Two-loop amplitudes for ttj production at the LHC

with Simon Badger, Matteo Becchetti, Colomba Brancaccio and Simone Zoia
(arxiv:2412.13876)

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tt+jet production at the LHC

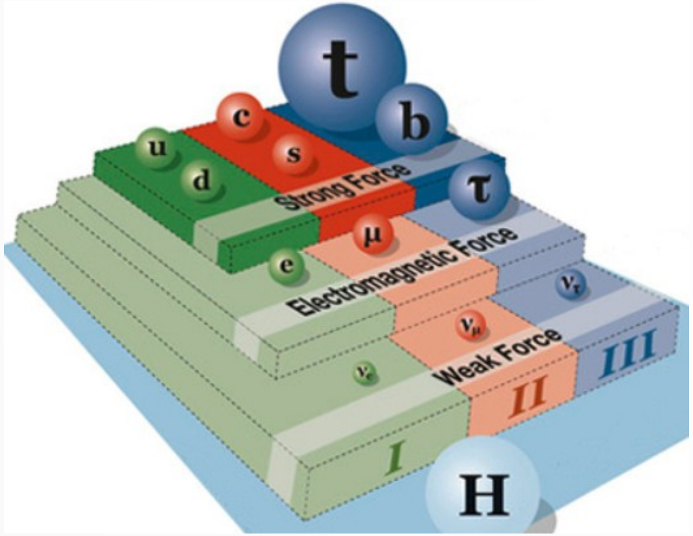
Top-quark: heaviest particle in SM, largest Yukawa coupling, decays before hadronizing

⇒ precision tests of SM and window to new physics

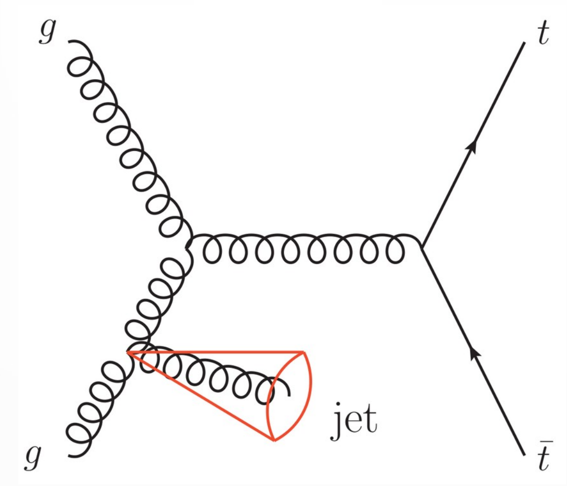
⇒ abundantly produced at the LHC, mainly through pp→tt production

⇒ large fraction of tt events accompanied by additional *jets* (~40% for p_{Tj}>40 GeV)

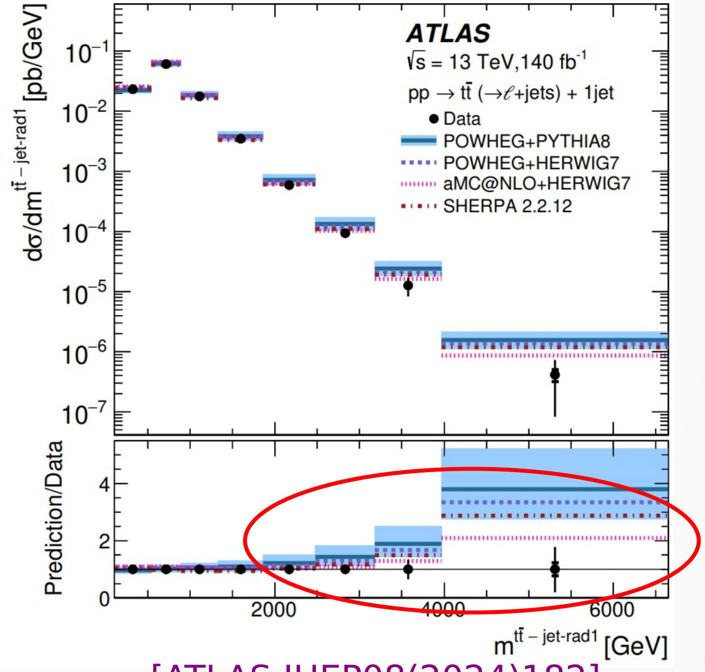
⇒ tt+jets measurements available from ATLAS and CMS



https://webific.ific.uv.es/web/en/content/measuring-pole-mass-top-quark-high-precision



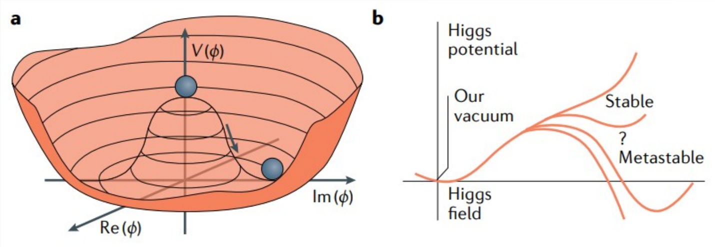
pp→ttj



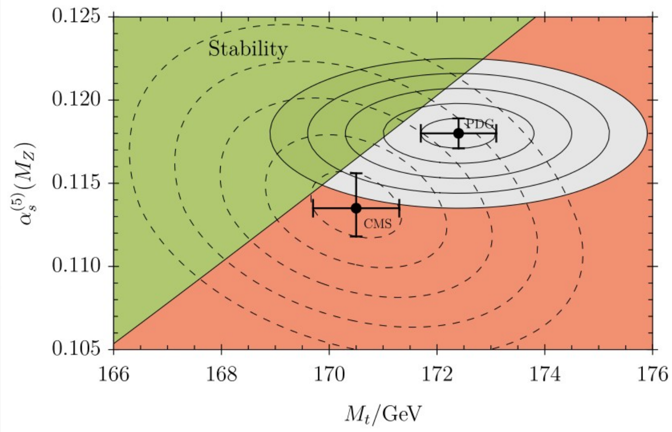
[ATLAS;JHEP08(2024)182]

tt+jet production at the LHC

m_{top} measurements ⇒ assess EW vacuum stability



[Bass,DeRoeck,Kado;Nat Rev Phys 3,608–624(2021)]

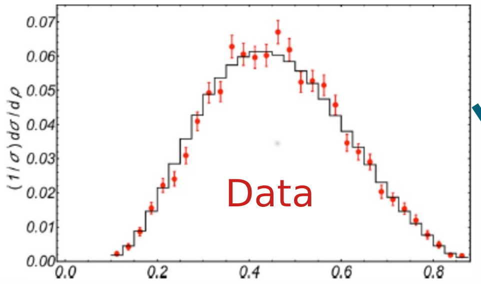


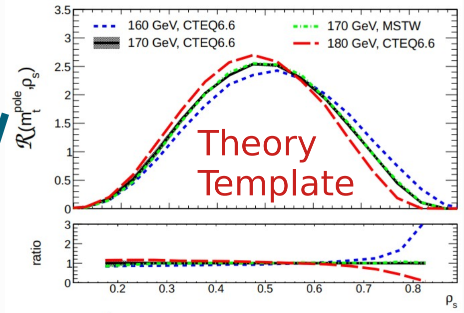
[Hiller,Hohne,Litim,Steudtner;2401.08811]

m_{top} extraction with ttj final state

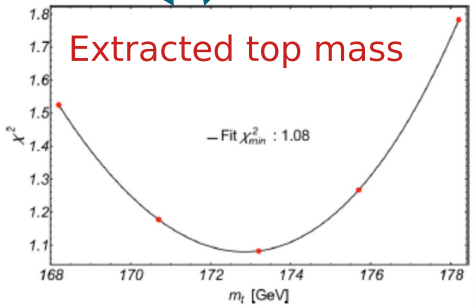
$$\mathcal{R}(m_t^{\text{pole}}, \rho_s) = \frac{1}{\sigma_{t\bar{t}+1\text{-jet}}} \frac{d\sigma_{t\bar{t}+1\text{-jet}}}{d\rho_s}(m_t^{\text{pole}}, \rho_s) \qquad \rho_s = \frac{2m_0}{\sqrt{s_{t\bar{t}j}}}.$$

[Alioli,Fernandez,Fuster,Irles,Moch,Uwer,Vos;1303.6415]
[Alioli,Fuster,Garzelli,Gavardi,Irles,Melini,Moch,Uwer,Vos;2202.07975]





$$\chi^2(m_t) = \sum_i \frac{(x_i^d - x_i^{th})^2}{\sigma_i^2}$$



Theoretical predictions for pp→ttj

$$d\sigma = \underbrace{d\sigma^{(0)}}_{\text{LO}} + \underbrace{\alpha_s\, d\sigma^{(1)}}_{\delta\text{NLO}} + \underbrace{\alpha_s^2\, d\sigma^{(2)}}_{\delta\text{NNLO}} + \ \ldots$$

NLO QCD (stable top) [Dittmaier,Uwer,Weinzierl;(2007,2008)]

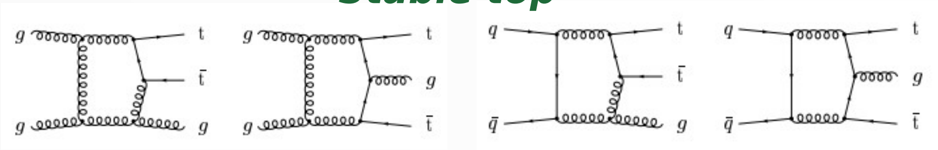
NLO QCD (NWA) [Melnikov,(Scharf),Schulze;(2010,2011)]

NLO QCD (off-shell) [Bevilacqua,**HBH**,Kraus,Worek;(2015,2016)]

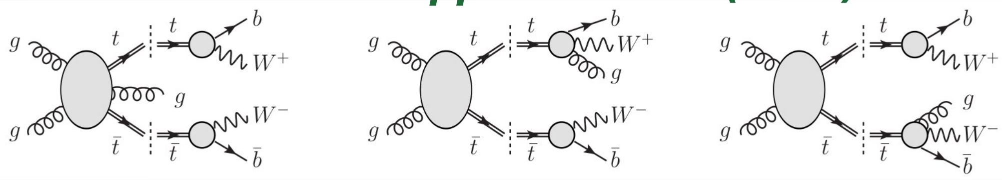
NLO QCD + PS [Kardos,Papadopoulos,Trocsanyi;(2011)][Alioli,Moch,Uwer;(2011)][Czakon,**HBH**,Kraus,Worek;(2015)]

Merged NLO QCD+EW (stable top) [Gutschow,Lindert,Schonherr;(2018)]

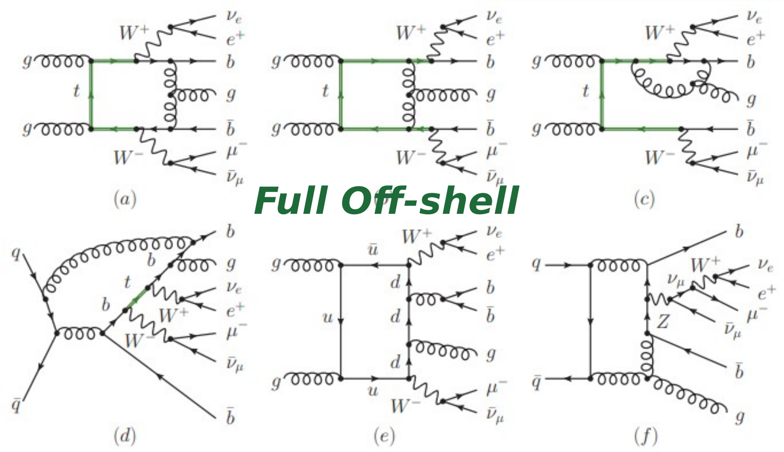
Stable top



Narrow-width Approximation (NWA)

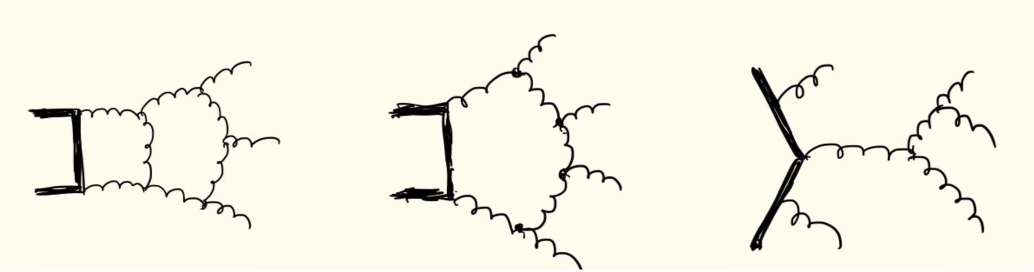


Full Off-shell



NNLO QCD corrections: ingredients

δNNLO:



Double-Virtual (VV) | Real-Virtual (RV) | Double-Real (RR)

NNLO subtraction scheme to cancel IR singularities

applied to 2→3 process: STRIPPER, Antenna Subtraction, q_T-subtraction

[Czakon(2010)][Gehrmann,Gehrmann-de Ridder,Glover(2005)][Catani,Grazzini(2007)]

One-loop amplitudes highly automatized: OpenLoops,Recola,MadLoop,Helac-1loop,etc

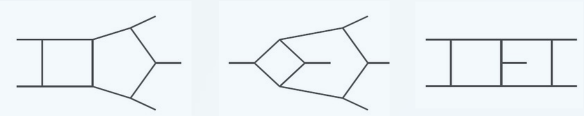
Remaining bottleneck: two-loop five-point amplitudes

$$\text{loop amplitude} = \sum (\text{rational coefficients}) \times (\text{integral/special functions})$$

highly non-trivial for multi-scale process!!

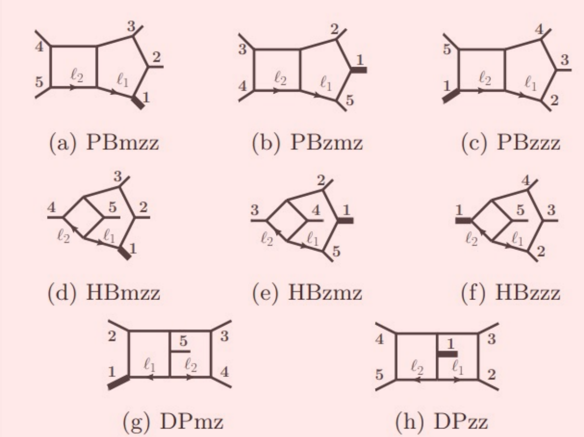
State-of-the art: two-loop five-point integrals

massless



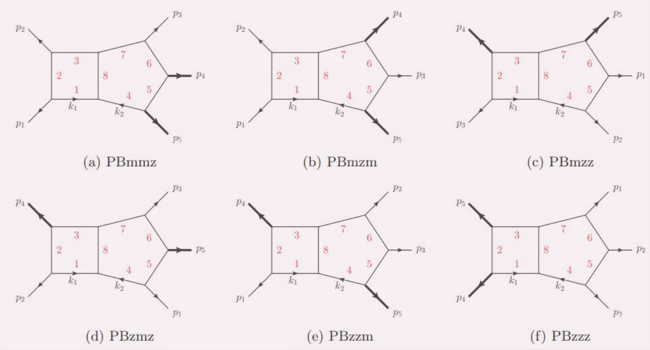
[Abreu,Chicherin,Dixon,Gehrmann,Henn, Herrmann,LoPresti,Papadopoulos,Page, Sotnikov,Tomassini,Wasser,Wever,Zeng, Zhang,Zoia(2015-2020)]

one external mass



[Abreu,Canko,Chicherin,Ita,Kardos,Moriello, Page,Papadopoulos,Smirnov,Sotnikov,Syrrakos, Tomassini,Tschernow,Wever,Zeng,Zoia (2015-2023)]

two external masses (planar)

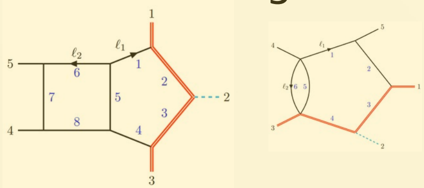


[Jiang,Liu,Xu,Yang(2024)]
[Abreu,Chicherin,Sotnikov,Zoia(2024)]

pentagon functions for massless and one-external mass

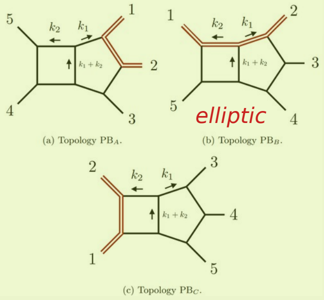
[Gehrmann,Henn,Lo Presti(2018)] [Chicherin,Sotnikov(2020)][Chicherin, Sotnikov,Zoia(2021)][Abreu,Chicherin,Ita, Page,Sotnikov,Tschernow,Zoia(2023)]

pp→ttH: leading colour, N_f



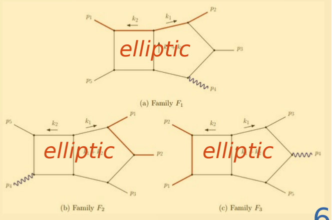
[Febres Cordero,Figueiredo,Kraus, Page,Reina(2023)]

pp→ttj: leading colour



[Badger,Becchetti,Chaubey, Marzucca(2022)]
[Badger,Becchetti,Giraudo, Zoia(2024)]
[Becchetti,Dlapa,Zoia(2025)]

pp→ttW: leading colour



[Becchetti,Canko, Chestnov,Peraro, Pozzoli,Zoia(2025)]

State-of-the art: two-loop five-point amplitudes

massless
all full colour

- $pp \to \gamma\gamma\gamma$ [Abreu,Page,Pascual,Sotnikov(2020)][Chawdhry,Czakon,Mitov,Poncelet(2021)] [Abreu,De Laurentis,Ita,Klinkert,Page,Sotnikov(2023)]
- $pp \to \gamma\gamma j$ [Agarwal,Buccioni,von Manteuffel,Tancredi(2021)][Chawdhry,Czakon,Mitov,Poncelet(2021)] [Badger,Brönnum-Hansen,Chicherin,Gehrmann,**HBH**,Henn,Marcoli,Moodie,Peraro,Zoia(2021)]
- $pp \to \gamma jj$ [Badger,Czakon,**HBH**,Moodie,Peraro,Poncelet,Zoia(2023)]
- $pp \to jjj$ [Abreu,Febres Cordero,Ita,Page,Sotnikov(2021)][De Laurentis,Ita,Klinkert,Sotnikov(2023)] [Agarwal,Buccioni,Devoto,Gambuti,von Manteuffel,Tancredi(2023)][De Laurentis,Ita,Sotnikov(2023)]

1 external mass

- $pp \to Wbb$ (leading colour, massless b) [Badger,**HBH**,Zoia(2021)][**HBH**,Poncelet,Popescu,Zoia(2022)]
- $pp \to Wjj$ (leading colour) [Abreu,Febres Cordero,Ita,Klinkert,Page,Sotnikov(2022)] [De Laurentis,Ita,Page,Sotnikov(2025)]
- $pp \to Hbb$ (full colour, massless b) [Badger,**HBH**,Krys,Zoia(2021)] [Badger,**HBH**,Poncelet,Wu,Zhang,Zoia(2024)]
- $pp \to W\gamma j$ (leading colour) [Badger,**HBH**,Krys,Zoia(2022)]
- $pp \to W/Z+bb$ (leading colour, massive b) [Buonocore,Devoto,Kallweit,Mazzitelli,Rottoli,Savoini(2022)] [Mazzitelli,Sotnikov,Wiesemann(2024)]
  massification of massless amplitude
- $pp \to W\gamma\gamma$ (analytic l.c., numerical s.l.c.) [Badger,**HBH**,Wu,Zhang,Zoia(2024)]

State-of-the art: two-loop five-point amplitudes

massless
all full colour

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massification of massless amplitude

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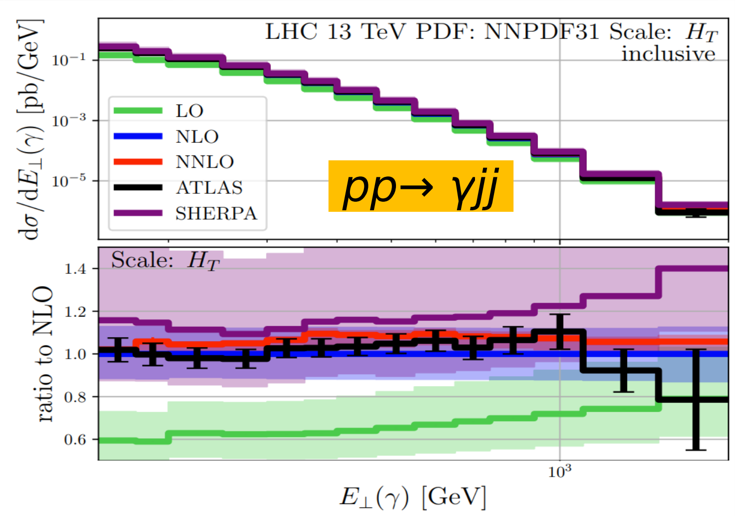
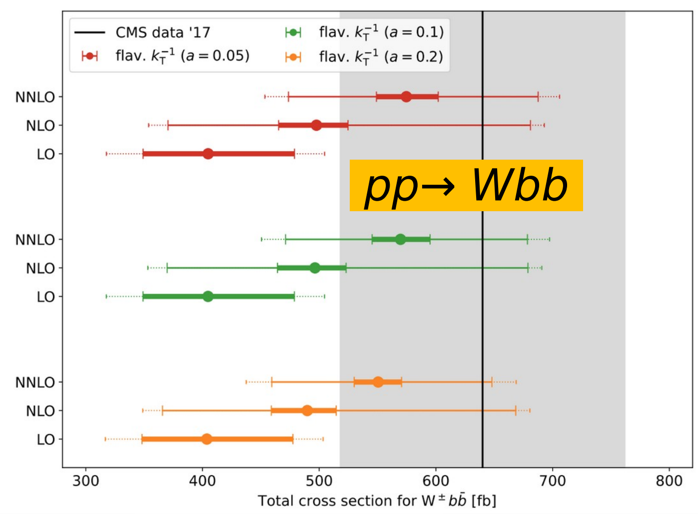
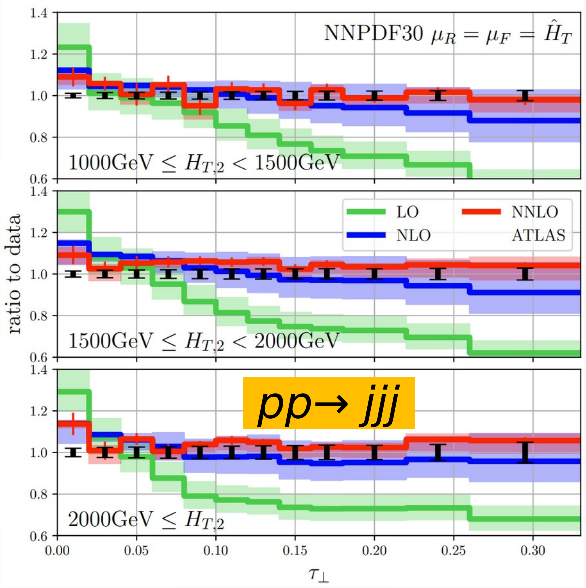
State-of-the art:
NNLO QCD for 2→3 process



[Badger,Czakon,**HBH**,Moodie,Peraro, Poncelet,Zoia(2023)]



[**HBH**,Poncelet,Popescu,Zoia(2022)]



[Czakon,Mitov,Poncelet(2021)][Chen,Gehrmann, Glover,Huss,Marcoli(2022)]Alvarez,Cantero, Czakon,Llorente,Mitov,Poncelet(2023)]

pp→γγγ, pp→γγj, pp→Wbb, pp→Zbb

[Chawdhry,Czakon,Mitov,Poncelet(2019)]
[Kallweit,Sotnikov,Wiesemann(2020)]
[Czakon,Mitov,Poncelet(2020)]
[Buonocore,Devoto,Kallweit,Mazzitelli,Rottoli, Savoini(2022)]
[Mazzitelli,Sotnikov,Wiesemann(2024)]
[Buccioni,Chen,Feng,Gehrmann,Huss,Marcolli(2025)]

pp→ttW, pp→ttH (using approximated two-loop amplitudes)

[Buonocore,Devoto,Grazzini,Kallweit,Mazzitelli,Rottoli,Savoini(2022)]
[Catani,Devoto,Grazzini,Kallweit,Mazzitelli,Savoini(2020)]

Two-loop amplitudes for pp→ttj

⇒ Increase in complexities due to external and internal masses

$$\vec{x} = (d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2)$$

$$d_{ij} = p_i . p_j$$

$$\mathcal{A}^{(2)} = \int d^d k_1 d^d k_2 \sum_{i=1}^{N_{\text{diag}}} \frac{\mathcal{N}_i(\{k\},\{p\})}{\mathcal{D}_i(\{k\},\{p\})} = \sum_i c_i(\vec{x},\epsilon)\, \mathrm{MI}_i(\vec{x},\epsilon)$$

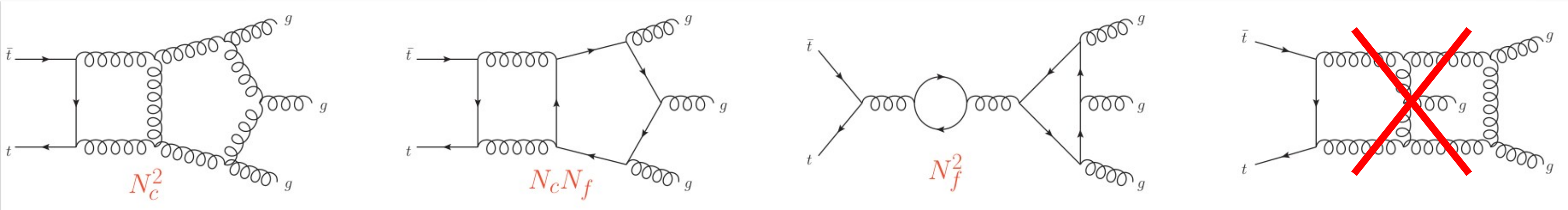
High-degree rational functions (6 kinematic variables + ϵ), massive spinors

Internal massive propagators, functions beyond polylogarithms

⇒ Two scattering processes: 0→ttggg and 0→ttqqg

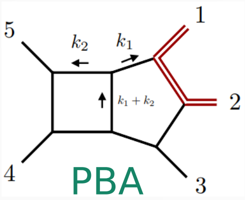
⇒ Work in the leading colour approximation: only planar diagrams

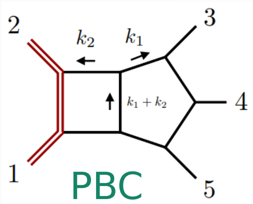
$$\mathcal{A}^{(L)}_{\text{LC}}(1_{\bar{t}}, 2_t, 3_g, 4_g, 5_g) = \sqrt{2}\, \bar{g}_s^3\, n^L \sum_{\sigma \in Z_3} \left(t^{a_{\sigma(3)}} t^{a_{\sigma(4)}} t^{a_{\sigma(5)}}\right)_{i_2}^{\bar{i}_1} A^{(L)}(1_{\bar{t}}, 2_t, \sigma(3)_g, \sigma(4)_g, \sigma(5)_g)$$



Two-loop master integrals for leading colour pp→ttj

[Badger,Becchetti,Chaubey,Marzucca(2022)][Badger,Becchetti,Giraudo,Zoia(2024)]





"polylogarithmic"

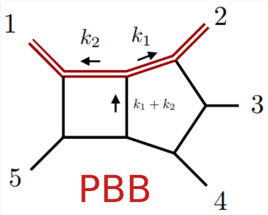
MIs satisfy the following differential equations in a canonical form [Henn(2013)]

$$\mathrm{d}\vec{g}(\vec{x},\epsilon) = \epsilon\,\mathrm{d}A(\vec{x})\cdot\vec{g}(\vec{x},\epsilon) \qquad \text{where} \qquad \mathrm{d}A(\vec{x}) = \sum_i c_i \,\mathrm{dlog}\, W_i(\vec{x})$$

letters

Solution in terms of iterated integrals

$$\vec{g}(\vec{x},\epsilon) = \sum_{k\geq 0}\epsilon^k \vec{g}^{(k)}(\vec{x}) \quad \Rightarrow \quad \vec{g}^{(k)}(\vec{x}) = \int \mathrm{d}A(\vec{x})\cdot\vec{g}^{(k-1)}(\vec{x}) + \text{const.}$$



→ elliptic curves
→ nested square roots

DEs take the following form

$$\mathrm{d}\vec{g}_B(\vec{x},\epsilon) = \sum_{k=0}^{2}\epsilon^k\,\Omega^{(k)}(\vec{x})\cdot\vec{g}_B(\vec{x},\epsilon)$$

MIs in the elliptic sector → start at ϵ^4

as close to canonical as possible

$$\Omega^{(k)}(\vec{x}) = \sum_i A_i^{(k)}(\vec{x})\,\mathrm{dlog}W_i(\vec{x}) + \sum_j B_j^{(k)}\omega_j(\vec{x})$$

Q-linearly dependent non-logarithmic one forms

$$\omega(x,y) = \omega_x(x,y)\,\mathrm{dx} + \omega_y(x,y)\,\mathrm{dy}$$

Canonical DEs recently available
[Becchetti,Dlapa,Zoia(2025)]

Numerical solutions obtained using *generalised power series expansion* method [Moriello(2020)]

→ DiffExp[Hidding(2020)],AMFlow[Liu,Ma(2020)],SeaSyde[Armadillo,etal(2022)],LINE[Prisco,Tramontano(2025)]

A basis of special functions: *pentagon functions* approach

[Gehrmann,Henn,Lo Presti(2018)][Chicherin,Sotnikov(2020)][Chicherin,Sotnikov,Zoia(2021)]
[Abreu,Chicherin,Ita,Page,Sotnikov,Tschernow,Zoia(2023)]

Why? analytic cancellation of UV+IR poles
simplification of finite remainders
efficient numerical evaluation

Inputs: ✔ canonical DEs
✔ high-precision boundary values of MIs at a random (boundary) point (AMFlow[Liu,Ma(2020)])

- Use the components of ϵ-expansion of MIs as special functions

$$\vec{g}(\vec{x},\epsilon) = \sum_{w=0}^{4} \epsilon^{w}\, \vec{g}^{(k)}(\vec{x})$$

NNLO computation requires MIs evaluated up to w=4

- Starting from canonical DEs, write MIs in terms of *Chen iterated integrals*

$$\vec{g}^{(2)}(\vec{x}) = \sum_{i_1 i_2} M_{i_1 i_2}[W_{i_1}, W_{i_2}]_{\vec{x}_0}(\vec{x})\, \vec{b}^{(0)}(\vec{x}_0) + \sum_{i} M_i [W_i]_{\vec{x}_0}(\vec{x})\, \vec{b}^{(1)}(\vec{x}_0) + \vec{b}^{(2)}(\vec{x}_0)$$

A basis of special functions: *pentagon functions* approach

[Gehrmann,Henn,Lo Presti(2018)][Chicherin,Sotnikov(2020)][Chicherin,Sotnikov,Zoia(2021)]
[Abreu,Chicherin,Ita,Page,Sotnikov,Tschernow,Zoia(2023)]

- At *symbol* level, extract independent MI components (Gaussian elimination + shuffle relations)

  → preferred ordering: lower weight functions over higher weight ones
  one-loop functions over two-loop ones

$$\{g_i^{(0)}\} \to \{1\}, \qquad \{g_i^{(1)}\} \to \{f_k^{(1)}\}, \qquad \{g_i^{(2)}\} \cup \{f_i^{(1)} \times f_j^{(1)}\} \to \{f_k^{(2)}\}, \qquad \text{etc.}$$

$\{f_i^{(w)}\}$ → algebraically independent and irreducible special functions

- <u>Ansatz</u>: MI components are polynomials in $\{f_i^{(w)}\}$ + ζ_2 and ζ_3 (up to weight 4)

$$g_i^{(2)}(\vec{x}) = \sum_j \alpha_j f_j^{(2)}(\vec{x}) + \sum_{j\leq k} \beta_{jk} f_j^{(1)}(\vec{x}) f_k^{(1)}(\vec{x}) + \gamma\zeta_2$$

α,β: fixed from symbol-level analysis
γ: fixed by evaluation at x_0 + rationalization

A basis of special functions: non-canonical DE

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]

DEs for the MI components $\vec{g}^{(w)}(\vec{x})$ are

$$\mathrm{d}\vec{g}^{(w)}(\vec{x}) = \Omega^{(0)}(\vec{x})\,\vec{g}^{(w)}(\vec{x}) + \Omega^{(1)}(\vec{x})\,\vec{g}^{(w-1)}(\vec{x}) + \Omega^{(2)}(\vec{x})\,\vec{g}^{(w-2)}(\vec{x})$$

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$$\mathrm{d}\vec{g}^{(w)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{g}^{(w)}}(\vec{x}) + \cancel{\Omega^{(1)}(\vec{x})\,\vec{g}^{(w-1)}}(\vec{x}) + \cancel{\Omega^{(2)}(\vec{x})\,\vec{g}^{(w-2)}}(\vec{x})$$

w=0 $\qquad \mathrm{d}\vec{g}^{(0)}(\vec{x}) = 0$ $\longrightarrow$ numerical evaluations at random points using AMFlow determine which $g_i^{(w)}(\vec{x}) = 0$

$\longrightarrow$ $\vec{g}^{(0)}(\vec{x}) = \text{constant}$

A basis of special functions: non-canonical DE

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]

DEs for the MI components $\vec{g}^{(w)}(\vec{x})$ are

$$\mathrm{d}\vec{g}^{(w)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{g}^{(w)}(\vec{x})} + \Omega^{(1)}(\vec{x})\,\vec{g}^{(w-1)}(\vec{x}) + \cancel{\Omega^{(2)}(\vec{x})\,\vec{g}^{(w-2)}(\vec{x})}$$

w=0 $\quad \mathrm{d}\vec{g}^{(0)}(\vec{x}) = 0 \longrightarrow \vec{g}^{(0)}(\vec{x}) = \mathrm{constant}$

w=1 $\quad \mathrm{d}\vec{g}^{(1)}(\vec{x}) = \sum_i A_i^{(1)}\,\mathrm{dlog}W_i(\vec{x})\,\vec{g}^{(0)} \longrightarrow$ canonical, can be solved in terms of iterated integrals

A basis of special functions: non-canonical DE

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]

DEs for the MI components $\vec{g}^{(w)}(\vec{x})$ are

$$\mathrm{d}\vec{g}^{(w)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{g}^{(w)}(\vec{x})} + \Omega^{(1)}(\vec{x})\,\vec{g}^{(w-1)}(\vec{x}) + \cancel{\Omega^{(2)}(\vec{x})\,\vec{g}^{(w-2)}(\vec{x})}$$

w=0 $\quad \mathrm{d}\vec{g}^{(0)}(\vec{x}) = 0 \longrightarrow \vec{g}^{(0)}(\vec{x}) = \text{constant}$

w=1 $\quad \mathrm{d}\vec{g}^{(1)}(\vec{x}) = \sum_i A_i^{(1)} \mathrm{dlog} W_i(\vec{x})\,\vec{g}^{(0)} \longrightarrow$ canonical, can be solved in terms of iterated integrals

w=2 $\quad \mathrm{d}g_{15}^{(2)}(\vec{x}) = \frac{1}{24}\left[\underbrace{12g_{103}^{(1)}(\vec{x}_0) + 8g_{110}^{(1)}(\vec{x}_0) + 4g_{111}^{(1)}(\vec{x}_0) + 3g_{118}^{(1)}(\vec{x}_0) - 48g_{63}^{(1)}(\vec{x}_0)}_{=0}\right]\omega_2(\vec{x}) + \cdots$

$\downarrow$

$$\mathrm{d}\vec{g}^{(2)}(\vec{x}) = \sum_i A_i^{(2)} \mathrm{dlog} W_i(\vec{x})\,\vec{g}^{(1)} \longrightarrow \text{canonical}$$

A basis of special functions: non-canonical DE

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]

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canonical, can be solved in terms of iterated integrals

w=3 $\quad$ same as for w=2

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canonical, can be solved in terms of iterated integrals

w=3 $\quad$ same as for w=2

w=4 $\quad g_i^{(4)}(\vec{x}) = f_k^{(4*)}$

non-polylogarithmic special functions

MIs of elliptic sectors

A basis of special functions: non-canonical DE

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]

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w=3 $\quad$ same as for w=2

w=4 $\quad g_i^{(4)}(\vec{x}) = f_k^{(4*)}$

non-polylogarithmic special functions

MIs of elliptic sectors

$$g_{15}(\vec{x},\epsilon) = \frac{1}{48} + \frac{\epsilon}{24}\left[f_1^{(1)} - 2f_2^{(1)} + 2f_4^{(1)} - 2f_6^{(1)}\right] - \frac{\epsilon^2}{48}\left[7f_1^{(2)} + 7f_3^{(2)} + \frac{25}{4}\zeta_2 - \frac{15}{4}\left(f_1^{(1)}\right)^2 + 15 f_1^{(1)} f_2^{(1)} - 8\left(f_2^{(2)}\right)^2 + \cdots\right]$$
$$+ \frac{\epsilon^3}{8}\left[f_5^{(3)} + \frac{820}{9}\zeta_3 + \frac{7}{12}\zeta_2 f_2^{(1)} - \left(f_2^{(1)}\right)^3 + 5\left(f_3^{(1)}\right)^2 f_5^{(1)} + 6 f_1^{(1)} f_2^{(1)} f_6^{(1)} + \cdots\right] + \epsilon^4 f_1^{(4*)} + \mathcal{O}(\epsilon^5)$$

non-polylogarithmic

A basis of special functions: non-canonical DE

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]

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w=2 $\quad \mathrm{d}\vec{g}^{(2)}(\vec{x}) = \sum_i A_i^{(2)} \mathrm{dlog} W_i(\vec{x})\,\vec{g}^{(1)}$

canonical, can be solved in terms of iterated integrals

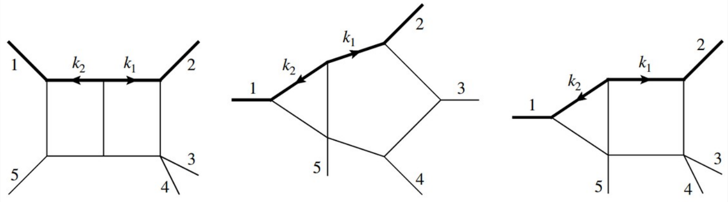
w=3 $\quad$ same as for w=2

w=4 $\quad g_i^{(4)}(\vec{x}) = f_k^{(4*)}$

non-polylogarithmic special functions

MIs of elliptic sectors

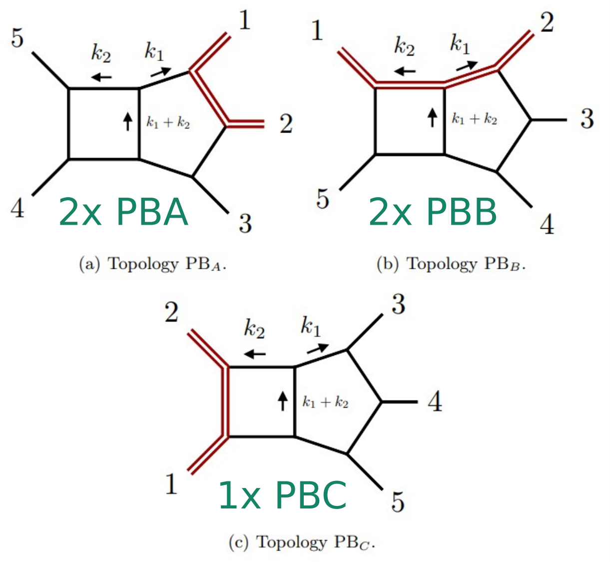
PBB MIs where ϵ^4 components are not written in terms of iterated integrals



#15 #19,20 #35,36,37

Function basis for 2-loop gg→ttg leading colour amplitude

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]



Special functions contributing to $A^{(2)}\left(1_{\bar{t}}, 2_t, 3_g, 4_g, 5_g\right)$

transcendental weight	1	2	3	4	4*	all
# of functions	6	8	45	166	12	237

polylogarithmic non-polylogarithmic

Numerical evaluation: construct DEs for special functions

$$\boxed{\mathrm{d}\vec{f}(\vec{x}) = M(\vec{x}) \cdot \vec{f}(\vec{x})} \qquad M(\vec{x}) = \sum_i A_i \,\mathrm{dlog} W_i(\vec{x}) + \sum_j B_j \,\omega_j(\vec{x})$$

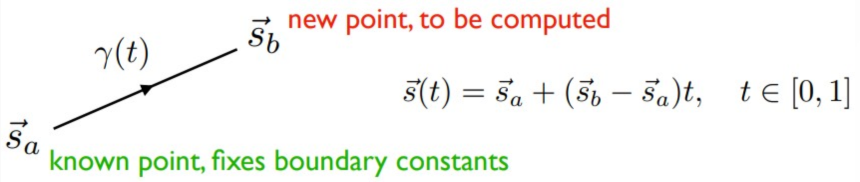
solved with generalised power series method

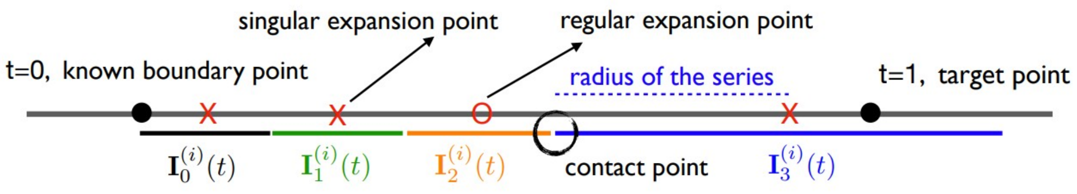
→ DiffExp[Hidding(2020)], LINE[Prisco,Tramontano(2025)]

Function basis for 2-loop gg→ttg leading colour amplitude

[Badger,Becchetti,Brancaccio,**HBH**,Zoia(2024)]

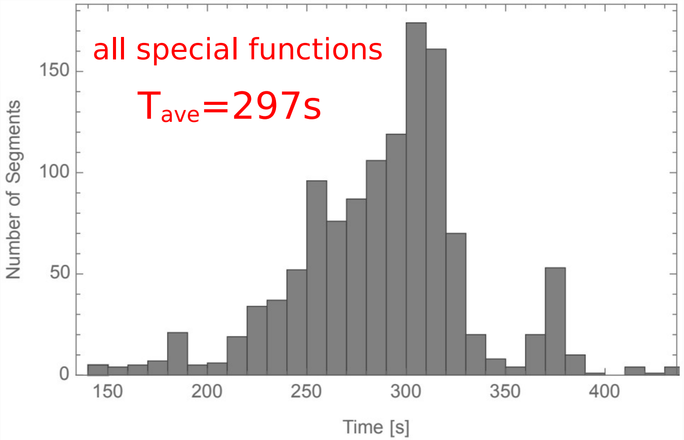
Performance analysis of numerical evaluation → study evaluation time per segment (T)

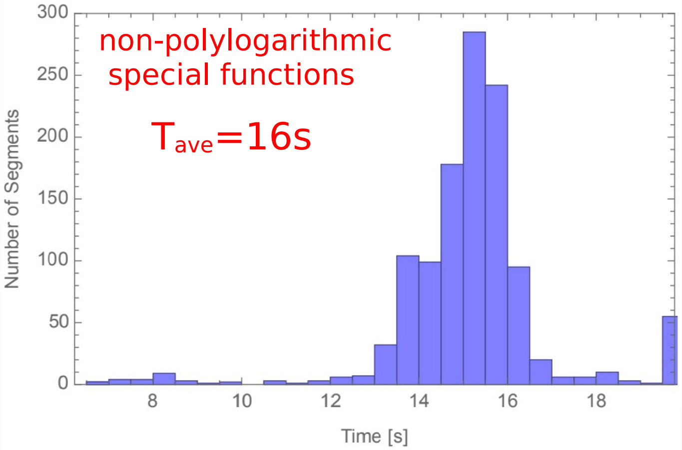




F. Morrielo, Power series and computation of multi-loop/multi-leg integrals (Zoomplitudes 2020)

~1K segments from a random sample of points (s_{45} channel)





polylogarithmic special functions can be evaluated efficiently à la PentagonFunctions++

[Gehrmann,Henn,Lo Presti(2018)]
[Chicherin,Sotnikov(2020)]
[Caron-Huot,Henn(2014)]

Helicity amplitude construction

Massive spinor prescriptions are known: [Kleiss,Stirling(1985)][Rodrigo(2005)][Arkani-Hamed,Huang,Huang(2017]

Our strategy: expand massive spinor structure in terms of form factors and fix gluon helicities
see also [Buccioni,Kreer,Liu,Tancredi(2023)]

$$\mathcal{A}^{(L)h_3h_4h_5} = \sum_{i=1}^{4} \Gamma_i \, \mathcal{G}_i^{(L)h_3h_4h_5}$$

$$\mathcal{G}_i^{(L)h_3h_4h_5} = \sum_{i=1}^{4} \left(\Gamma^\dagger\Gamma\right)^{-1}_{ij} \boxed{\left[\Gamma_j^\dagger \, \mathcal{A}^{(L)}\right]^{h_3h_4h_5}}$$

"contracted helicity amplitude"

$$\Gamma_1 = m_t^2 \bar{u}_2 v_1$$
$$\Gamma_2 = m_t \bar{u}_2 \not{p}_3 v_1$$
$$\Gamma_3 = m_t \bar{u}_2 \not{p}_4 v_1$$
$$\Gamma_4 = \bar{u}_2 \not{p}_3 \not{p}_4 v_1$$

straightforward to square the amplitude or to attach decays

$\left[\Gamma_j^\dagger \, \mathcal{A}^{(L)}\right]^{h_3h_4h_5}$ is constructed using four-dimensional projector method [Peraro,Tancredi(2019,2020)]

$$\left[\Gamma_i^\dagger \, \mathcal{A}^{(L)}\right] = \sum_{j=1}^{8} \mathcal{T}_j \, \mathcal{F}_{i;j}^{(L)}$$

$$\mathcal{F}_{i;j}^{(L)} = \sum_{j=1}^{8} \left(\mathcal{T}^\dagger\mathcal{T}\right)^{-1}_{jk} \left[\mathcal{T}_k^\dagger \Gamma_i^\dagger \, \mathcal{A}^{(L)}\right]$$

$$\mathcal{T}_1 = \varepsilon_3 \cdot p_1 \; \varepsilon_4 \cdot p_1 \; \varepsilon_5 \cdot p_1$$
$$\vdots$$
$$\mathcal{T}_8 = \cdots$$

fixing gluon helicities

$$\left[\Gamma_i^\dagger \, \mathcal{A}^{(L)}\right]^{h_3h_4h_5} = \sum_{j=1}^{8} \mathcal{T}_j^{h_3h_4h_5} \, \mathcal{F}_{i;j}^{(L)}$$

*sum over spin and polarisation states are implied

Analytic computation framework

$$A^{(2)}(\{p\},\epsilon) = \sum_i (\text{Feynman diagram})_i$$

$\downarrow$ map loop numerators to $\mathcal{I}$

$$A^{(2),h}(\{p\},\epsilon) = \sum_i c_i^h(\{p\},\epsilon)\,\mathcal{I}_i(\{p\},\epsilon)$$

$\downarrow$ IBP reduction

$$A^{(2),h}(\{p\},\epsilon) = \sum_i d_i^h(\{p\},\epsilon)\,\mathrm{MI}_i(\{p\},\epsilon)$$

$\downarrow$ map to special functions

$\downarrow$ subtract UV/IR poles

$\downarrow$ ϵ expansion

$$F^{(2),h}(\{p\}) = \sum_i \boxed{e_i^h(\{p\})}\,\mathrm{mon}_i(f) + \mathcal{O}(\epsilon)$$

QGRAF[Nogueira] FORM[Vermaseren,etal]
FiniteFlow[Peraro(2019)] LiteRed[Lee]

- Input: MIs in terms of special functions
- Compute $e_i^h(\{p\})$ numerically over finite fields
- Analytic reconstruction from numerical samples

Reconstructed in $\dfrac{\#\text{ points } \times \text{ eval. time}}{\#\text{ cores}}$

guess denominators using letters
linear relations among coefficients
univariate partial fraction decomposition

- Optimised IBP relations from **NeatIBP** [Wu,etal(2023)]
  → solved numerically over finite fields
  → improved evaluation time and RAM usage
- Rational parametrisation of external kinematics
  → momentum twistor variables $(s_{34}, t_{12}, t_{23}, t_{45}, t_{51}, x_{5123})$

$$\langle 3|1|4] = s_{34}(t_{23} - t_{45} + t_{12}x_{5123})$$

Analytic results and numerical evaluation

Analytic reconstruction summary:

	MIs	SFs	#1	#2	#3	#4	# of points
$[\Gamma_1^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	$\boxed{404/393}$	314/303	291/280	291/0	44/40	44/0	137076
$[\Gamma_2^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	398/389	30[illegible]	[illegible]	[illegible]	[illegible]	[illegible]/0	89624
$[\Gamma_3^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	411/402	32[illegible]	[illegible]	[illegible]	[illegible]	[illegible]/0	161482
$[\Gamma_4^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	420/411	326/317	304/299	304/0	58/54	56/0	179838

N/D: max polynomial degrees in the numerator/denominator

mass-renormalised amplitude all-order ϵ dependence

finite remainder

linear relations

denominator guessing

partial fraction in x_{5123} (no x_{5123} dependence)

more denominator guessing

independent helicities evaluated simultaneously

$$\vec{h} = (+++, ++-, +-+)$$

mom. twistor variables

$$(s_{34}, t_{12}, t_{23}, t_{45}, t_{51}, x_{5123})$$

Numerical evaluation of the hard functions from analytic finite remainders

$$\mathcal{H}^{(0)} = \overline{\sum} \mathcal{F}^{(0)*}\mathcal{F}^{(0)}$$

$$\mathcal{H}^{(1)} = 2\,\mathrm{Re}\overline{\sum} \mathcal{F}^{(0)*}\mathcal{F}^{(1)}$$

$$\mathcal{H}^{(2)} = 2\,\mathrm{Re}\overline{\sum} \mathcal{F}^{(0)*}\mathcal{F}^{(2)} + \overline{\sum} \mathcal{F}^{(1)*}\mathcal{F}^{(1)}$$

$$d_{12} = \frac{4602}{57095}, \quad d_{23} = \frac{217}{8151}, \quad d_{34} = -\frac{8513}{67193},$$

$$d_{45} = \frac{7}{22}, \quad d_{15} = -\frac{14291}{77626}, \quad m_t^2 = \frac{1701}{90164},$$

$$\alpha_s = 0.118$$

$$\mu_R = 17$$

$$d_{ij} = p_i . p_j$$

PRELIMINARY

	$gg \to t\bar{t}g$
$\mathcal{H}^{(0)}$	440.908
$\mathcal{H}^{(1)}$	-1377.071
$\mathcal{H}^{(2)}$	1981.615

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$[\Gamma_3^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	411/402	321/312	299/293	299/0	57/54	54/0	161482
$[\Gamma_4^\dagger \mathcal{A}^{(2),N_c^2}]^{\vec{h}}$	420/411	326/317	304/299	304/0	58/54	56/0	179838

mass-renormalised amplitude
all-order ϵ dependence

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Summary

- ✔ Address the main bottleneck in deriving pp→ttj prediction at NNLO QCD
  → two-loop five-point amplitude
- ✔ Construction of special function basis for the case of non-canonical DEs
- ✔ Derived analytic expressions for gg→ttg helicity amplitudes at two loops (leading colour)
  → functional reconstruction method from finite-field samples
  → numerical evaluation of hard functions entering NNLO QCD computation

Outlook

- ➢ Efficient numerical evaluation of special functions for a large set of phase-space points
  → large scale cluster evaluation + robust control of numerical accuracy
- ➢ Analytic expressions for the qq→ttg subprocess
- ➢ NNLO QCD cross sections

Back-up slides

Rational phase-space parametrisation

[Badger,Becchetti,Chaubey,Marzucca,Sarandrea(arXiv:2201.12188)]

$$p_1 = q_1 + q_2, \qquad p_2 = q_3 + q_4, \qquad p_3 = q_5, \qquad p_4 = q_6, \qquad p_5 = q_7,$$

$$q_1 \cdot q_2 = q_3 \cdot q_4, \qquad \langle q_2 q_5 \rangle = 0, \qquad [q_2 q_5] = 0, \qquad \langle q_4 q_5 \rangle = 0, \qquad [q_4 q_5] = 0.$$

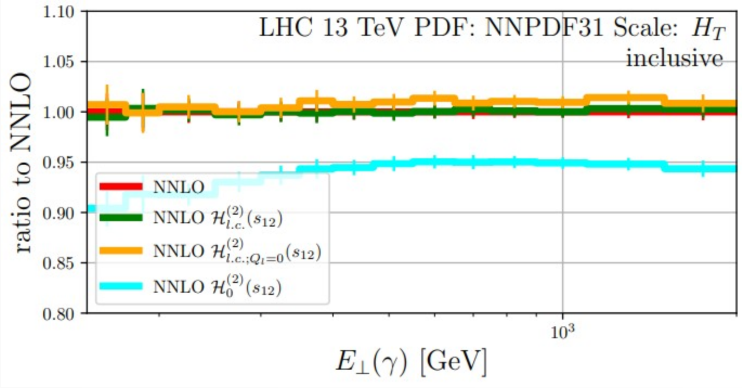
$$\begin{aligned}
s_{34} &= (p_3 + p_4)^2, \\
t_{12} &= s_{12}/s_{34}, \\
t_{23} &= (s_{23} - m_t^2)/s_{34}, \\
t_{45} &= s_{45}/s_{34}, \\
t_{15} &= (s_{15} - m_t^2)/s_{34}, \\
x_{5123} &= -\frac{\langle 5|p_1 p_{45}|3\rangle}{\langle 53\rangle s_{12}}.
\end{aligned}$$

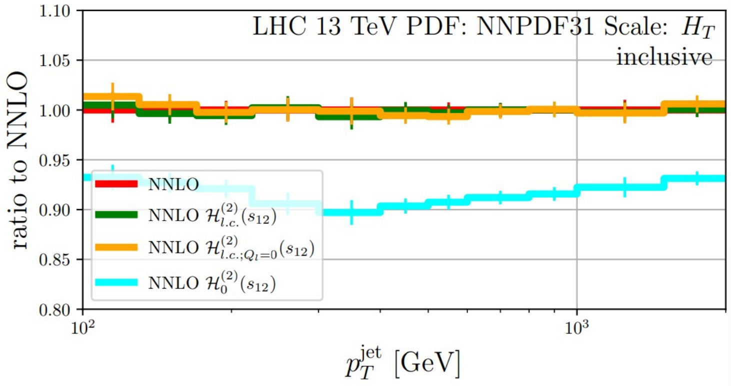
$$\begin{aligned}
d_{12} &= \frac{s_{34}t_{12}}{2t_{45}}\Big(t_{45} + 2t_{45}(-1+t_{51})x_{5123} \\
&\quad + 2t_{12}t_{45}x_{5123}^2 - 2(t_{51} + (-1+t_{12})x_{5123})(t_{23} + t_{12}x_{5123})\Big), \\
d_{23} &= \frac{s_{34}t_{23}}{2}, \\
d_{34} &= \frac{s_{34}}{2}, \\
d_{45} &= \frac{s_{34}t_{45}}{2}, \\
d_{15} &= \frac{s_{34}t_{51}}{2}, \\
m_t^2 &= \frac{s_{34}t_{12}}{t_{45}}\Big(t_{23}(t_{51} + (-1+t_{12})x_{5123}) \\
&\quad + x_{5123}(t_{45} + t_{12}t_{51} - t_{45}t_{51} + t_{12}(-1+t_{12}-t_{45})x_{5123})\Big).
\end{aligned}$$

Subleading colour contribution

pp→γjj

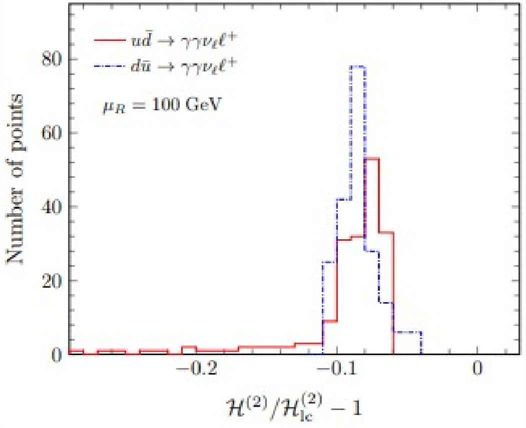
[Badger,Czakon,**HBH**,Moodie,Peraro,Poncelet,Zoia(2023)]

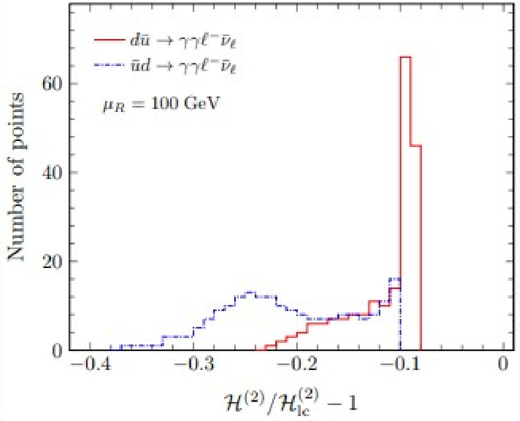




pp→Wγγ

[Badger,**HBH**,Wu,Zhang,Zoia(2024)]





Optimising analytic reconstruction

$$\mathcal{F}^{(2)} = \sum_i r_i(\{p\})\, \mathrm{mon}_i(f)$$

need to reduce the complexity of the rational coefficients!!

- set one of the kinematic variables to one ($s_{12} = 1$ or $x_1 = 1$)
- find linear relations among coefficients and reconstruct the simpler ones

$$\sum_i y_i r_i = 0,$$

we can supply ansatz →

$$\sum_i y_i r_i + \sum_j \tilde{y}_j \tilde{r}_j = 0,$$

- guess the denominators → from letters
- on-the-fly univariate partial fraction → one fewer variable, lower degrees

$$f(x,y) = \frac{y^4 + 13xy^2 + x^2}{(y-x)(y+x)^2}, \quad \text{ansatz} \Rightarrow f(x,y) = \frac{u_{110}(x)}{y-x} + \frac{u_{210}(x)}{y+x} + \frac{u_{220}(x)}{(y+x)^2} + r(x) + v_1(x)y$$

- factor matching: letters, spinor products/strings
- reconstructed expressions can be further simplifed: *MultivariateApart, pfd-parallel*