Unexpected Symmetries in Kerr Black Hole Scattering

Mao Zeng, University of Edinburgh

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In collaboration with Dogan Akpinar, Graham Brown, Fernando Febres Cordero, Riccardo Gonzo, Manfred Kraus, Guanda Lin, Michael Ruf, Alexander Smirnov

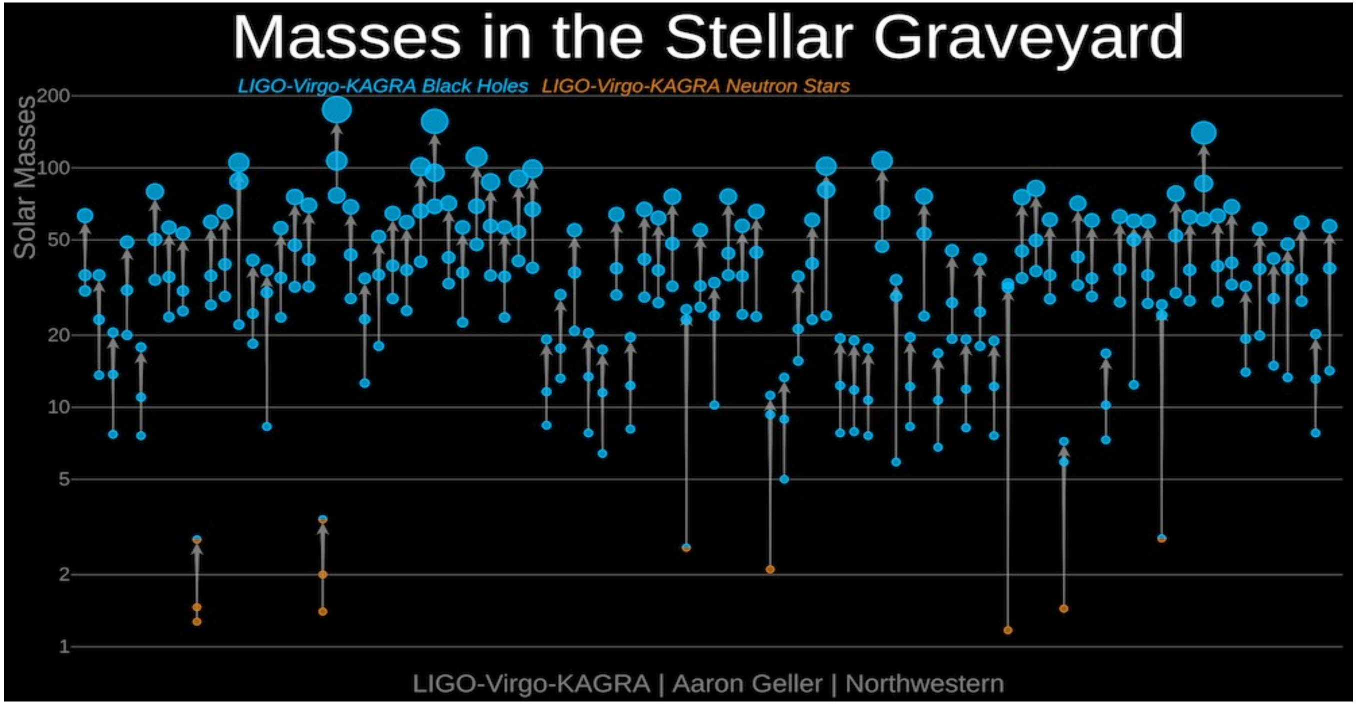
Outline

- Background
- **2-loop amplitudes** for Kerr BH scattering
- **Classical observables** from Dirac brackets
- **Unexpected symmetries** in various limits

  (1) spin shift symmetry

  (2) hidden integrability of orbit motion
- Conclusion

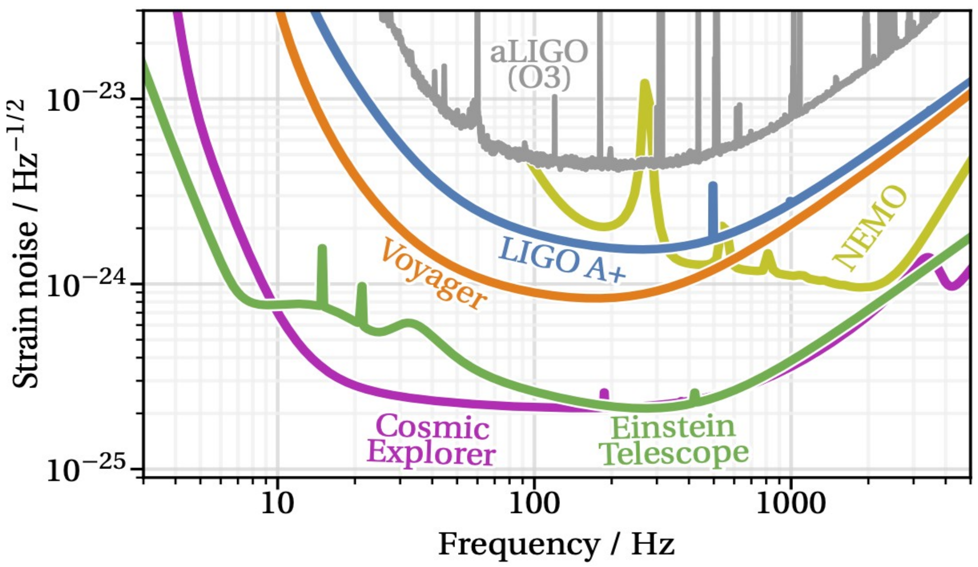
Gravitational wave science in full force



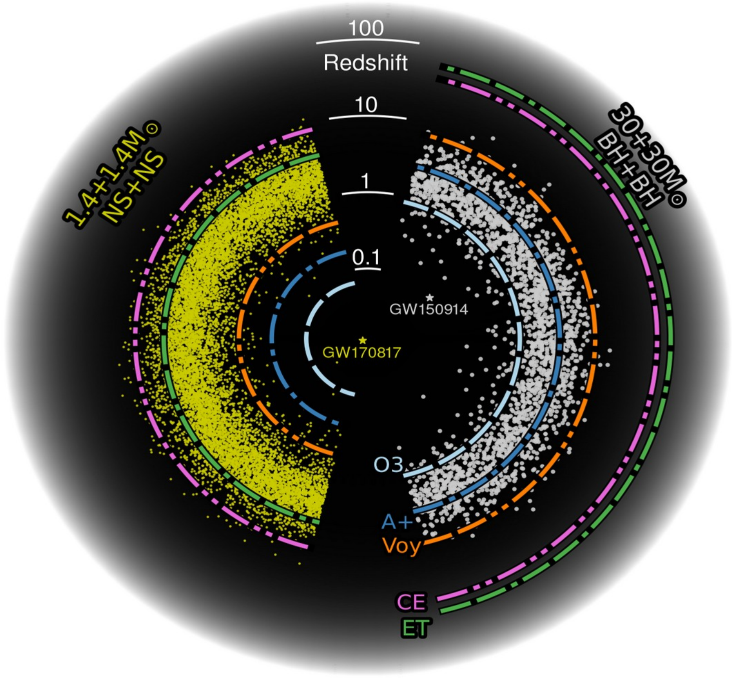
- **90 events** in above plot from LIGO-Vrigo-KABRA O3b run, late 2021; expect **more than 200** at the end of O4 run October this year!

Future detectors

- Ongoing LIGO A+ upgrade, Einstein Telescope, Cosmic Explorer, LISA, TianQin...
- Orders of magnitude increase in sensitivity. **Need improved theoretical modeling, including in inspiral perturbative regime – up to O(G^7).**



https://cosmicexplorer.org/sensitivity.html

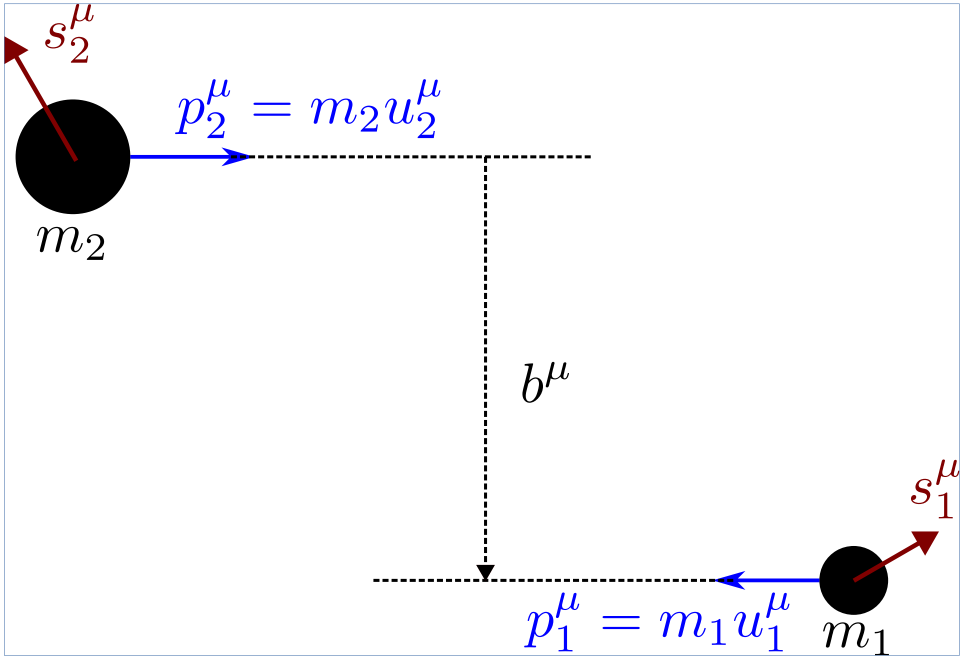


QFT is a powerful tool for GW physics

- **Post-Newtonian expansion** via non-relativistic general relativity (NRGR) [Goldberger, Rothstein, ’06], an EFT inspired by NRQCD.
- **Post-Minkowskian expansion** via on-shell scattering amplitudes, representing BH / NS as massive particles. Exploded in last ~7 yrs. [Numerous authors, many in the audience]
- Often produces results beyond the best from direct classical GR (and vice versa).
- Our contribution (+ongoing) – **O(G^3S^4) spinning binary dyanmics** from 2-loop amplitudes. Accidental discovery of **new hidden symmetries & integrability.**

Problem statement

- What are the impulse & spin kick in the **scattering of two BHs** with the following initial conditions at $t = -\infty$? What are the symmetries?



- Relevance to GW detectors: Scattering observables can be used to infer **bound-orbit spinning dynamics** via analytic continuation [Kalin, Porto, ’19. Gonzo, Shi, ’23, ’24] or matching with two-body Hamiltonian [e.g. Bern, Luna, Roiban, Shen, MZ, ’20. Febres Cordero, Kraus, Lin, Ruf, MZ, ’22. Jakobsen, Mogull, ’22]

Some amplitudes approaches *and tradeoffs*

- Massive spinor-helicity amplitudes w/ spin exponentiation [Arkani-Hamed, Huang, Huang, ’17…] : Tensor rank grows with spin; integral reduction challenging beyond one loop
- Non-transverse higher-spin fields [Bern, Luna, Roiban, Shen, MZ, ’20. Alaverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Teng, ’24]: Care needed to separate spin magnitude changing effects to target ordinary conservative dynamics; not yet done beyond 1-loop
- Finite-spin massive particles [Holstein, Ross, ’07. Vaidya, ’14. Maybee, O’Connell, Vines, ’19. Damgaard, Haddad, Helset, ’19. Aoude, Haddad, Helset, ’20 …]
  - Most straightforward option for ≥2 loop calculations
  - Spin-*j* accesses $O(S^{2j})$ in perturbative spin expansion. E.g. spin-2 access the quartic order.
  - Originally plagued by $\hbar^2 S^2$ **Casimir ambiguity,** resolved by spin-interpolation *(next slides)*
- (Outside amplitudes) PMEFT with spin [Liu, Porto, Yang], WQFT with N=2 SUSY [Jakobsen, Mogull, Plefka, Steinhoff, ’21], or bosonic oscillators [Haddad, Jakobsen, Mogull, Plefka, ’24]. Generalized Wilson lines [Bonocore, Kulesza, Pirsch, ’25]

Status of O(G^3) spin effects

- High spin orders reached at 1 loop, e.g. [Bern, Kosmopoulos, Luna, Roiban, Teng, ’22. Aoude, Haddad, Helset, ’22, ’23. Bohnenblust, Cangemi, Johansson, Pichini, ’24. Bohnenblust, Ita, Kraus, Schlenk, ’25 …]
- Only a handful of results exist at 2 loops / O(G^3), most only to O(S^2), e.g. *Conservative and dissipative results from WQFT* [Jakobsen, Mogull, ’22]. *All-order-in-spin radiation reaction* [Alessio, Di Vecchia, ’22]. *Energy loss from PMEFT* [Riva, Vernizzi, Wong ’22]. *Also O(G^4S^1)* [Jakobsen, Mogull, Plefka, Sauer, Xu, ’23].
- We performed the first 2-loop amplitude calculations of conservative spinless-spinning binary dynamics, first at order S^2 [Febres Cordero, Krauss, Lin, Ruf, MZ, ’22. Akpinar, Febres Cordero, Kraus, Ruf, MZ, ’24], then at unprecedented order O(S^4) [Akpinar, Febres Cordero, Kraus, Smirnov, MZ, ’25].
- Physically **7th-post-Minkowskian order,** as $S \leq Gm$.

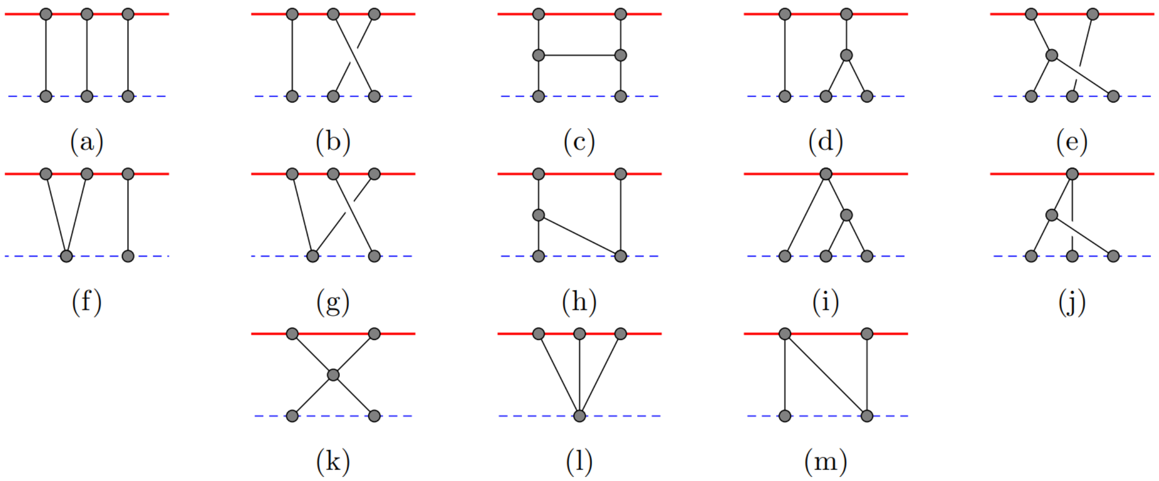
Kerr black hole = minimally coupled massive spinning particles

- Observed as early as e.g. [Vaidya, '14]. (Extra non-minimal coupling oeprators needed for neutron stars.) Klein-Gorden for *s*=0, Proca for *s*=1, finally Fierz-Pauli for *s*=2:

$$\begin{aligned}\mathcal{L}_{\text{Fierz}-\text{Pauli}} = g^{\mu\nu} g^{\alpha\beta} g^{\omega\sigma} \Big\{ & \frac{1}{2}(\nabla_\omega H_{\mu\alpha})(\nabla_\sigma H_{\nu\beta}) \\ & - (\nabla_\mu H_{\alpha\omega})(\nabla_\beta H_{\nu\sigma}) + (\nabla_\mu H_{\nu\alpha})(\nabla_\beta H_{\omega\sigma}) \\ & - \frac{1}{2}(\nabla_\mu H_{\alpha\beta})(\nabla_\nu H_{\omega\sigma}) \Big\} \\ & - \frac{1}{2} m_1^2 g^{\mu\nu} g^{\alpha\beta} \big(H_{\mu\alpha} H_{\nu\beta} - H_{\mu\nu} H_{\alpha\beta}\big) \ ,\end{aligned}$$

2-loop amplitude calculation

- Caravel – automated numerical generalized unitarity beyond 1 loop. [Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, ’20]
  - Cuts from Berends-Giele recursion; checked against double copy + dimensional reduction to massive particles [Bern, Cheung, Roiban, Shen, Solon, MZ, ’19. Johansson, Ochirov, ’19. Bautista, Guevara, ’19. Chiodaroli, Johansson, Pichini, ’21.]



Left: diagrams in integrand ansatz for 2-loop scattering of massive particles

- Soft expansion [Parra-Martinez, Ruf, MZ, ’20] followed by IBP reduction
  - Integrals with up to rank 13 and 10 “dots”, using private FIRE [Smirnov, MZ]

Translation: polarizations → classical spin

- Many ways to derive the translation, e.g. spinning generalization of the observable formalism [Maybee, O'Connell, Vines, '19]. Incoming Wavepacket centered around *k*, with a single polarization state *i* for simplicity,

$$\int_q e^{iq\cdot b}\phi(k+q)a_i^\dagger(k+q)$$

KMOC kernel: $\mathcal{K} = \mathcal{S}^\dagger \mathcal{O}\mathcal{S} - \mathcal{O}$

- Change in observable $\mathcal{O}$ after scattering: $\Delta\mathcal{O} = \langle \mathrm{in}|\mathcal{K}|\mathrm{in}\rangle$. For spin-1:

$$\epsilon_i^{*\rho}(k+q)\,\mathcal{K}_{\rho\mu}\,\epsilon_i^{\mu}(k) = \epsilon_i^{*\nu}(k)\,\Lambda_\nu{}^\rho(k+q,k)\mathcal{K}_{\rho\mu}(k,q)\,\epsilon_i^{\mu}(k)$$

(No sum over *i*)

$$= \epsilon_i^{*\nu}(k)\,\widetilde{\mathcal{K}}_{\nu\mu}(k,q)\,\epsilon_i^{\mu}(k)$$

Lorentz boost to same frame

- Then rewritten as polynomials of Pauli-Lubanski **spin vector** for initial state:

$$\mathrm{S}_\mu = \epsilon_{\mu\nu\rho\sigma}k^\nu\epsilon_i^{*\rho}\epsilon_i^\sigma\,,$$ *identified with classcial spin*

Spin interpolation

[Akpinar, Febres Cordero, Kraus, Ruf, MZ, '24]

- 2-loop calculation made easy with low spin=0, 1, 2 massive particles, but $\hbar^2 S^2$ artificially mixed with quantum-suppressed spinless terms.

- Fix Casimir terms by ***spin interpolation*** - Keep the quantum part of amplitude and extract the ***gradient*** with respect to S^2.

- Generally, write down universal ansatz with formal spin operators, and match to amplitudes with various spin representations.

- Need to translate formal spin operators to any representation. Care needed at higher spin orders [Akpinar, Febres Cordero, Krauss, Smirnov, MZ, '25]

$$\text{e.g.} \quad \mathrm{sym}(S^2)^2 \equiv \eta_{\mu\nu}\eta_{\rho\sigma} S^{(\mu}S^{\nu}S^{\rho}S^{\sigma)} \neq (\mathrm{sym}\, S^2)^2 = [s(s+1)]^2$$

Effortless extraction of observables

- Scattering amplitudes have $1/\hbar$ divergence in the classical limit. Appropriate **subtraction of singularities** needed, as in colliders.

- Many formalisms to deal with this.
  - Non-relativistic EFT
  - KMOC with spin
  - Eikonal exponent

- (Conjecturally) simple, all-order formalism for conservative dynamics
  - Radial action from *amplitude-action correspondence*
  - Observables from iterated *Dirac brackets*

“Finite” amplitude = radial action

- *Amplitude-action correspondence* first proposed in O(G^4) calculation of spinless dynamics [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, MZ, ’21].

- The radial action

$$I_r = 2\int_{r_{\min}}^{\infty} p_r \, dr$$

  is identified with the “finite part” of amplitude. Particularly simple up to 2 loops [Bern, Gatica, Herrmann, Luna, MZ, ’21]. Assume this is unchanged w/ spin.

- Equivalent to *“exponential representation of S-matrix”* [Damgaard, Planté, Vanhove, ’21. Damgaard, Hansen, Planté, Vanhove, ’23]

$$S = \exp(iN)$$

Observables from Dirac brackets

- With the S-matrix as a matrix exponential, $S = \exp(iN)$,

$$\begin{aligned}\Delta O &= \langle \mathrm{out}|\mathcal{O}|\mathrm{out}\rangle - \langle \mathrm{in}|\mathcal{O}|\mathrm{in}\rangle \\ &= \langle \mathrm{in}|\left(e^{-iN}\mathcal{O}e^{iN} - \mathcal{O}\right)|\mathrm{in}\rangle \\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\underbrace{\left[-iN,\ldots[-iN,\mathcal{O}]\right]}_{n\ \mathrm{times}}\end{aligned}$$

exponent has finite classical limit

(Also: Riccardo's talk)

- ***Dirac brackets*** [Dirac, 1950] as applied in the Hanson-Regge spherical top [1974] give the correct classical limits of the quantum commutators [Gonzo, Shi, '24. Kim, Kim, Lee, '24]

- Previous proposals for brackets worked at only 1 loop / O(G^2). [Bern, Luna, Roiban, Shen, MZ, '21. Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola, White, '21. Luna, Moynihan, O'Connell, Ross, '23. Gatica, '23. Jakobsen, Mogull, Plefka, Steinhoff, '21]

Brackets for spinning objects

[Hanson, Regge, 1974. Gonzo, Shi, '24. Kim, Kim, Lee, '24]

- Worldline description of free spinning object with coordinates b^μ and frame attached to the body, $\Lambda^\mu{}_\nu$. Conjugate momenta: velocity & spin tensor, u^μ, $S^{\mu\nu}$.

- Starting point: Poisson brackets:

$$\{b_i^\mu, v_i^\nu\}_{\text{P.B.}} = \frac{\eta^{\mu\nu}}{m_i}, \quad (30)$$

$$\{S_i^{\mu\nu}, S_i^{\alpha\beta}\}_{\text{P.B.}} = S_i^{\mu\alpha}\eta^{\nu\beta} - S_i^{\mu\beta}\eta^{\nu\alpha} - (\mu \leftrightarrow \nu)$$

From Gonzo, Shi, '24

- Constraints: $b \cdot u_i = 0\,, u_i^2 = 1,\ u_{i\mu}S_i^{\mu\nu} = 0,\ \Gamma_i^{0\mu} = u_i^\mu$.

- Constraints violated by Poisson brackets but fixed by **Dirac brackets.**

Validation of $O(G^3S^4)$ result

- Agreement with the aligned-spin scattering angle in **post-Newtonian results** [Bautista, Khalil, Sergola, Kavanagh, Vines, '24] through $O(G^3S^4)$.
- Conservative observables from Dirac brackets agree with our previous $O(S^2)$ results from **KMOC** [Febres Cordero, Krauss, Lin, Ruf, MZ, '22] and with another group using **WQFT** [Jakobsen, Mogull, '22].
- Using all-order-in-spin radiation reaction amplitudes [Alessio, Di Vecchia, '22] and acting with Dirac brackets, obtain **cancellation of high-energy divergences** in observables through $O(S^4)$, previoiusly observed through $O(S^2)$ using WQFT [Jakosen, Mogull, '22].
- Stringents checks provide confidence for our partly conjectural setup.

Surprise in the amplitudes

- Look at finite amplitude for spinning-spinless scattering up to O(S^2).

$$\underbrace{c^{(0,1)}\mathbb{1}}_{\mathcal{O}(S^0)}+\underbrace{c^{(1,1)}(s_1\cdot\ell)}_{\mathcal{O}(S^1)}+\underbrace{\boxed{c^{(2,1)}}(s_1\cdot q)^2+c^{(2,2)}q^2(s_1\cdot u_2)^2+\boxed{c^{(2,3)}}q^2 s_1^2}_{\mathcal{O}(S^2)}$$

- In (either) **probe limit**, a striking relation $\boxed{c^{(2,1)}=-c^{(2,3)}}$. Broken by 1SF & non-minimal coupling deviations from Kerr.

(Talk by Andres Luna)

- Previous seen at 1-loop [Aoude, Haddad, Helset, '22], identified as ***spin shift symmetry:*** $s_i^\mu \to s_i^\mu + \xi q^\mu$. [Bern, Kosmopoulos, Luna, Roiban, Teng, '22]

- We confirm the symmetry to **O(s^4)** at 2 loops (& for QED, "root-Kerr") - crucial new data connecting the symmetry with spinning ***probe BH.***

Could it be integrability?

[Akpinar, Brown, Gonzo, MZ, in progress]

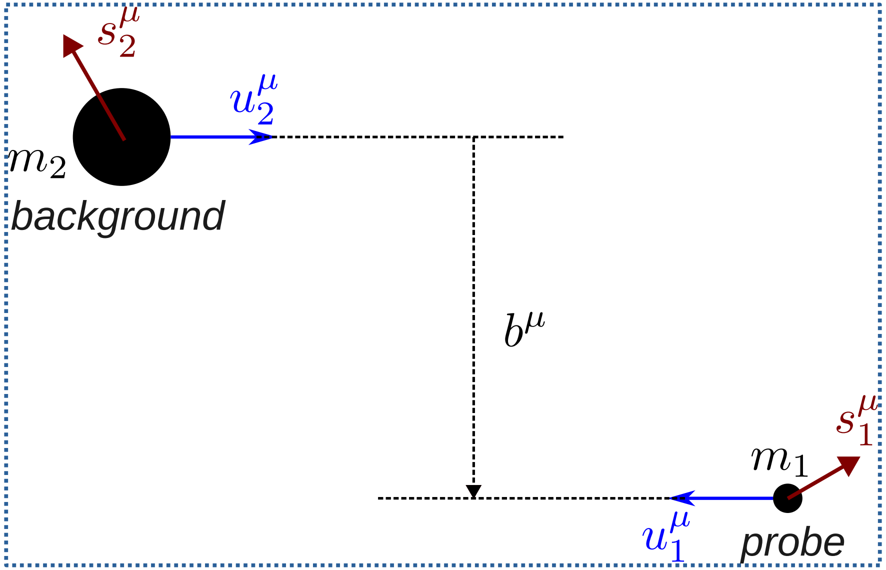
- Spinless probe in Schwarzschild background has as many conserved quantities as *d.o.f.*: *energy, mass, azimuthal & total angular momentum.*
- For Kerr, Latitudinal angular momentum lost, but there is a conserved **Carter's constant** making the motion integrable [Carter, 1968]

$$K = p_\theta^2 + (ap_t + p_\phi)^2 + a^2(m^2 - p_t^2)\cos^2(\theta) + p_\phi^2 \cot^2(\theta)$$

- Extends to a **spinning probe** to linear [Rüdiger, '81, '83] and quadratic orders [Compère, Druart, Vines, '23] in probe spin; NOT for probe neutron star.
- Dirac brackets are ideal for testing **asymptotic conservation laws**

$$\boxed{\{I_r, \mathcal{O}\} = 0} \implies \Delta\mathcal{O} = \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{\{I_r, \ldots \{I_r,}_{n \text{ times}} \mathcal{O}\}\} = 0$$

Could it be integrability?



- Let's test conserved quanties in an ***on-shell gauge invariant way.*** Do they remain unchanged after scattering, i.e. commute with I_r?

$y \equiv u_1 \cdot u_2$

$\varepsilon^\mu \equiv \epsilon^{\mu\nu\rho\sigma} b_\nu u_{1\rho} u_{2\sigma}$

- Conserved quantities at $r \to \infty$: [Justin Vines talk, Nordita 2023]
  - Azimuthal angular momentum $|s_2| L/m_1 = \varepsilon \cdot s_2 - y\, s_1 \cdot s_2 + s_1 \cdot u_2\, s_2 \cdot u_1$
  - Generalized Carter's constant $\mathcal{Q}^{(2)} = -\varepsilon \cdot \varepsilon - 2y(\varepsilon \cdot s_2) - (y^2 - 1) s_2^2 - (u_1 \cdot s_2)^2$
  - Rüdiger's first invariant $Q_Y = \varepsilon \cdot s_1 + y\, s_1 \cdot s_2 - s_1 \cdot u_2\, s_2 \cdot u_1$

Where integrability holds

[Akpinar, Brown, Gonzo, MZ, in progress]

- Used both new 2-loop data and the treasure trove of existing 1-loop data.

- **(Conjectural)** All orders in *G (checked through LO, NLO, NNLO)*, Kerr background to all orders in spin *(checked cases up to quartic order)*, through quartic order in **probe** spin.
  This goes beyond established quadratic-in-probe-spin integrability [Compere, Druart, Vines, '23].

- **(Proven by us)** NLO (1-loop) in *G*, *spinless-spinning binary,* through quartic order in spin (+ subset of higher-order terms), no restriction to probe limit.
  This goes beyond established 1.5 PN / perturbative 2PN integrability [Tanay, Stein, Ghersi, '20].

Unexplained rich symmetries

- Results indicate that the **spin-shift symmetry** is always *accompanied* by further symmetries from **integrability** but *not explained* by the latter!
- Integrability places no constraints on s_1^2, s_2^2 terms in I_r that commute with all conserved quantities. Only constrained by spin-shift $s_i^\mu \to s_i^\mu + \xi q^\mu$.
- Integrability constraints are infinitesimal symmetries of I_r in **impact parameter b space;** spin shift in **transverse momentum q space.** Analgous to (ordinary + dual) conformal symmetry in *N*=4 sYM?
- How constraining is (spin-shift symmetry + integrability)? Preliminary: **Can fix all observables from aligned-spin scattering angle alone.**

Conclusion

- First calculation **$O(G^3S^4)$ conservative dynamics** of binary Kerr BH using scattering amplitudes from 2-loop numerical generalized unitarity.
- Effortless extraction of classical observables beyond the aligned-spin limit: **Radial action** from finite part of amplitude + **Dirac brackets.**
- Accidental discovery of **spin-shift symmetry** beyond one loop – led to discovery of further symmetries associated with **integrability**.
- **New on-shell approach** to hidden symmetries in classical systems.
- **Future work:** (1) further calculations to check symmetries. (2) decrypt the symmetries and make them manifest.

Thank you!

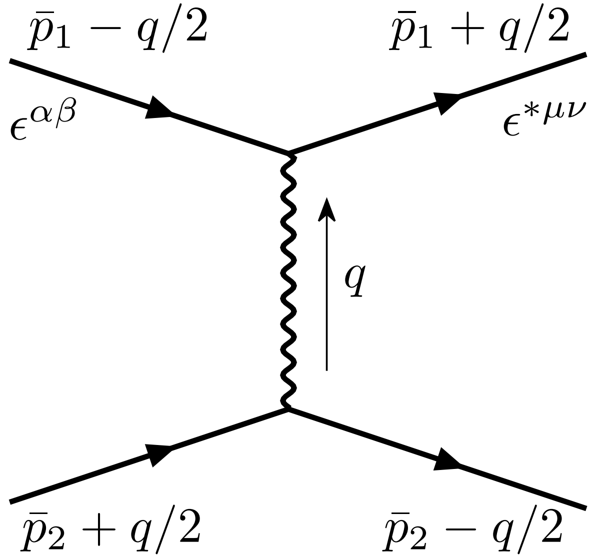
backup slides on next pages

Form factor decomposition

- Crossing + Parity $\Rightarrow$ 9 form factors in massive scalar-tensor scattering.

$$T_1^{\mu\nu\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} \ ,$$
$$T_2^{\mu\nu\alpha\beta} = \bar{p}_2^{\mu}\bar{p}_2^{\nu}\bar{p}_2^{\alpha}\bar{p}_2^{\beta} \ ,$$
$$T_3^{\mu\nu\alpha\beta} = \bar{p}_2^{\mu}\eta^{\nu\alpha}\bar{p}_2^{\beta} \ ,$$
$$T_4^{\mu\nu\alpha\beta} = 2\bar{p}_2^{\nu}\bar{p}_2^{\alpha}\bar{p}_2^{[\mu}q^{\beta]} \ ,$$
$$T_5^{\mu\nu\alpha\beta} = 2\eta^{\nu\alpha}\bar{p}_2^{[\mu}q^{\beta]} \ ,$$

$$T_6^{\mu\nu\alpha\beta} = q^{\mu}\bar{p}_2^{\nu}\bar{p}_2^{\alpha}q^{\nu} \ ,$$
$$T_7^{\mu\nu\alpha\beta} = q^{\mu}\eta^{\nu\alpha}q^{\beta} \ ,$$
$$T_8^{\mu\nu\alpha\beta} = 2q^{\mu}\bar{p}_2^{[\nu}q^{\alpha]}q^{\beta} \ ,$$
$$T_9^{\mu\nu\alpha\beta} = q^{\mu}q^{\nu}q^{\alpha}q^{\beta} \ ,$$



When conservation laws hold

[Akpinar, Brown, Gonzo, MZ, in progress]

Order	Integrable?	Works beyond probe limit?
$O(G^2 s_1^4 s_2^4)$	✔	✕
Only one spin, $O(G^2 s_1^{11} s_2^0)$, “naive terms” [1]	✔	✔
same as above, but “extra terms” [1]	✕	N/A
$O(G^3 s_1^4 s_2^0)$ [2]	✔	✕
$O(G^3 s_1^2 s_2^2)$, total 2 orders in spin [3]	✔	✕

[1] Bohnenblust, Cangemi, Johansson, Pichini, ’24. [2] Akpinar, Febres Cordero, Kraus, Smirnov, MZ, ’25. [3] (deduced from) [Jakobsen, Mogull, ’22]