Simplicity in Amplitudes

Ruth Britto
Trinity College Dublin

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Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

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Stories of amplitudes begin in N=4 Super-Yang-Mills

- QCD with light quarks ~ gluons and gluinos. [see Dixon TASI 96]
    - At tree level, only adjust color factors
    - Supersymmetry tricks can be used in QCD!
- N=4 SYM is fascinating in its own right [see Elvang-Huang 13]
    - Many symmetries and dualities in D=4 and planar limit
    - Explore a rich theory
- It's been called the *hydrogen atom* of gauge theories

Amplitudes in planar N=4 SYM have plenty of structure

- Inspiration to tackle real-world scattering problems
- Use amplitudes to contemplate quantum field theory very broadly!

Examples of structure

- Duality with twistor string theory *[Witten]*
  - new ways to compute tree amplitudes *[Cachazo, Svrček, Witten; Roiban, Spradlin, Volovich]*
- Restricted set of master integrals
- Positive Grassmannian geometry *[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]*
- Cluster algebra *[Golden, Goncharov, Spradlin, Vergu, Volovich; Drummond, Foster, Gürdoğan]*

These ideas all transfer to other theories.

Structure → Simplicity → Shortcuts

Illustration: the Parke-Taylor formula

$$A_{\text{tree}}^{\text{MHV}}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) = i\frac{\langle jk\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}$$

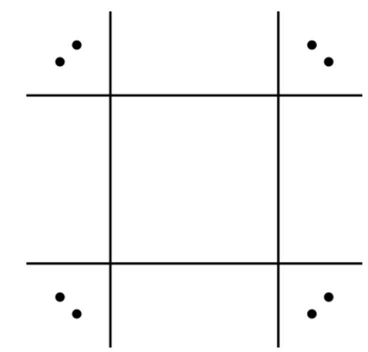
- Exposes basic singularities
- Supported on a line in twistor space
- Becomes a building block for other helicity amplitudes in on-shell methods

Back to my roots:
1-loop amplitudes
in N=4 SYM!

1 loop in N=4 SYM: Background

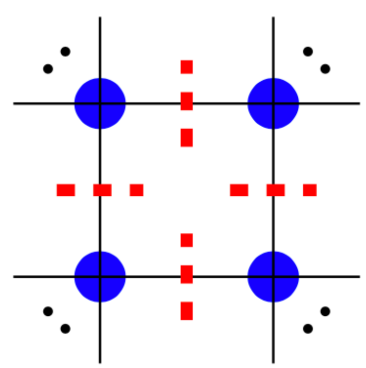
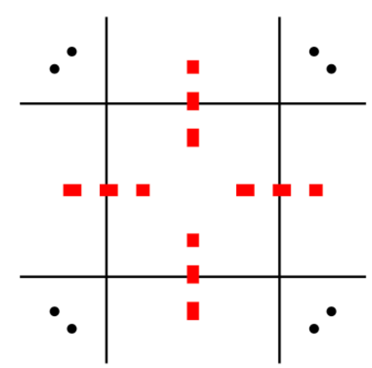
(state of the art in 2004)

- Master integrals are boxes.

$$A^{1-\text{loop}} = \sum_{\text{boxes } B} c_B$$  $$+ \mathcal{O}(\epsilon)$$

[Bern, Dixon, Kosower]

- Their coefficients are easily obtained from quadruple cuts.

 $= c_B$ 

[RB, Cachazo, Feng]

Developments in super-amplitudes

[reviewed in Elvang-Huang 13]

- Work directly in the full N=4 theory, instead of going towards QCD

- Motivated largely by new duality with Wilson loops

- New uses of dual superconformal symmetry and momentum twistors

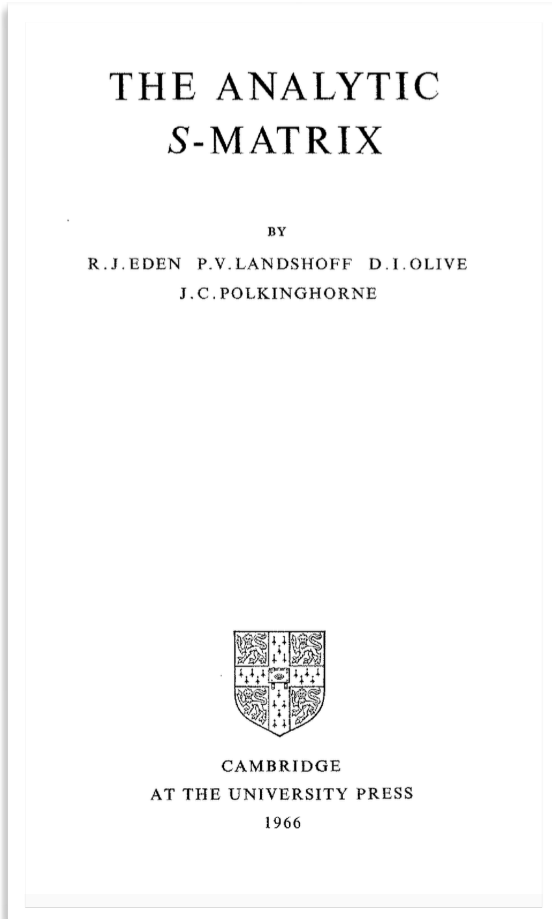
- Grassmannian geometry, amplituhedron

Recent progress

- Many more loops!
- Mainly MHV sector, some NMHV
- Master integrals: no-triangle, chiral pentagons, etc.
- IR divergences are an issue!
  - dimensional regularization, mass reg., etc.
  - regularization tends to spoil symmetries
  - one approach is to work on integrands and delay the problem

[See reviews, e.g. Snowmass ’22 by Arkani-Hamed, Dixon, McLeod, Spradlin, Trnka, Volovich; SAGEX ch. 5 by Papathanasiou]

Control the space of functions!



THE ANALYTIC
S-MATRIX

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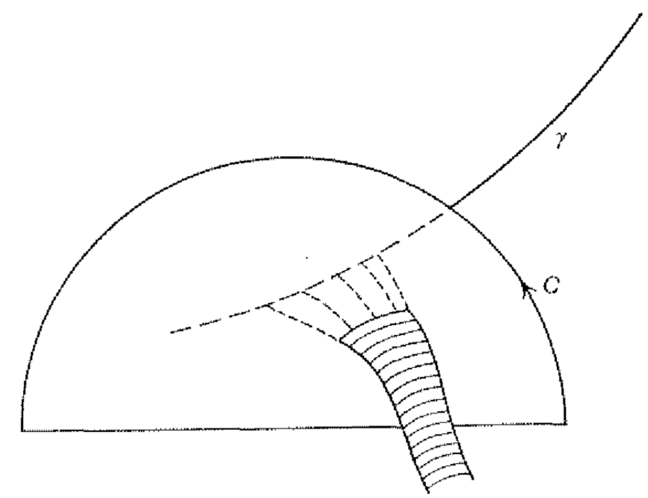


Fig. 2.6.5. A real Landau curve γ, showing that if it were singular in a certain limit its attached singular surface would be a horn of singularities penetrating the contour C, and could therefore be removed.

Analytic properties provide huge constraints — but on what functions?

This is the key idea underlying on-shell methods, like unitarity, generalized unitarity, and BCFW recursion.

Example: box integrals

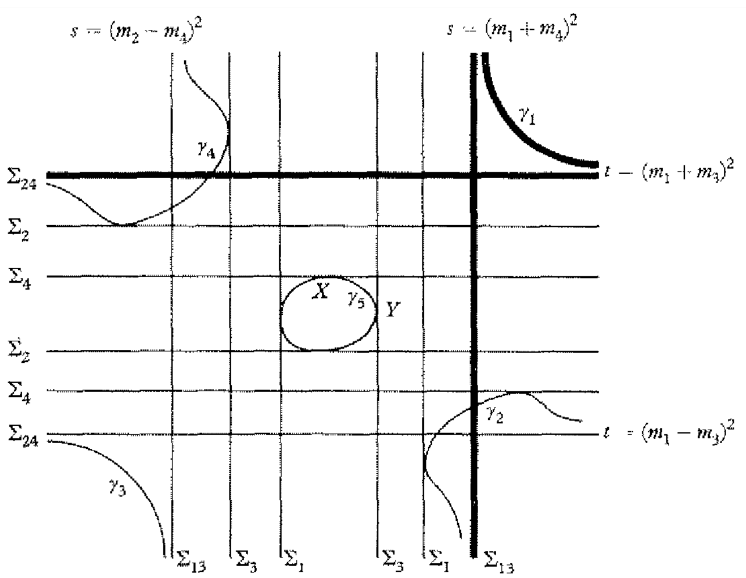
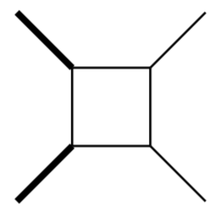


Fig. 2.4.4. The real section of the surfaces of singularity for the square Feynman graph. The curves correspond to the leading singularities, and the straight lines to lower order singularities from reduced diagrams. Heavy lines denote curves that are singular for positive α when conditions stated in the text are satisfied.



$$= \frac{2r_\Gamma}{st}\frac{1}{\epsilon^2}\left[\frac{1}{2}(-s)^{-\epsilon}+(-t)^{-\epsilon}-\frac{1}{2}(-p_1^2)^{-\epsilon}-\frac{1}{2}(-p_2^2)^{-\epsilon}\right]$$

$$-\frac{2r_\Gamma}{st}\left[-\frac{1}{2}\log\left(\frac{s}{p_1^2}\right)\log\left(\frac{s}{p_2^2}\right)+\frac{1}{2}\log^2\left(\frac{s}{t}\right)\right.$$

$$\left.+\mathrm{Li}_2\left(1-\frac{p_1^2}{t}\right)+\mathrm{Li}_2\left(1-\frac{p_2^2}{t}\right)\right]+\mathcal{O}(\epsilon).$$

[Bern, Dixon, Kosower]

- Analytic continuation still important!
- But also: different mass configurations, control of divergences
- Crucial: finding (poly)logarithms

Successes for polylogarithms

Polylogs are iterated integrals.

$$G(a_1, \ldots, a_n, z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

Symbols map them to tensor algebra.

$$dF = F_i \, d \log R_i \implies \mathcal{S}(F) = F_i \otimes R_i$$

$$\mathcal{S}[G(a,b,c)] = -\, b \otimes (a-c) + (a-c) \otimes (a-b) - (a-c) \otimes b - (b-c) \otimes (a-b) + (b-c) \otimes (a-c) - (a \otimes (a-b)) + a \otimes b + b \otimes (a-b)$$

Symbols encode relations.

$$\mathcal{S}\left[\mathrm{Li}_2(-z) + \mathrm{Li}_2(1-z) - \frac{\pi^2}{6} + \log(z)\log(z+1)\right] = z \otimes (1-z) + (1-z) \otimes z - z \otimes (1-z) - (1-z) \otimes z = 0$$

Many Feynman integrals are polylogarithmic.

Symbol bootstrap

for polylogarithmic amplitudes

1. Determine alphabet
2. Determine words
3. Determine coefficients

- Steinmann relations
- Extended Steinmann
- Cluster adjacency
- Landau singularities
- Differential equations
- Diagrammatic coaction

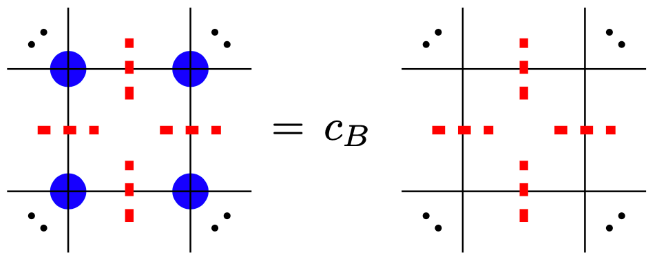
[See reviews, e.g. Snowmass '22 by Arkani-Hamed, Dixon, McLeod, Spradlin, Trnka, Volovich; SAGEX ch. 5 by Papathanasiou]

1-loop amplitudes, n legs

- MHV amplitudes have 1m and 2me boxes, coefficients all A^{tree}.
  [Bern, Dixon, Dunbar, Kosower]

- NMHV amplitudes have all boxes except 4m, coefficients are R-invariants
  [Drummond, Henn, Korchemsky, Sokatchev; Elvang, Freedman, Kiermaier;]

- Beyond NMHV, coefficients are functions of R-invariants; also pursue amplituhedron ideas sorted by helicity



- Suppose we want to use dimensional regularization.
- Suppose we want to work with arbitrary helicities.
- What is then the optimal presentation of the amplitude?

<u>Approach from two angles</u>:

1. Coaction (via symbol)
2. Geometry and cluster algebra

Divergent parts

$$\mathcal{S}[F^{1\mathrm{m}}_{i,i+1,i+2,i+3}] = -\frac{1}{\epsilon^2} + \frac{\otimes\, x^2_{i,i+2} - \otimes x^2_{i,i+3} + \otimes x^2_{i+1,i+3}}{\epsilon} + \mathcal{S}_2[F^{1m}_{i,i+1,i+2,i+3}] + \mathcal{O}(\epsilon)$$

$$\mathcal{S}[F^{2\mathrm{me}}_{i,i+1,j,j+1}] = \frac{\otimes\, x^2_{i,j} - \otimes x^2_{i,j+1} - \otimes x^2_{i+1,j} + \otimes x^2_{i+1,j+1}}{\epsilon} + \mathcal{S}_2[F^{2\mathrm{me}}_{i,i+1,j,j+1}] + \mathcal{O}(\epsilon)$$

$$\mathcal{S}[F^{2\mathrm{mh}}_{i,i+1,i+2,k}] = -\frac{1}{2\epsilon^2} + \frac{\otimes\, x^2_{i,i+2} - \otimes x^2_{i,k} + 2 \otimes x^2_{i+1,k} - \otimes x^2_{i+2,k}}{2\epsilon} + \mathcal{S}_2[F^{2\mathrm{mh}}_{i,i+1,i+2,k}] + \mathcal{O}(\epsilon)$$

$$\mathcal{S}[F^{3\mathrm{m}}_{i,i+1,j,k}] = \frac{\otimes\, x^2_{i,j} - \otimes x^2_{i,k} - \otimes x^2_{i+1,j} + \otimes x^2_{i+1,k}}{2\epsilon} + \mathcal{S}_2[F^{3\mathrm{m}}_{i,i+1,j,k}] + \mathcal{O}(\epsilon)$$

$$\mathcal{S}[F^{4\mathrm{m}}_{i,j,k,l}] = \mathcal{S}_2[F^{4\mathrm{m}}_{i,j,k,l}] + \mathcal{O}(\epsilon)$$

Universal IR behavior

$$\mathrm{Div}\, A^{1-\mathrm{loop}} = \left[-\frac{1}{\epsilon^2} \sum_r (-x^2_{r,r+2})^{-\epsilon} \right] A^{\mathrm{tree}}$$

Implies linear relations among box coefficients.

$$\mathcal{S}_2[F^{1\mathrm{m}}_{i,i+1,i+2,i+3}] = \frac{x^2_{i+1,i+3}}{x^2_{i,i+3}} \otimes \left(x^2_{i,i+3} - x^2_{i+1,i+3}\right) + \frac{x^2_{i,i+2}}{x^2_{i,i+3}} \otimes \left(x^2_{i,i+3} - x^2_{i,i+2}\right)$$
$$- x^2_{i,i+2}x^2_{i+1,i+3} \otimes x^2_{i,i+2}x^2_{i+1,i+3} + x^2_{i,i+3} \otimes x^2_{i,i+2}x^2_{i+1,i+3}x^2_{i,i+3}$$

$$\mathcal{S}_2[F^{2\mathrm{me}}_{i,i+1,j,j+1}] = \frac{x^2_{i+1,j}x^2_{i,j+1}}{x^2_{i,j}x^2_{i+1,j+1}} \otimes \left(x^2_{i,j+1}x^2_{i+1,j} - x^2_{i,j}x^2_{i+1,j+1}\right)$$
$$+ \frac{x^2_{i,j}}{x^2_{i+1,j}} \otimes \left(x^2_{i+1,j} - x^2_{i,j}\right) + \frac{x^2_{i+1,j+1}}{x^2_{i,j+1}} \otimes \left(x^2_{i,j+1} - x^2_{i+1,j+1}\right)$$
$$+ \frac{x^2_{i,j}}{x^2_{i,j+1}} \otimes \left(x^2_{i,j+1} - x^2_{i,j}\right) + \frac{x^2_{i+1,j+1}}{x^2_{i+1,j}} \otimes \left(x^2_{i+1,j} - x^2_{i+1,j+1}\right)$$
$$- x^2_{i,j} \otimes x^2_{i,j} + x^2_{i,j+1} \otimes x^2_{i,j+1} + x^2_{i+1,j} \otimes x^2_{i+1,j} - x^2_{i+1,j+1} \otimes x^2_{i+1,j+1}$$

$$\mathcal{S}_2[F^{4\mathrm{m}}_{i,j,k,l}] = \frac{x^2_{i,j}x^2_{k,l}}{x^2_{i,k}x^2_{j,l}} \otimes \frac{x^2_{i,k}x^2_{j,l} + x^2_{j,k}x^2_{i,l} - x^2_{i,j}x^2_{k,l} - \sqrt{\lambda}}{x^2_{i,k}x^2_{j,l} + x^2_{j,k}x^2_{i,l} - x^2_{i,j}x^2_{k,l} + \sqrt{\lambda}}$$
$$+ \frac{x^2_{j,k}x^2_{i,l}}{x^2_{i,k}x^2_{j,l}} \otimes \frac{x^2_{i,k}x^2_{j,l} - x^2_{j,k}x^2_{i,l} + x^2_{i,j}x^2_{k,l} - \sqrt{\lambda}}{x^2_{i,k}x^2_{j,l} - x^2_{j,k}x^2_{i,l} + x^2_{i,j}x^2_{k,l} + \sqrt{\lambda}}$$

Four types of letters.

$$
\begin{aligned}
\mathcal{S}_2[F^{1\mathrm{m}}_{i,i+1,i+2,i+3}] = {} & \frac{x^2_{i+1,i+3}}{x^2_{i,i+3}} \otimes \left(x^2_{i,i+3} - x^2_{i+1,i+3} \right) + \frac{x^2_{i,i+2}}{x^2_{i,i+3}} \otimes \left(x^2_{i,i+3} - x^2_{i,i+2} \right) \\
& - x^2_{i,i+2} x^2_{i+1,i+3} \otimes x^2_{i,i+2} x^2_{i+1,i+3} + x^2_{i,i+3} \otimes x^2_{i,i+2} x^2_{i+1,i+3} x^2_{i,i+3}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_2[F^{2\mathrm{me}}_{i,i+1,j,j+1}] = {} & \frac{x^2_{i+1,j} x^2_{i,j+1}}{x^2_{i,j} x^2_{i+1,j+1}} \otimes \left(x^2_{i,j+1} x^2_{i+1,j} - x^2_{i,j} x^2_{i+1,j+1} \right) \\
& + \frac{x^2_{i,j}}{x^2_{i+1,j}} \otimes \left(x^2_{i+1,j} - x^2_{i,j} \right) + \frac{x^2_{i+1,j+1}}{x^2_{i,j+1}} \otimes \left(x^2_{i,j+1} - x^2_{i+1,j+1} \right) \\
& + \frac{x^2_{i,j}}{x^2_{i,j+1}} \otimes \left(x^2_{i,j+1} - x^2_{i,j} \right) + \frac{x^2_{i+1,j+1}}{x^2_{i+1,j}} \otimes \left(x^2_{i+1,j} - x^2_{i+1,j+1} \right) \\
& - x^2_{i,j} \otimes x^2_{i,j} + x^2_{i,j+1} \otimes x^2_{i,j+1} + x^2_{i+1,j} \otimes x^2_{i+1,j} - x^2_{i+1,j+1} \otimes x^2_{i+1,j+1}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_2[F^{4\mathrm{m}}_{i,j,k,l}] = {} & \frac{x^2_{i,j} x^2_{k,l}}{x^2_{i,k} x^2_{j,l}} \otimes \frac{x^2_{i,k} x^2_{j,l} + x^2_{j,k} x^2_{i,l} - x^2_{i,j} x^2_{k,l} - \sqrt{\lambda}}{x^2_{i,k} x^2_{j,l} + x^2_{j,k} x^2_{i,l} - x^2_{i,j} x^2_{k,l} + \sqrt{\lambda}} \\
& + \frac{x^2_{j,k} x^2_{i,l}}{x^2_{i,k} x^2_{j,l}} \otimes \frac{x^2_{i,k} x^2_{j,l} - x^2_{j,k} x^2_{i,l} + x^2_{i,j} x^2_{k,l} - \sqrt{\lambda}}{x^2_{i,k} x^2_{j,l} - x^2_{j,k} x^2_{i,l} + x^2_{i,j} x^2_{k,l} + \sqrt{\lambda}}
\end{aligned}
$$

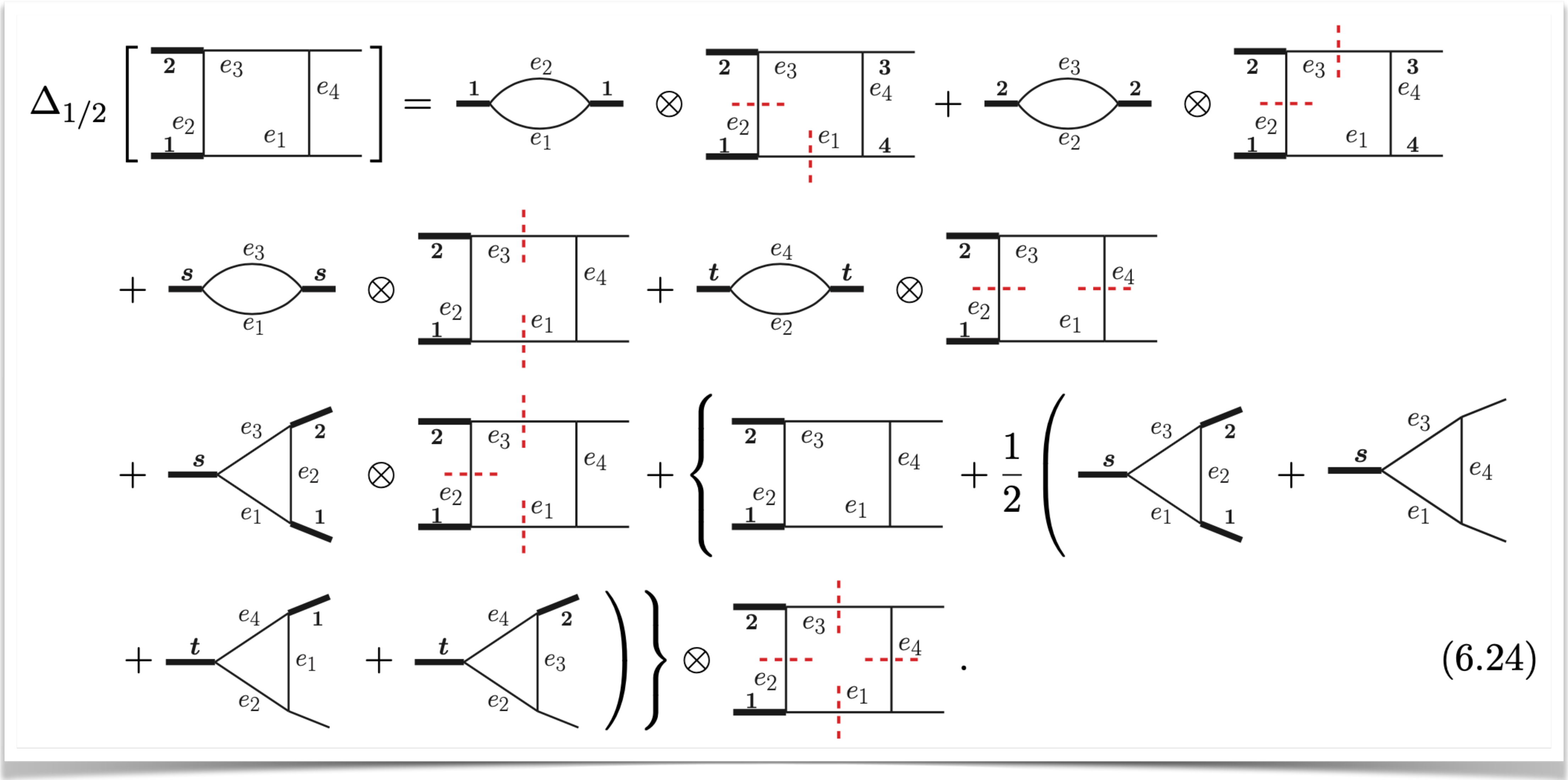
Four types of letters.

$$
\begin{aligned}
\mathcal{S}_2[F^{2\mathrm{mh}}_{i,i+1,i+2,k}] = {} & \frac{x^2_{i+1,k}}{x^2_{i,k}} \otimes \left(x^2_{i,k} - x^2_{i+1,k}\right) + \frac{x^2_{i+1,k}}{x^2_{i+2,k}} \otimes \left(x^2_{i+2,k} - x^2_{i+1,k}\right) \\
& -2x^2_{i+1,k} \otimes x^2_{i+1,k} - \frac{1}{2}x^2_{i,i+2} \otimes x^2_{i,i+2} + \frac{1}{2}x^2_{i+2,k} \otimes x^2_{i+2,k} + \frac{1}{2}x^2_{i,k} \otimes x^2_{i,k} \\
& -x^2_{i+1,k} \otimes x^2_{i,i+2} - x^2_{i,i+2} \otimes x^2_{i+1,k} + x^2_{i,k} \otimes x^2_{i+1,k} + x^2_{i+2,k} \otimes x^2_{i+1,k} \\
& +\frac{1}{2}x^2_{i+2,k} \otimes x^2_{i,i+2} - \frac{1}{2}x^2_{i+2,k} \otimes x^2_{i,k} + \frac{1}{2}x^2_{i,k} \otimes x^2_{i,i+2} \\
& +\frac{1}{2}x^2_{i,i+2} \otimes x^2_{i+2,k} - \frac{1}{2}x^2_{i,k} \otimes x^2_{i+2,k} + \frac{1}{2}x^2_{i,i+2} \otimes x^2_{i,k}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}_2[F^{3\mathrm{m}}_{i,i+1,j,k}] = {} & \frac{x^2_{i,k}x^2_{i+1,j}}{x^2_{i,j}x^2_{i+1,k}} \otimes \left(x^2_{i,k}x^2_{i+1,j} - x^2_{i,j}x^2_{i+1,k}\right) + \frac{x^2_{i,j}}{x^2_{i+1,j}} \otimes \left(x^2_{i+1,j} - x^2_{i,j}\right) + \frac{x^2_{i+1,k}}{x^2_{i,k}} \otimes \left(x^2_{i,k} - x^2_{i+1,k}\right) \\
& -\frac{1}{2}x^2_{i,j} \otimes x^2_{i,j} + \frac{1}{2}x^2_{i,k} \otimes x^2_{i,k} - \frac{1}{2}x^2_{i+1,k} \otimes x^2_{i+1,k} + \frac{1}{2}x^2_{i+1,j} \otimes x^2_{i+1,j} \\
& +\frac{1}{2}x^2_{i,j} \otimes x^2_{i,k} + \frac{1}{2}x^2_{i,j} \otimes x^2_{j,k} - \frac{1}{2}x^2_{i,k} \otimes x^2_{i,j} - \frac{1}{2}x^2_{i,k} \otimes x^2_{j,k} + \frac{1}{2}x^2_{j,k} \otimes x^2_{i,j} - \frac{1}{2}x^2_{j,k} \otimes x^2_{i,k} \\
& -\frac{1}{2}x^2_{i+1,j} \otimes x^2_{i+1,k} - \frac{1}{2}x^2_{i+1,j} \otimes x^2_{j,k} + \frac{1}{2}x^2_{i+1,k} \otimes x^2_{i+1,j} \\
& +\frac{1}{2}x^2_{i+1,k} \otimes x^2_{j,k} - \frac{1}{2}x^2_{j,k} \otimes x^2_{i+1,j} + \frac{1}{2}x^2_{j,k} \otimes x^2_{i+1,k}
\end{aligned}
$$

Coaction: algebraic structure

[Brown]



[Abreu, RB, Duhr, Gardi]

Contains all symbol information.

What about physical amplitudes?

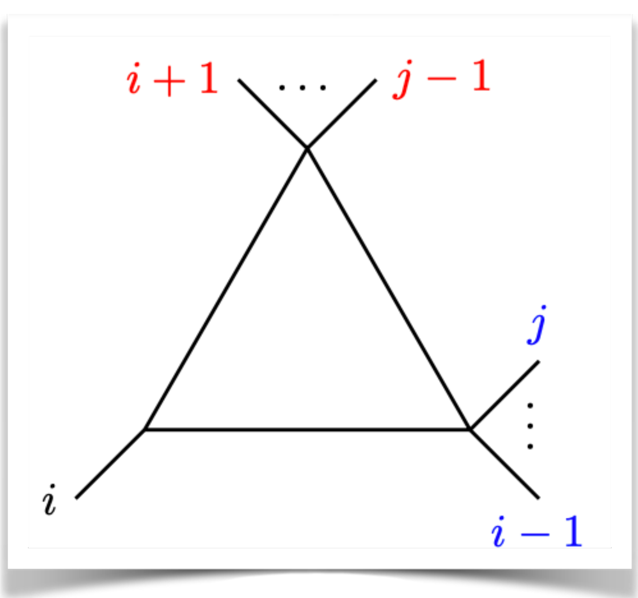
Combine box symbols with their coefficients

- First entries are Mandelstam invariants x_{ij}^2
- Second entries are of the form x_{ij}^2, $x_{ij}^2 - x_{i+1,j}^2$, $x_{ij}^2 x_{kl}^2 - x_{ik}^2 x_{jl}^2$ (except 4-mass)
- The letters $x_{ij}^2 x_{kl}^2 - x_{ik}^2 x_{jl}^2$ come only with unique boxes and their coefficients. Same for 4-mass.
- The letters $x_{ij}^2 - x_{i+1,j}^2$ drop out in the sum.
- Remaining symbol words at weight 2 are of the form $x_{ij}^2 \otimes x_{kl}^2$. What are their coefficients?

[Angelopoulou, RB, Parisi in preparation]

Absence of $x_{ij}^2 - x_{i+1,j}^2$

- Not cluster variables—but cluster structure not proven in general
- Not compatible with dual superconformal symmetry—but we're in full dim reg *[see Prlina, Spradlin, Stankowicz, Stanojevic, Volovich]*
- Bottom-up proof from BCFW-type momentum shift, for relation found from dual superconformal considerations *[Brandhuber, Heslop, Travaglini; Angelopoulou, RB, Parisi]*

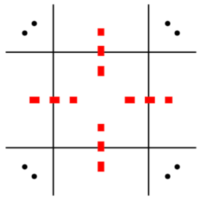


$$\sum_{k=i+2}^{j-1} c_{i,i+1,k,j} = \sum_{l=j+1}^{i-1} c_{i,i+1,j,l}$$

Geometric interpretation as two dissections of boundary of loop amplituhedron (NMHV, conj. beyond)

Coefficients of $x_{ij}^2 \otimes x_{kl}^2$

- For 4 distinct indices: either 0 or a single box c_{ijkl}
- For 2 distinct indices, $x_{ij}^2 \otimes x_{ij}^2$: same as max cut, or exactly 1/2 max cut in the case of 2-particle invariants
    - Max cut means the contribution from the leading Landau singularity
    - It is useful to separate this part.



- For 3 distinct indices: generically combination of 3m coefficients $\to$ 0 for MHV, 2 R-invariants for NMHV; exposes cluster adjacency

Cluster adjacency

- A principle for N=4 SYM amplitudes with various conjectured manifestations *[Golden, Goncharov, Spradlin, Vergu, Volovich; Drummond, Foster, Gürdoğan; many others]*

- Is the amplitude assembled from “cluster variables” in patterns constrained by cluster algebra?

- We are exploring a conjecture by Gürdoğan and Parisi, based on amplituhedron, of cluster adjacency of poles of coefficients and max-cut Landau singularities—and more generally, adjacency of all the letters and coefficients term by term.

- Checked now for NMHV beyond max-cut. The coefficients of extra terms involve two R-invariants, exposing cluster properties clearly. Prospects beyond NMHV.

What’s the best basis?

- A very general question.
- We have specialized bases for N=4 SYM.
- In dim reg, we look for canonical bases.
- Diagrammatic coaction uses diagrams in different dimensions.
- In our 1-loop calculation here, we like boxes, but could omit the parts with symbol letters that drop out in the end.
- Are there other answers? Open question for diagrammatic coaction.

Summary

- Planar N=4 SYM continues to claim prime importance in the field of amplitudes.
- There is much more to understand even at 1 loop!

Future directions

- Formulate and use diagrammatic coaction beyond 1-loop integrals; more loops and/or full amplitudes
- Seek relations beyond polylogarithms - understand **how** to control the function space
- Seek cluster algebra and amplituhedron geometry for all helicities
- As always: apply lessons of N=4 SYM to other theories!