Excursion-set approach for Primordial Black Holes: small-scale clustering and merger rates

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Spatial clustering

Constraints on PBHs formations

- Microlensing
- Cosmic Microwave Background
- Limits to their merger rates

Effect of (sizable) clustering

- Change merger rates of PBH binaries
- Modify the formation of cosmological structures
- Relax bounds set by CMB and microlensing

$\longrightarrow$ Evolution of PBH clustering involves complicated non-linear dynamics, the initial amount of clustering can in principle produce drastic effects on the subsequent evolution.

$\longrightarrow$ No consensus (yet) on the subject

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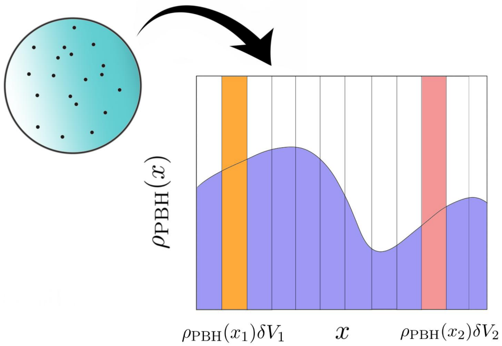
Study of two-point correlations

- Poisson model and bias theory
- Press-Schechter formalism
- Peak theory
- Excursion-set formalism

In this talk:

- Focus on large Gaussian density fluctuations
- Derive explicit expressions for the initial two-point statistics of PBHs...
- using the excursion-set formalism...
- in order to account for "cloud-in-cloud"...
- and exclusion effects at short separation scales

- Random Poisson point process
- Continuum limit $\longrightarrow$ one postulates the existence of a field $\rho_{\rm PBH}(\mathbf{x})$



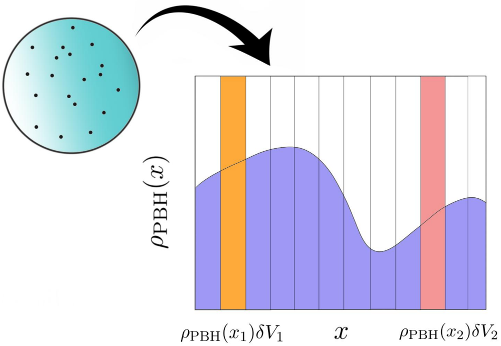
Characterizing clustering: Poisson model and bias theory

- Random Poisson point process
- Continuum limit $\longrightarrow$ one postulates the existence of a field $\rho_{\mathrm{PBH}}(\mathbf{x})$
- Joint probability to form PBHs:

$$\delta P = \rho_{\mathrm{PBH}}(\boldsymbol{x}_1)\delta V_1 \rho_{\mathrm{PBH}}(\boldsymbol{x}_2)\delta V_2$$

- Auto-correlation function

$$\xi_{\mathrm{PBH}}(r) = \left\langle \frac{[\rho_{\mathrm{PBH}}(\boldsymbol{x}+\boldsymbol{r}) - n][\rho_{\mathrm{PBH}}(\boldsymbol{x}) - n]}{n^2} \right\rangle$$



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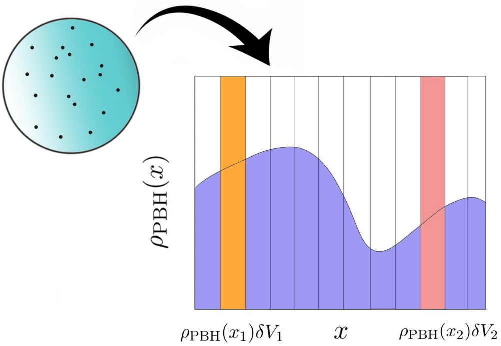
$$\xi_{\rm PBH}(r) = \left\langle \frac{[\rho_{\rm PBH}(\boldsymbol{x}+\boldsymbol{r}) - n][\rho_{\rm PBH}(\boldsymbol{x}) - n]}{n^2} \right\rangle$$

- Average joint probability to form PBHs

$$\delta P(r) = n^2 \left[1 + \xi_{\rm PBH}(r)\right] \delta V_1 \delta V_2$$



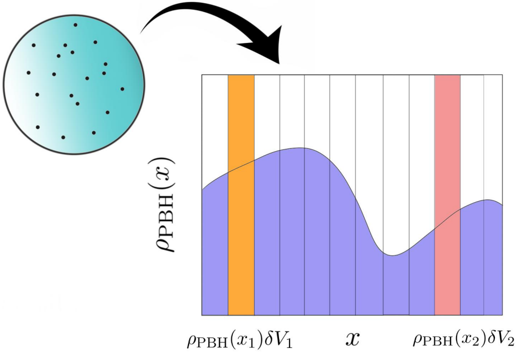
Characterizing clustering: Poisson model and bias theory

Technical difficulties

- A perturbation theory for the density field $\delta = \delta\rho/\rho$ after inflation
- A bias b relating $\rho_{\rm PBH}$ to δ

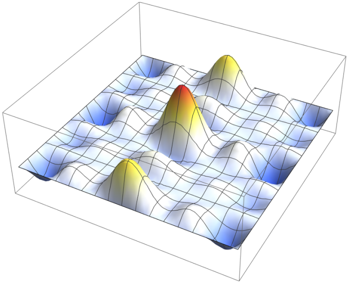
Limitations

- "Cloud-in-cloud"
- "Pointlike" treatment $\longrightarrow$ short range behaviour
- Estimating the bias is difficult



Characterizing clustering: "Press-Schechter"-inspired [Ali-Haïmoud, 2018]

- A Gaussian field $\delta(\boldsymbol{x})$
- A threshold of formation $\delta_{\rm c}$



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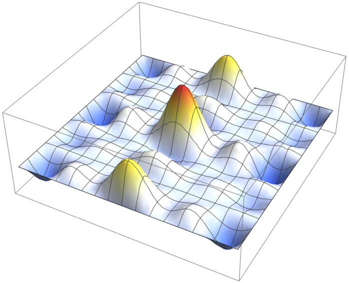
Characterizing clustering: “Press-Schechter”-inspired [Ali-Haïmoud, 2018]

- A Gaussian field $\delta(\boldsymbol{x})$
- A threshold of formation δ_c
- Probability to for a PBH around $\boldsymbol{x}_1$:

$$P_1 = \int_{\delta_c}^{\infty} \frac{e^{-\delta^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}\,\mathrm{d}\delta = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\delta_\mathrm{c}}{\sqrt{2}\sigma}\right)$$



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- Probability to form a pair of PBHs

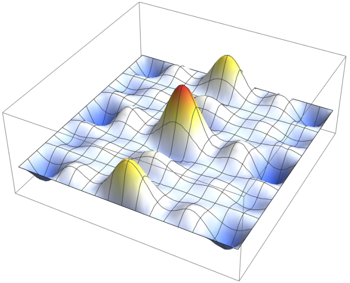
$$P_2(r) = \int\int_{\delta_c}^{\infty} \frac{\mathrm{d}^2\boldsymbol{\delta}}{2\pi \det \Sigma} e^{-\frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}\Sigma\boldsymbol{\delta}}$$

with

$$\Sigma = \begin{pmatrix} \sigma^2 & S_r \\ S_r & \sigma^2 \end{pmatrix}$$

- Auto-correlation function

$$1 + \xi_{\mathrm{PBH}}(r) = \frac{P_2(r)}{P_1^2}$$



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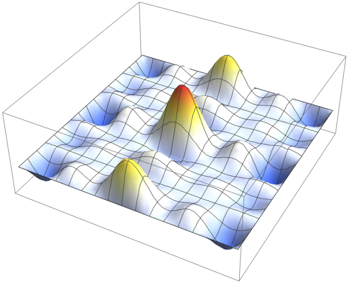
Characterizing clustering: “Press-Schechter”-inspired [Ali-Haïmoud, 2018]

Technical difficulties

- A perturbation theory for the density field δ
- An estimation of the density threshold δ_c (based on numerical relativity and/or analytical considerations)

Limitations

- "Cloud-in-cloud"
- Short-range behaviour: no natural description of the exclusion effects



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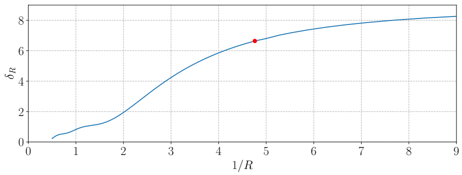
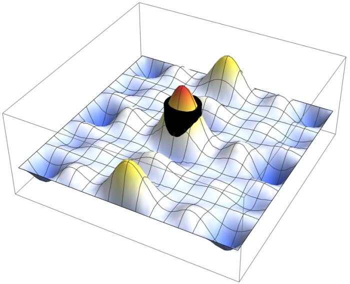
Coarse-graining procedure and "excursion-set" model





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Excursion-set formalism [Bond et al. 1991]

- Multi-scale analysis $\longleftrightarrow$ stochastic trajectories for δ_R
- Gravitationally bound structures $\longleftrightarrow$ crossing of the threshold $\delta_c(R)$
- PBHs $\longleftrightarrow$ first-crossing problem P_{FPT}
- No "cloud-in-cloud" issues $\longleftrightarrow$ structures hierarchy are taken into account

- Langevin equation for δ_{R}

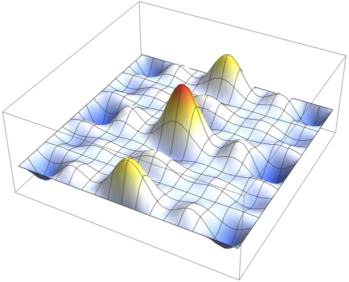
$$\frac{\mathrm{d}\delta_{\mathrm{R}}}{\mathrm{d}S} = \eta(S)$$

- in terms of of the variance S

$$S(R) = \int_0^\infty \mathcal{P}_\delta(k) W^2(k, R)\, \mathrm{d}\ln k \propto 1/R$$

Joint probability in excursion-set

- Consider a realization of $\delta = \delta\rho/\rho$ and 2 points separated by r
- On scales $\gg r$, one sees $\sim$ same perturbation
- On scales $\ll r$, one sees independent perturbations



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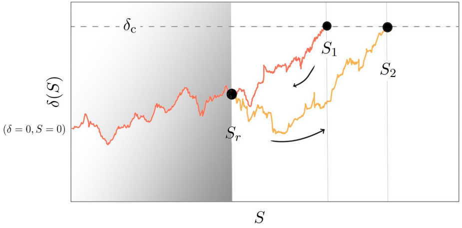

- Translation into Langevin trajectories
- Joint probability to form a pair of PBHs with masses S_1, S_2

$$\mathcal{P}(S_1, S_2; r) = \int_{-\infty}^{\delta_c(S_r)} \mathrm{d}\delta_r \, P(\delta_r, S_r) P_{\mathrm{FPT}}(S_1|\delta_r, S_r) P_{\mathrm{FPT}}(S_2|\delta_r, S_r)$$

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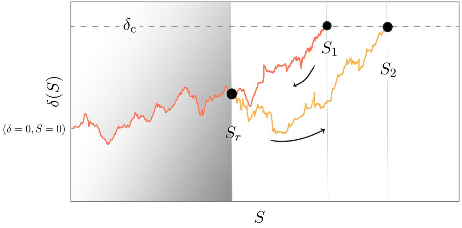

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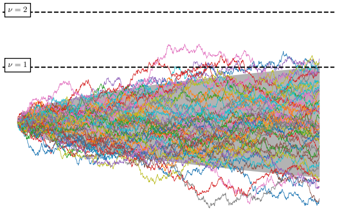
- Common past $\longrightarrow$ correlations due to individual structures embedded in larger super-structures

- Probability to form one PBH

$$P_1 = n = \mathrm{erfc}\left(\nu/\sqrt{2}\right), \qquad \nu \equiv \delta_c/\sigma$$



Auto-correlation function (scale-invariant threshold δ_c) [Auclair, Blachier 2024]

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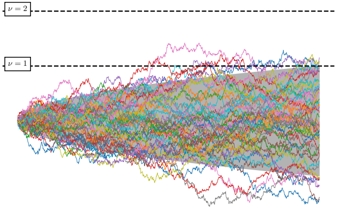

- Marginalized joint probability

$$P_2(r) = \int \int_{S_r}^{\sigma^2} \mathcal{P}(S_1, S_2; r)\, \mathrm{d}S_1\, \mathrm{d}S_2$$

- Auto-correlation function

$$1 + \xi_{\mathrm{PBH}}(r) = P_2(r)/P_1^2$$

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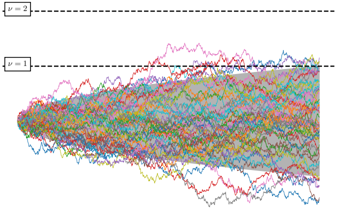

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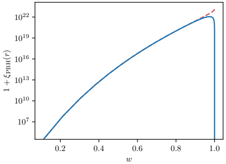
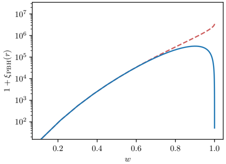
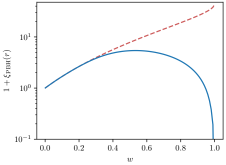
$$w \equiv S_r/\sigma^2 \sim \begin{cases} 0 & \text{for} \quad r \to \infty \\ 1 & \text{for} \quad r \to 0 \end{cases}$$



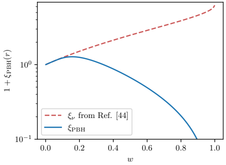
(a) $\nu = 10$



(b) $\nu = 2$



(c) $\nu = 2$



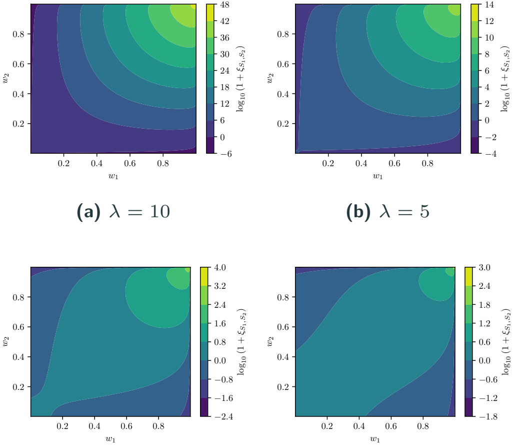
(d) $\nu = 1$

Pairwise correlation function [Auclair, Blachier 2024]

- Consider two PBHs with masses M_1, M_2
- Excess probability to find them at distance r

$$1 + \xi_{S_1 S_2} = \frac{\mathcal{P}(S_1, S_2; r)}{P_{\mathrm{FPT}}(S_1) P_{\mathrm{FPT}}(S_2)}$$

- Study of the correlations directly at the level of pairs of PBHs with specified mass
- $w_i = S_i / S_r$



(a) $\lambda = 10$ **(b)** $\lambda = 5$

(c) $\lambda = 2$ **(d)** $\lambda = 1$

Intermediate summary

What does the excursion-set formalism probe?

- PBH collapse $\longrightarrow$ first-passage time of a Langevin trajectory
- Clustering $\longrightarrow$ joint first-passage times of two trajectories with common past
- Effective exclusion effects on small scales $\xi_{\mathrm{PBH}} \to -1$, and cloud-in-cloud
- Access to the probability distribution of the mass-ratio

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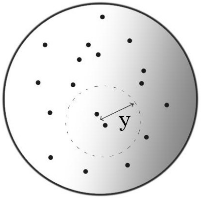
How it can be further used?

- From "abstract" $\longrightarrow$ actual physical scenarii (typical clustering length, etc)
- From "initial clustering" $\longrightarrow$ estimation of the merger rates of PBHs binaries

$$\mathrm{d}\mathcal{R} = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\boldsymbol{L}\,\mathscr{P}\left(x, y; m_1, m_2; \boldsymbol{L}\right) \delta^D\left[t - \tau(r_a(x), L)\right]$$

Setup: two-body channel

- Binaries of mass M_1, M_2 separated by x
- Within a volume of radius y devoid of any other PBHs
  $\longrightarrow$ evolve as a genuine 2-body configuration
- $+$ tidal torques from other PBHs/matter inhomogeneities
  $\longrightarrow$ provides angular momentum $\boldsymbol{L}$ to the binary



Angular momentum

- Incorporating excursion-set within "Tidal Torque Theory" (TTT) [Peebles, 1969]

$$\boldsymbol{L}_{\rm R}(\boldsymbol{x}) = \bar{\rho} a^4 \int_{V(R)} {\rm d}^3\boldsymbol{x}\, [1 + \delta(\boldsymbol{x})]\, (\boldsymbol{x} - \boldsymbol{x}_{\rm cm}) \times \boldsymbol{u}(\boldsymbol{x})$$

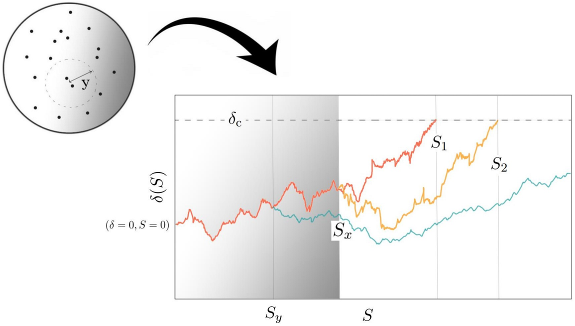
- The joint probability factorizes [Susa, Saski & Tanaka, 1994]

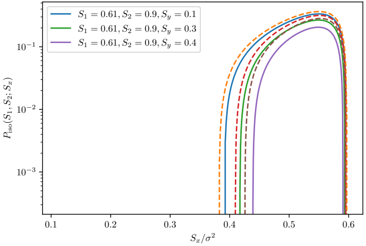
$$P(\delta_R, \boldsymbol{L}_R) = P(\delta_R) P(\boldsymbol{L}_R)$$

- Dipolar approximation $y \gtrsim x$
- $P_{\rm iso}$ = probability of forming "isolated" binaries

$$P_{\rm iso}(S_1, S_2; x, y) = 1 + \xi_{S_1 S_2}(x) - F(\mathcal{W}, w_{\rm tot}, x, y)$$

- Relevance of the pairwise correlation function





Conclusion

Summary

- PBH collapse $\longrightarrow$ first-passage time of a Langevin trajectory
- Clustering $\longrightarrow$ joint first-passage times of two trajectories with common past
- Effective exclusion effects on small scales $\xi_{\mathrm{PBH}} \to -1$, and cloud-in-cloud
- Access to the probability distribution of the mass-ratio
- Sketch of a possible refinement of the current computation of merger rates

Possible future directions

- From "abstract" $\longrightarrow$ actual physical scenarii
- Possibility of letting the threshold $\delta_{\mathrm{c}}(S)$ scale dependent
- Beyond Gaussian initial conditions $\longrightarrow$ Stochastic trees *with C. Animali, P. Auclair, V. Vennin*
- Weakly non-Gaussian $\longrightarrow$ excursion-set with coloured noise

Explicit formulae

- Pairwise correlation function

$$1+\xi_{12}(x)=\frac{\mathrm{e}^{\lambda^2(w_{\rm tot}-1)}}{(1-w_1w_2)^{3/2}\lambda}\sqrt{\frac{\mathcal{W}}{\pi}}\left[1+\sqrt{\pi\mathcal{W}}\lambda\left(\frac{1}{2\lambda^2\mathcal{W}}+1\right)e^{\lambda^2\mathcal{W}}\,\mathrm{erf}\left(\lambda\sqrt{\mathcal{W}}\right)\right]$$

with

$$w_{\rm tot}\equiv w_1+w_2 \quad \text{and} \quad \mathcal{W}\equiv\frac{(1-w_1)(1-w_2)}{1-w_1w_2}$$

- Marginalized correlation function

$$1+\xi_{\rm PBH}(r)=\frac{1}{n^2}\frac{\sqrt{2}\,\mathrm{e}^{-\frac{\nu^2}{2w(r)}}}{\sqrt{\pi w(r)}}\int_0^\infty \sinh\left(\frac{\nu}{w(r)}x\right)\mathrm{erfc}^2\left[\frac{x}{\sqrt{2(1-w(r))}}\right]\mathrm{e}^{-\frac{x^2}{2w(r)}}\,\mathrm{d}x$$

- Probability of forming "isolated" binaries

$$P_{\rm iso}=1+\xi_{12}(x)-\frac{4}{\sqrt{\pi}}\frac{e^{\lambda^2(w_{\rm tot}-1)}}{\lambda^2\left[\mathcal{W}(1-w_1w_2)\right]^{3/2}}\int \mathrm{d}u\,u^2e^{-u^2/\mathcal{W}}\Big\{e^{2\lambda u}T\left[(u+\Upsilon)\,Q_{xy}^{(1)},Q_{xy}^{(2)}\right] - e^{-2\lambda u}T\left[(u-\Upsilon)\,Q_{xy}^{(1)},Q_{xy}^{(2)}\right]\Big\}.$$

with $T(z,a)=$ Owen's T-function