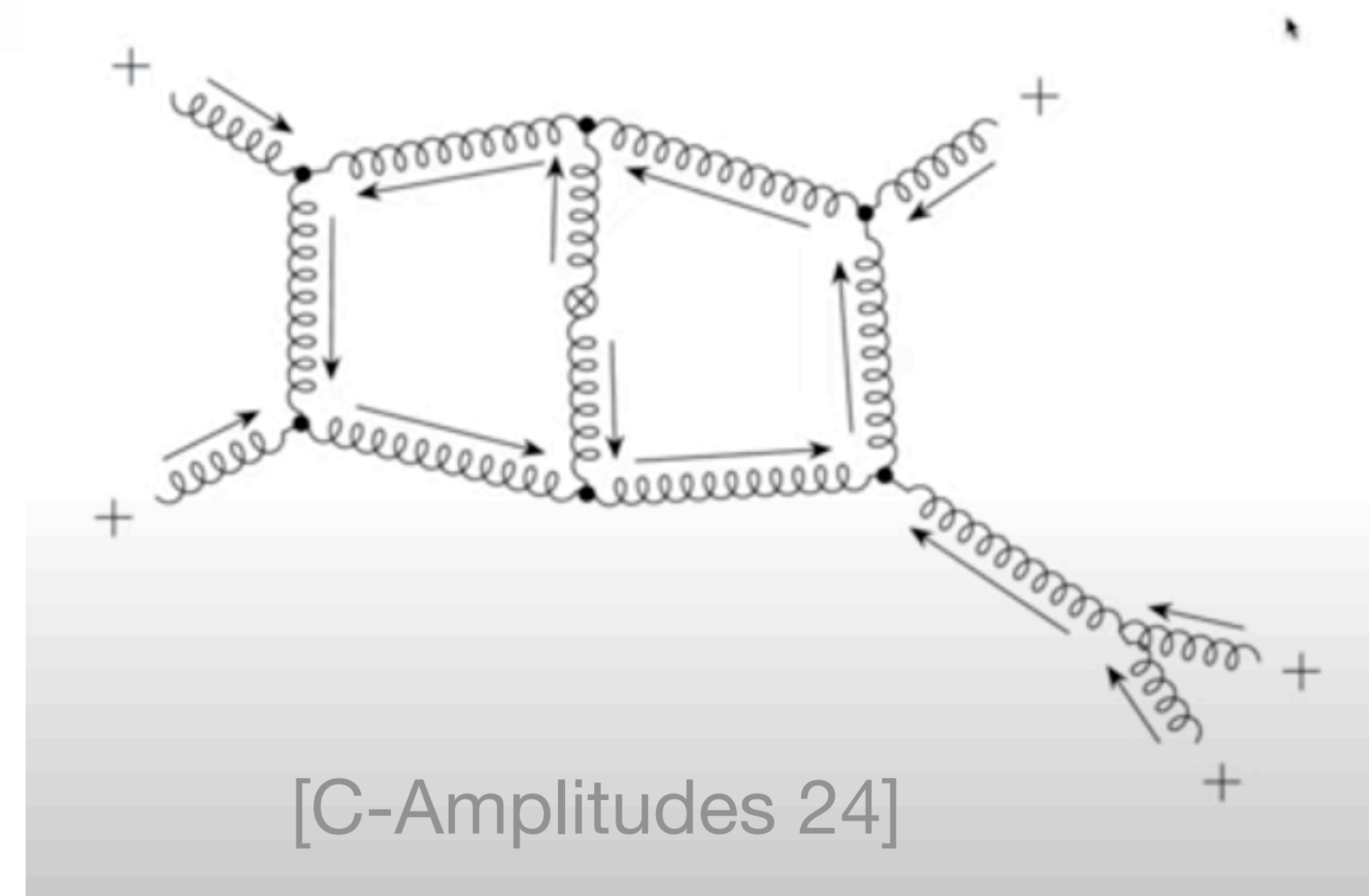
(- -) vertex in form factor



Form factor: add (copies of) $\text{tr}(B^2)(x_0)$ as a local operator

For $q = 0$, $\int dx \text{tr}(B^2)(x) \dots$

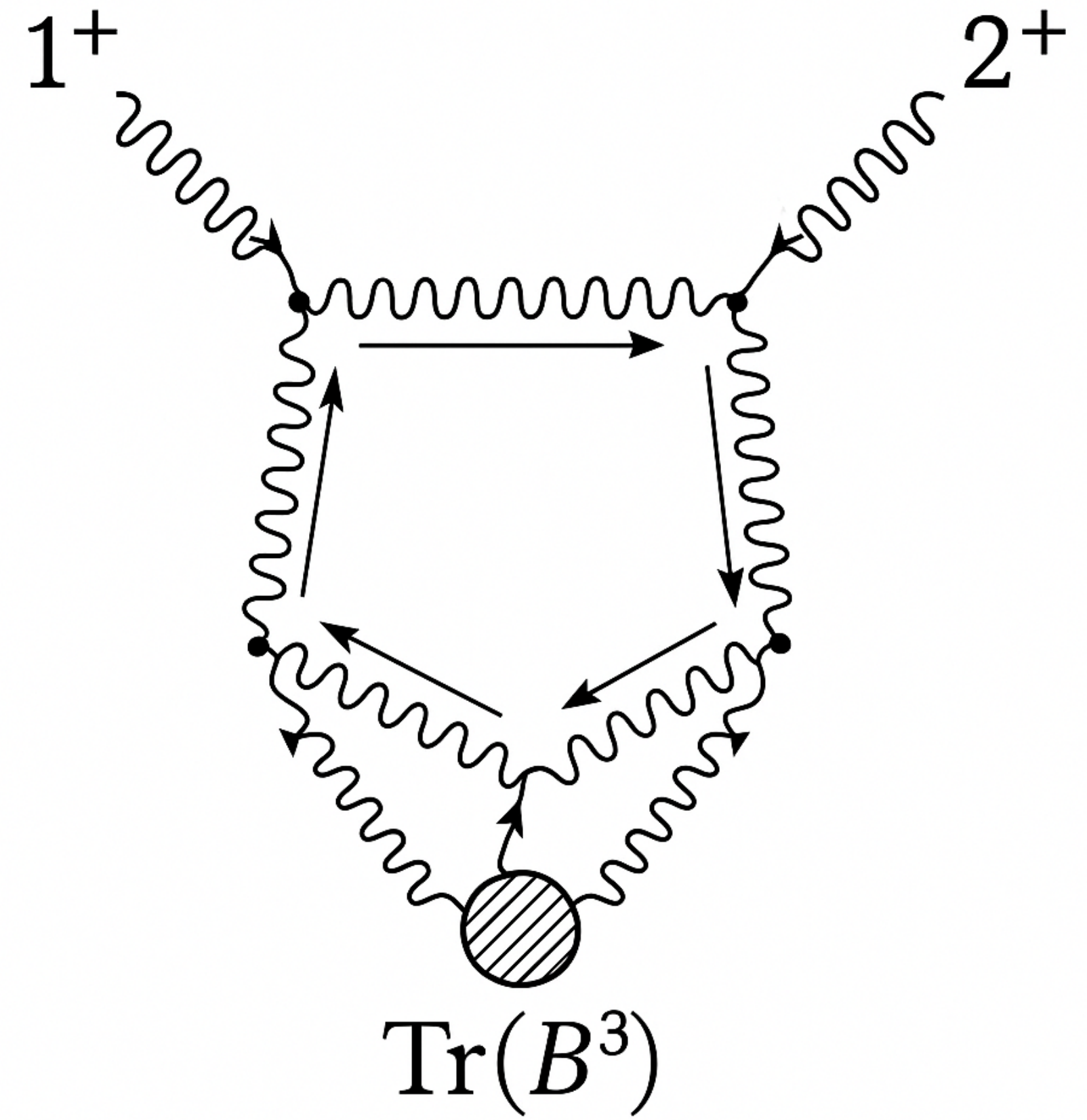
- With just one $\text{Tr}(B^2)$:
- tree-level MHV
 - 1-loop 1-minus
 - 2-loop all-plus



Higher loop form factors for YM theory

- $tr(B^n)$ form factors in SDYM compute $tr(F_-^n)$ YM form factors
- e.g. ASD part of 3-gluon 1-Higgs effective interaction (int. out top quark) in this th

$$\mathcal{L}_{int} \supset \phi f_{abc} F_-^a F_-^b F_-^c$$



Non-factorizing contributions to collinear limits

$$J^a[\tilde{\lambda}_1](z_1) J^b[\tilde{\lambda}_2](z_2) |_{3-loop} \supset \frac{[12]^3}{\langle 12 \rangle^2} b_{ab}^{(3),d_1 d_2 d_3} : \tilde{J}_{d_1} \tilde{J}_{d_2} \tilde{J}_{d_3} : (z_2)$$

The expression for b is complicated but now known

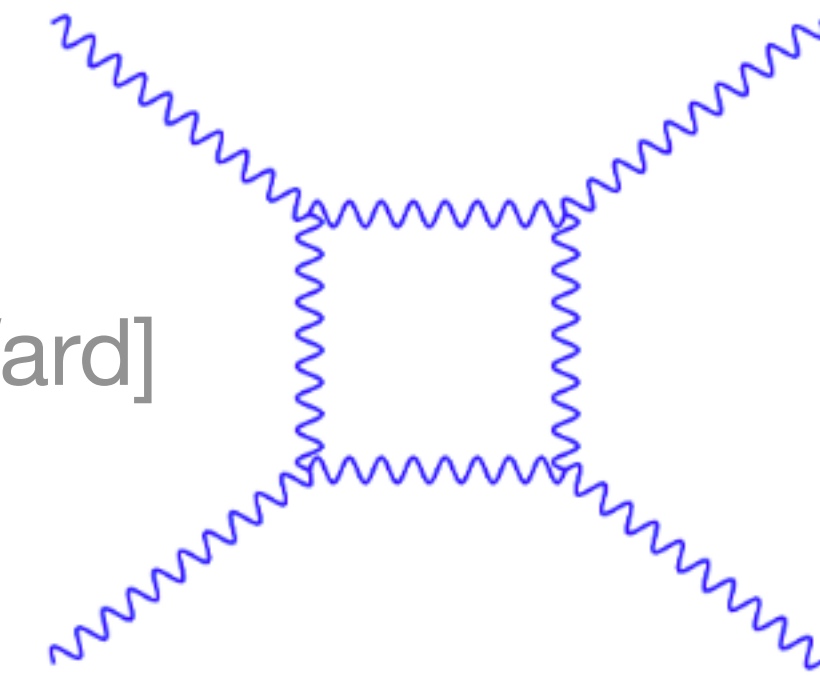
Self-dual Yang-Mills is classically integrable

Many ways to see this, including tower of conserved currents & twistor space uplift

[Bardeen '96]:

Quantum non-conservation of integrability currents from 1-loop amplitude

[Ward]



[Costello]: This diagram is a gauge anomaly diagram in six dimensions, of the twistor space theory

Obstruction to (quantum) integrability \leftrightarrow Non-existence of (quantum) twistorial uplift

$$\mathrm{Tr}_{\mathrm{adj}}(X^4) - \mathrm{Tr}_R(X^4) = \lambda_{\mathfrak{g},R}^2 \mathrm{Tr}_{\mathrm{fun}}(X^2)^2$$

$X \in \mathfrak{g}$

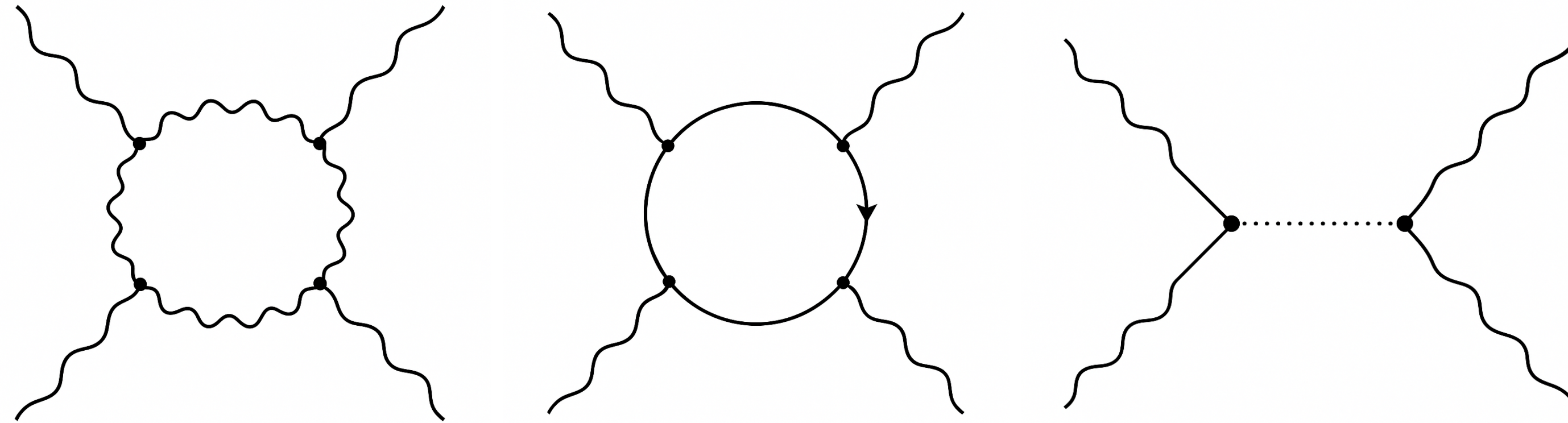
[Okubo]

Add judicious massless matter to cancel 6d anomaly and restore integrability

Quantum mechanically, a 6d gauge anomaly furnishes an obstruction to 4d integrability

[Costello-Li, Costello]

$$X \in \mathfrak{g} \quad \text{Tr}_{\text{adj}}(X^4) - \text{Tr}_R(X^4) = \lambda_{\mathfrak{g},R}^2 \text{Tr}_{\text{fun}}(X^2)^2$$



Anomaly may be cancelled by:

R. Bittleston found many interesting families of examples

1. Supersymmetrization
2. Green-Schwarz mechanism (for certain \mathfrak{g})
3. Addition of certain real fermionic matter (for certain \mathfrak{g}, R)
4. Suitable combination of points 2, 3

Today's twistorial QFTs: self-dual YM theories coupled to various matter sectors

- $SU(2)$ SDYM w/ $N_f = 8$, $SU(3)$ SDYM w/ $N_f = 9$
- $SU(N)$ SDYM w/ $8F \oplus 8\bar{F} \oplus \wedge^2 F \oplus \wedge^2 \bar{F}$
- $SU(2)$, $SU(3)$, $SO(8)$, $E_{6,7,8}$ SDYM w/ fourth-order “axion” ρ
- $SU(N)$ SDYM w/ $N_f = N$ and ρ
- $SO(N)$ SDYM w/ $N_f = N - 8$ and ρ
- etc.

see also: [Bittleston-Costello-Zeng]

Part 2: 2d chiral CFTs control form factors

- We have guaranteed infinite number of conserved currents at quantum level
 - Adding matter deforms their OPEs (order by order in \hbar)
- Complexified momenta $p_{\alpha\dot{\alpha}} = \lambda_{\alpha}\tilde{\lambda}_{\dot{\alpha}}$, holomorphic collinear limits $\langle ij \rangle \rightarrow 0$
 - $\lambda_{\alpha} = (1, z), \tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2)$ $z_{ij} \equiv z_i - z_j = \langle ij \rangle$

z is a 2d hol'c coordinate, 2d chiral algebra structure

[Costello-NMP]: 2d chiral algebra exists and is associative if twistorial uplift exists

Thm: Form factors in these theories = chiral algebra correlation functions

Rational, no branch cuts, poles fixed by collinear limits

OPE: L-loop form factor recursively determined by lower-loop/lower-point FFs

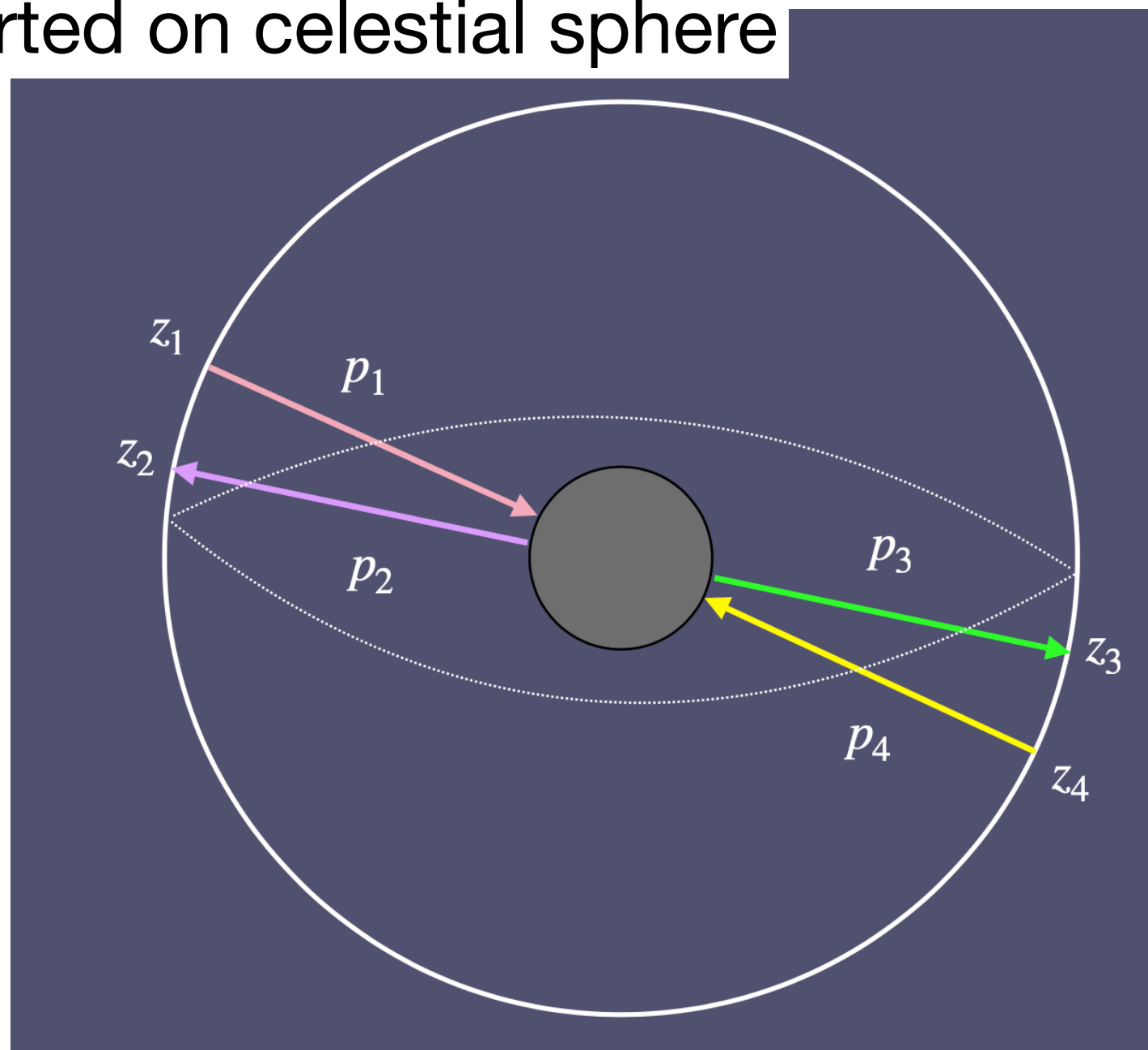
Algebraic: induction readily gives n-pt answer

2d chiral algebra	4d theory
conf. primary operators	massless on-shell states
OPEs	holomorphic collinear limits
conformal blocks (cf. CS/WZW)	local operators
correlation functions	form factors

Useful to think of chiral algebra as supported on celestial sphere

cf. [Guevara-Himwich-Pate-Strominger]

[Strominger]



Chiral algebra

+
-

Generator	Field	Scaling Dimension	Spin	Combined Dilatation	Weight
$J[t_1, t_2], t_i \geq 0$	\mathcal{A}	$-(t_1 + t_2)$	$1 - \frac{t_1+t_2}{2}$	1	0
$\tilde{J}[t_1, t_2], t_i \geq 0$	\mathcal{B}	$-(t_1 + t_2 + 2)$	$-1 - \frac{t_1+t_2}{2}$	0	1
$E[t_1, t_2], t_1 + t_2 \geq 1$	η	$-(t_1 + t_2)$	$-\frac{t_1+t_2}{2}$	0	1/2
$F[t_1, t_2], t_i \geq 0$	η	$-(t_1 + t_2 + 2)$	$-\frac{t_1+t_2}{2}$	1	1/2
$M_i[t_1, t_2], t_i \geq 0$	ψ_i	$-(t_1 + t_2 + 1/2)$	$\frac{1-t_1-t_2}{2}$	3/4	1/2
$\tilde{M}_j[t_1, t_2], t_i \geq 0$	ψ^i	$-(t_1 + t_2 + 3/2)$	$-\frac{1+t_1+t_2}{2}$	1/4	1/2

symmetries under $\mathfrak{sl}(2)_+ \times \mathfrak{sl}(2)_-$, rescalings of 6d twistor theory...

$$J_a[\tilde{\lambda}](z) = \sum_{n, m \in \mathbb{Z}_{\geq 0}} \frac{1}{n!m!} \tilde{\lambda}_1^n \tilde{\lambda}_2^m J_a[n, m](z)$$

momentum eigenstates

“conformally soft” modes

Form factors for these theories have been computed

- [C-NMP x2]:
 - tree-level MHV
 - 1-loop 1-minus (n pt!)
 - [Costello]:
 - 2-loop all-plus (n pt!)
- } chiral algebra

1-loop for SDYM + non-axion matter

$$\begin{aligned}
 & \left\langle \frac{1}{2} \text{tr}(B^2) \middle| \tilde{J}^{a_1}(z_1) J_{a_2}[\mu](z_2) \dots J_{a_n}[\mu](z_n) \right\rangle \\
 &= -\frac{1}{n96\pi^2} \sum_{\sigma \in S_n} \sum_{2 \leq i < j \leq n} \frac{[ij] \langle 1i \rangle^2 \langle 1j \rangle^2}{\langle ij \rangle \langle \sigma_1 \sigma_2 \rangle \langle \sigma_2 \sigma_3 \rangle \dots \langle \sigma_n \sigma_1 \rangle} \text{Tr}_{\mathfrak{g} \oplus \Pi R} (\mathbf{t}_{a_{\sigma_n}} \dots \mathbf{t}_{a_{\sigma_1}})
 \end{aligned}$$

- [Dixon-Morales]:
 - Verify 2-loop 4-pt amplitude
- } gen. unitarity
mass regulator

[Costello-NMP]:

If the 6d theory has a gauge/BRST anomaly, this chiral algebra fails to associate at one-loop

On the other hand, given a non-anomalous 6d theory, we *used* associativity to find some one-loop deformations of the OPE

(Verified in [Fernandez])

$$J_a[1,0](z_1)J_b[0,1](z_2) \sim \frac{1}{z_{12}^2} f_{ab}^c \tilde{J}_c[0,0](z_1) - \frac{1}{z_{12}} f_{ab}^c \partial_z \tilde{J}_c[0,0](z_1) \quad (++) \text{ one-loop splitting}$$
$$- CK^{fe} \frac{1}{z_{12}} (f_{ae}^c f_{bf}^d + f_{ae}^d f_{bf}^c) : J_c[0,0] \tilde{J}_d[0,0] : (z_1)$$

Conjecture: One may use knowledge of the 6d/2d coupling on twistor space + associativity to determine the chiral algebra to arbitrary order

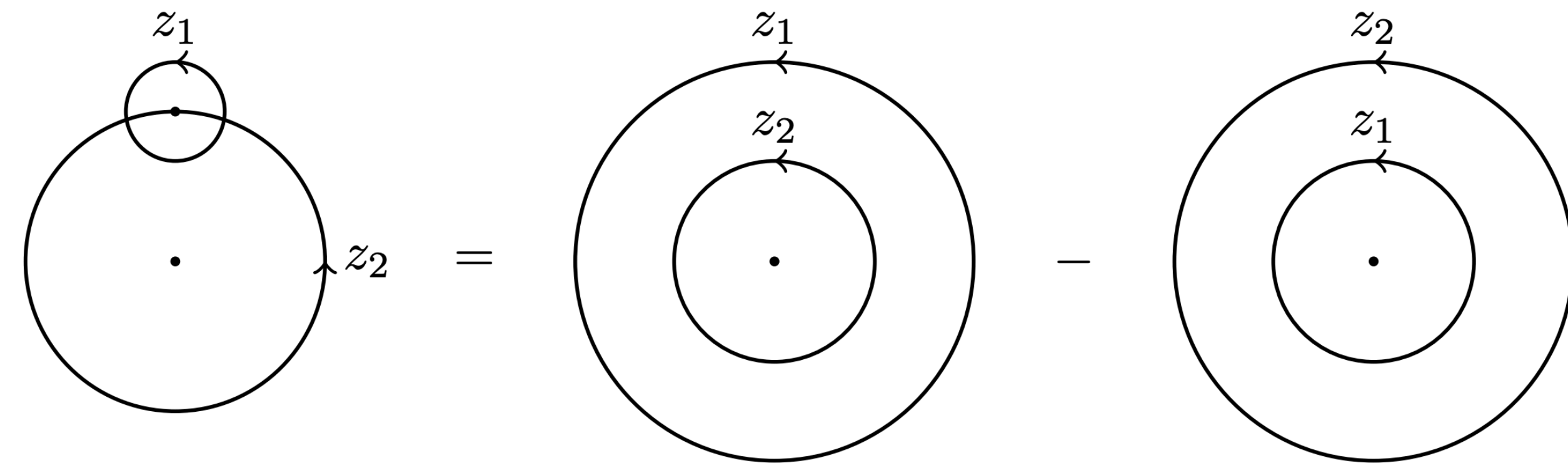
Part 4: Results at arbitrary loop order

We now know the chiral algebra (collinear splitting fns) to arbitrary loop order using constraints from symmetry and associativity [Fernandez-NMP]

To fix the OPE corrections

- Step 1: Determine the general form of the OPE corrections (6d/chiral algebra couplings, symmetries)
- Step 2: Determine conditions on the undetermined numerical coefficients from associativity
- Step 3: Solve those equations to fix the coefficients

$$\oint_{|z_2|=2} dz_2 z_2^l \left(\oint_{|z_{12}|=1} \phi_1(z_1) \phi_2(z_2) \right) \phi_3(0) = \oint_{|z_1|=2} dz_1 \phi_1(z_1) \left(\oint_{|z_2|=1} dz_2 z_2^l \phi_2(z_2) \phi_3(0) \right) - (-1)^{F_1 F_2} \oint_{|z_2|=2} dz_2 z_2^l \phi_2(z_2) \left(\oint_{|z_1|=1} dz_1 \phi_1(z_1) \phi_3(0) \right)$$



$$\phi_i(z) \phi_j(w) \sim \sum_{n \geq 1} \frac{\{\phi_i \phi_j\}_n(w)}{(z-w)^n}$$

$$\{ \{ \phi_1 \phi_2 \}_1 \phi_3 \}_{l+1} = \{ \phi_1 \{ \phi_2 \phi_3 \}_{l+1} \}_1 - (-1)^{F_1 F_2} \{ \phi_2 \{ \phi_1 \phi_3 \}_1 \}_{l+1}$$

To determine which diagrams contribute to a given OPE, we use:

- Interactions in the twistorial Lagrangian are only cubic in form

e.g. **SDYM + axion-like theory**

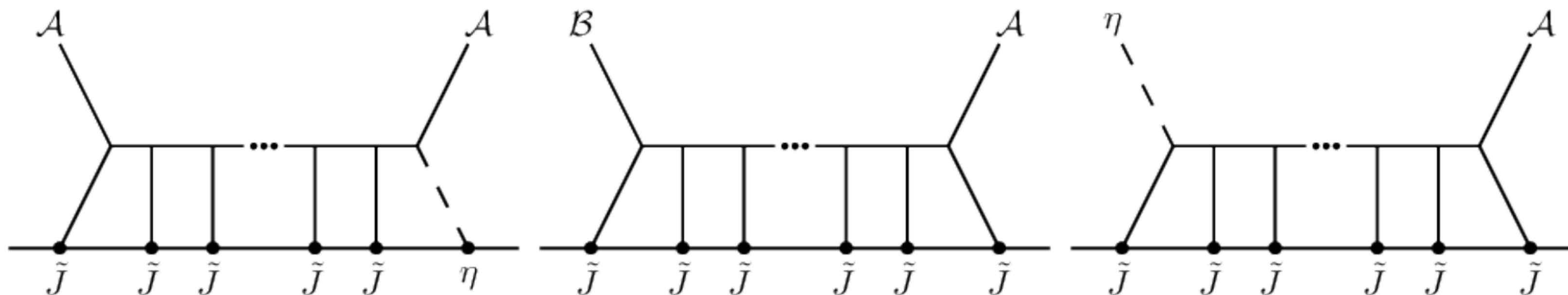
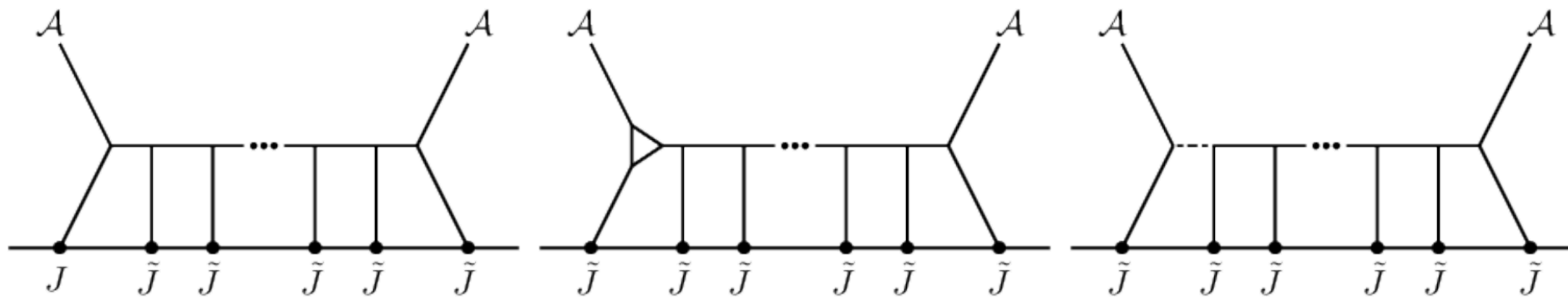
- Only two type of vertices: $\mathcal{B}\mathcal{A}^2$ and $\eta\mathcal{A}^2$

$$\hbar \rightarrow \alpha\hbar$$

$$\mathcal{B} \rightarrow \alpha\mathcal{B}$$

$$\eta \rightarrow \sqrt{\alpha}\eta$$

- Invariance under rescaling of \hbar



Here are some of the terms expressed using unknown coefficients.

$$\tilde{J}_a[t](z)J_b[r](0) \sim \frac{1}{z} \sum_{m \geq 1}^{\sum_{j=1}^{m+1} k_j = t+r-m} \sum_{k_j^i \geq 0} \hbar^m \underset{(t,r)}{f}^{(m)} [k_1, \dots, k_{m+1}]_{ab}^{i_1 \dots i_{m+1}} : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

$$E[t](z)J_b[r](0) \sim \frac{1}{z} \sum_{m \geq 1}^{\sum_{j=1}^{m+1} k_j = t+r-m-1} \sum_{k_j^i \geq 0} \hat{\lambda}_g \hbar^{m+\frac{1}{2}} \underset{(t,r)}{j}^{(m)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

$$F[t](z)J_b[r](0) \sim \sum_{m \geq 1}^{\sum_{j=1}^{m+1} k_j = t+r-m} \sum_{k_j^i \geq 0} \hat{\lambda}_g \hbar^{m+\frac{1}{2}} \left(\frac{1}{z^2} \underset{(t,r)}{k}^{(m)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} + \frac{1}{z} \underset{(t,r)}{l}^{(m)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} \hat{\partial}_1 \right) : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

Here are some of the terms expressed using unknown coefficients.

Pole order: fixed by matching combined dilatation symmetry

$$\frac{1}{z} \text{ and } \partial_z \text{ have combined dilatation} = 1$$

J, \tilde{J}, E, F have combined dilatation = 1, 0, 0, 1 respectively

$$\tilde{J}_a[t](z)J_b[r](0) \sim \frac{1}{z} \sum_{m \geq 1} \sum_{\substack{\sum_{j=1}^{m+1} k_j = t+r-m \\ k_j^i \geq 0}} \hbar^m f_{(t,r)}^{(m)} [k_1, \dots, k_{m+1}]_{ab}^{i_1 \dots i_{m+1}} : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

$$E[t](z)J_b[r](0) \sim \frac{1}{z} \sum_{m \geq 1} \sum_{\substack{\sum_{j=1}^{m+1} k_j = t+r-m-1 \\ k_j^i \geq 0}} \hat{\lambda}_g \hbar^{m+\frac{1}{2}} j_{(t,r)}^{(m)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

$$F[t](z)J_b[r](0) \sim \sum_{m \geq 1} \sum_{\substack{\sum_{j=1}^{m+1} k_j = t+r-m \\ k_j^i \geq 0}} \hat{\lambda}_g \hbar^{m+\frac{1}{2}} \left(\frac{1}{z^2} k_{(t,r)}^{(m)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} \right.$$

$$\left. + \frac{1}{z} l_{(t,r)}^{(m)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} \hat{\partial}_1 \right) : \prod_{j=1}^{m+1} \tilde{J}_{i_j}[k_j] :$$

- Use associativity to fix OPE coefficients in terms of the $f^{(m)}$ coefficients.
- We obtained a recursion relation for $f^{(m)}$ at arbitrary m .
- We used the recursion relation to find a closed-form expression for $f^{(1)}$, and a recursive expression for $f^{(m)}$ with $m > 1$.

$$K_{ab}^{i_1 \dots i_{m+1}} = -f_{aj_1}^{i_1} K^{j_1 j_2} f_{j_2 j_3}^{i_2} \dots f_{j_{2m-2} j_{2m-1}}^{i_m} K^{j_{2m-1} j_{2m}} f_{j_{2m} b}^{i_{m+1}}$$

$$\alpha(t, k) = t^2(k^1 + 1) - t^1(k^2 + 1) \quad \beta(t) = t^1 + t^2$$

$$\binom{(m)}{j}_{(t,r)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} = - \left(\frac{\alpha(t, k_1)}{\beta(t)} \right) \binom{(m)}{f}_{(t-1,r)} (k_1, \dots, k_{m+1}) K^{i_1 j} K_{j b}^{i_2 \dots i_{m+1}}$$

$$\binom{(m)}{k}_{(t,r)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} = - \left(\frac{\beta(k_1 + 1)}{\beta(t + 1)} \right) \binom{(m)}{f}_{(t,r)} (k_1, \dots, k_{m+1}) K^{i_1 j} K_{j b}^{i_2 \dots i_{m+1}}$$

$$\binom{(m)}{l}_{(t,r)} [k_1, \dots, k_{m+1}]_b^{i_1 \dots i_{m+1}} = - \binom{(m)}{f}_{(t,r)} (k_1, \dots, k_{m+1}) K^{i_1 j} K_{j b}^{i_2 \dots i_{m+1}}.$$

Some terms with single poles were also determined more formally by homotopy transfer methods [Zeng]

Recursive Expression for $f^{(m>1)}$

$$\begin{aligned}
 f_{(r^1, r^2)(t^1, t^2)}^{(m)} [k_1; \dots; k_{m+1}] = & - \sum_{j=1}^{t^1} f_{(r^1, r^2)(t^1-j, t^2)}^{(m-1)} [k_1; \dots; k_{m-1}; l] f_{(l^1, l^2)(1,0)}^{(1)} [k_m; (k_{m+1}^1 + 1 - j, k_{m+1}^2)] \\
 & + \sum_{j=1}^{t^1} f_{(r^1, r^2)(l^1, l^2)}^{(m-1)} [k_1; \dots; k_m] f_{(t^1-j, t^2)(1,0)}^{(1)} [l; (k_{m+1}^1 + 1 - j, k_{m+1}^2)] \\
 & - \sum_{j=1}^{t^2} f_{(r^1, r^2)(0, t^2-j)}^{(m-1)} [k_1; \dots; k_{m-1}; l] f_{(l^1, l^2)(0,1)}^{(1)} [k_m; (k_{m+1}^1 - t^1, k_{m+1}^2 + 1 - j)].
 \end{aligned}$$

Also a nice closed form solution [Zeng] in terms of Clebsch-Gordon coefficients, Wigner 6j symbols

$f^{(1)}$ two ways

$$\sum_{ac} = \left(\sum_{a=0}^{\min(m, l_1)} \sum_{c=1}^{\min(n, l_2+1)} \binom{l_1}{a} \binom{l_2}{c-1} - \sum_{a=1}^{\min(m, l_1+1)} \sum_{c=0}^{\min(n, l_2)} \binom{l_1}{a-1} \binom{l_2}{c} \right) \binom{m}{a} \binom{n}{c}$$

holomorphic integrals in twistor space
[Fernandez]

$$\mathcal{M}_1 = \sum_{ac} \frac{(m+n-a-c)!(a+c-1)!a!c!(r+m-a)!(s+n-c)!(1+k_1+k_2)!(1+l_1+l_2-a-c)!}{(m+n)!(1+m+r+n+s-a-c)!k_1!k_2!l_1!l_2!}$$

$$m(p, q; x, y; u) = \sum_{j=1}^{\text{Min}[u, x+1]} \frac{(p-j)!(1+x+y-j)!}{(1+p+q-j)!(1+x-j)!}$$

associativity [Fernandez-NMP]

$$\begin{aligned} f_{(r,t)}^{(1)}(k, l) &= \left(\frac{1}{16\pi^2} \right) \frac{(r^2+t^2)!(1+k^1+k^2)!}{k^1!k^2!l^2!} m(r^1+t^1, r^2+t^2, l^1, l^2; t^1) \\ &- \left(\frac{1}{16\pi^2} \right) \frac{t^2!(1+k^1+k^2-r^1-r^2)!}{(k^1-r^1)!(k^2-r^2)!l^2!} m(t^1, t^2, l^1, l^2; t^1) \theta(k^1-r^1) \theta(k^2-r^2) \\ &- \left(\frac{1}{16\pi^2} \right) \frac{r^1!(1+k^1+k^2)!}{k^1!k^2!(l^1-t^1)!} m(r^2+t^2, r^1, l^2, l^1-t^1; t^2) \theta(l^1-t^1). \end{aligned}$$

It turns out, these two expressions
are indeed equal!

Part 6: What's Next?

Summary

- Twistorial QFTs are a class of integrable, not-necessarily-SUSY 4d theories characterized by a local uplift to twistor space.
- Their basic observables are form factors. Form factors with a single local operator insertion are controlled, and computable, by a 2d chiral algebra
- We have shown that the chiral algebra OPEs can be fixed by symmetries and associativity. This effectively solves a small subsector of massless QCD with certain matter content.
- We have also extended the “chiral algebra bootstrap” to scattering in the presence of defects sourcing self-dual field configurations (tree-level MHV scattering in self-dual dyon). w/ N. Garner

Future directions

- Form factors with multiple local operator insertions. WIP w/ Costello & Morales
- Twistorial theories at large- N : single-trace formulas independent of matter?
- Other applications to (rational terms of) massless QCD amplitudes?

[Dixon-Morales x2]: Used twistorial anomaly condition to find relations among 1-loop n -gluon QCD subamplitudes

[Morales, to appear]: 2-loop $SU(2)$ & $SU(3)$ QCD amplitudes w/ arbitrary N_F via one-loop + tree + chiral algebra correlators

Thank you!