New Horizons in Primordial Black Hole Physics

Cogenesis of baryon and dark matter from ultra-light Primordial Black Hole

Nayan Das

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Collaborator : Debasish Borah

Department of Physics
Indian Institute of Technology Guwahati

Motivation

Cōɖǘ Ãōŋ̨Yɖ

ÅÁ Yg̈ɋYžƆY ÁƲ Ĺ$Ãõ ęÁ Ʋōɖ

CÁYę CÃ ̈žŋ̨Yɖō Ɔŋ̨ Áž̈ɨ ưɖōg̈ŋ̨ōŋ̨̈Áž̈ōɨ̈ɋ

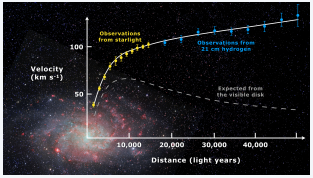
õɖÁɋΛƆŋ̨̈Áž̈ ōƆ f ɖōg̈ŋ̨ɨЉ ɋ Yɋɨōŋ̨Yɋ ęƆōŋ̨̈Yɖ̈žưɔ
bƆ ;ɨ̈ōƆǘ nÁɨ̈Y ōę CÃɔ
ƆƆ KgōĦÁɖōŋ̨̈Áž ÁƲ ; nɔ

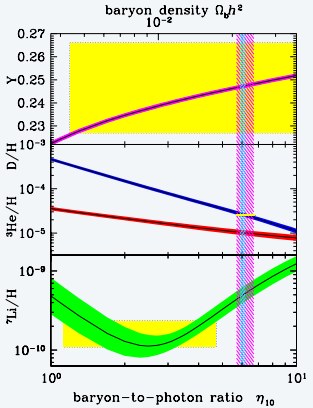
Ãōŋ̨Yɖ ōžŋ̨̈Љɋōŋ̨Yɖ ōęɨ ɋ ɋ Yɳɖɨ

ÅÁ Yg̈ɋYžƆY ÁƲ ōžŋ̨̈Љɋōŋ̨Yɖ ƲɖÁɋ ƆÁę ɋ ̈ƆЉɖōɨęʔ
ưō ɋ ɋ ōЉɖōɨęɔ

ÃYƆpōž̈ę ɋ ʔ ¹YĦŋ̨Áưžɣę̈ęʔ ɹ YƆǘЉC̈žY YɳƆɔ

øÁɨ̈Y ÁƲ ; n ƲÁɖ ɳpŸɖ ƆÁ ɋ ɋÁž Áɖ̈ưž̈ɋ





Ĥɋư ɋōŋō ưɖÁΛĦ

Primordial Black Hole

õ ; n <ōžˇ ŮÁḍ̨ǭ ǫΛϒ ɳ̨Á

ōϽ ƆÁ̈̀ōĤęϒ ÁƯ Ȟḍ̈́ǭ Áḍǫ̈́ōù Ágϒḍǫϒžę̈́ɳ̨̈́ϒę

ƀϽ ƆÁ̈̀ōĤęϒ ÁƯ ̈́žɻōɳ̨Áž ɹϒù̇ǫę

ƆϽ ƆÁ̈̀ōĤęϒ ÁƯ ɳ̨ÁĤÁ̈̀ÁưΌōù ǫϒƯϒƆɳ̨ę

ǫϽ ƆÁ̈̀ōĤęϒ ŮḍÁǭ ƀΛƀƀùϒ ƆÁ̈̀ę̈́Ážϒɳ̨Ɔɜ

nϒḍϒ? ɢϒ ōḍϒ ōưžÁęɳ̨̈́Ɔ ōƀÁΛɳ̨ŮÁḍ̨ǭ ōɳ̨̈́Áž ǭ ϒƆpōž̈́ę ǭɜ

N Ůḍϒϒ Ȟōḍ̨ō ǭ ϒɳ̨ϒḍę

ōϽ õ ; n ̈́žɳ̨̈́ōù ǭ ōęę? $\tilde{A}_{_j}$ɜ

ƀϽ sžɳ̨̈́ōù ŮḍōƆɳ̨̈́Ážōù ϒžϒḍưɨ ǫϒžę̈́ɳ̨ɨ? $\frac{?\,[\,(\check{e}_{_j})}{\phi(\check{e}_{_j})}$ɜ

Ĺϒ ƆÁžę̈́ǫϒḍ ǭÁžÁƆpḍÁǭ ōɳ̨̈́Ɔ ǭ ōęę ǫ̈́ęɳḍ̈́ƀΛɳ̨̈́Áž ōžǫ ɲϒḍÁ ęȞ̈́žɜ

õ ; n Ɔōžˇ Ȟù̇ōɨ ̈́žɳ̨ϒḍϒęɳ̨̈́žư ḍÁù̇ϒ ̈́žˇ ȞḍÁǫΛƆɳ̨̈́Áž ÁƯ ƆÁưϒžϒę̈́ęɜ SKǒKNɜKΔ Ï Δ Ï? NKΔTɜKTTγ Ï Ï

Evaporation of PBH

UnYd UÁdQ ōnÁž õ ; n Y Q nę ĎōdnCōYę QΛY nÁ ÷Ã YωYCn

ĈY Q <ōęęCōō KωYCn

nō cúžư YgōĎÁdōnÁž $\frac{\varrho^2 \mathring{A}}{\varrho \eta \varrho K} = \frac{1}{2} \frac{(K; \tilde{A}_{?\,[}; \varrho)}{\sqrt{Y^{\frac{K}{\tilde{A}^2_{\delta} / \tilde{A}_{?\,[}}}} \quad (\ 1)^{2\varrho}}$;

nō cúžư nY Q ĎYdōnΛdY $\check{e}_{?\,[} = \frac{\tilde{A}^2_{\delta}}{\tilde{A}_{?\,[}}$ $(K;\ \tilde{A}_{?\,[}; \varrho)$! fdYibÁQi UōCnÁdę sžnYudōnžư cdnK

$$\frac{\varrho \tilde{A}_{?\,[}}{\varrho \eta} = \frac{\tilde{A}^4_{\delta}}{\tilde{A}^2_{?\,[}}; \qquad = \frac{27}{4} \frac{u'_{;n}(\check{e}_{?\,[})}{480} \qquad \text{CKC}$$

Ãōęę YgÁiΛnÁž

$$\tilde{A}_{?\,[}(\eta) = \tilde{A}_{\_j} \quad 1 \quad \frac{3}{\tilde{A}^3_{\_j}} \frac{\tilde{A}^4_{\delta}}{(\eta \quad \eta_{\_j})}^{1/3} : \qquad \text{CNC}$$

Evaporation of PBH

nÁɢ ɣgɣɣ℘ Ĉ< ōęęʌ ǫ Ȟɳ℔Áʒę ǫ ōɨ ʒÁɳ bɣ ęɣ℔Ư ƆÁʒę℔ęɳɣʒɳ℘
sɳ℔ưʒÁɗɣę bōƆΰ℔ɗɣōƆɳ℔Áʒ ÁƯ ɣ ǫ ɨ℔ęę℔Áʒę Áʒ ɳpɣ ɳʌōʒɳʌ ǫ ęɳōɳɣ ÁƯ ; nɜ
Ĉɣ ǫ ℔ƄƆ℔ōęę℔Ɔō℔ ōęęʌ ǫ Ȟɳ℔Áʒ ɨ℔ę gō℔ǫ ōȞȞɗÁɲ℔ǫ ōɳɣ℔ɨ $f\tilde{A}_{\_j}; \eta\tilde{A}_{\_j} g_{\chi}\eta$ 0:5ɜ

Ãɣ ǫ Áɗɨ℔; Ʌɗǫɣʒ KɷɣƆɳ

øɣƆɣʒɳ ęɳʌǫ℔ɣę ęʌưưɣęɳ ɳpōɳ℔ʒƯÁɗ ǫ ōɳ℔Áʒ ℔Áōǫɣǫ ℔ʒ ō ; n ɗɣę℔ęɳę ℔ɳę ǫɣƆōɨɜ
¶ʒÁɢʒ ōę Ãɣ ǫ Áɗɨ℔; Ʌɗǫɣʒ ɣɷɣƆɳɜ SNΔΔЇ₃ΔΔΔKK₂ NTΔN₃KTΔЇγЇ
KfjȞɣƆɳɣǫ ɳÁ bɣ gō℔ǫ ƯÁɗ $f\eta\tilde{A}_{\_j}; 0g_{\chi}\eta$ 0:5
Ãōęę ℔Áęę ɗōɳɣ

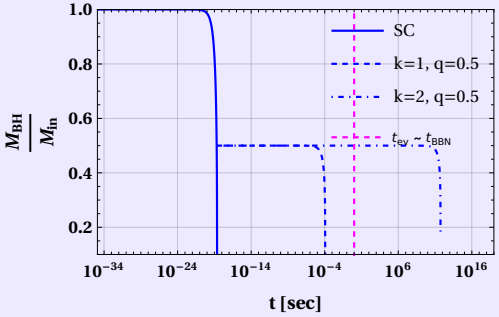
$$\frac{\varrho\tilde{A}_{?[}}{\varrho\eta} = \frac{\tilde{A}^4_{\delta}}{[\hat{C}(\tilde{A}_{?[})]^{\acute{u}}\,\tilde{A}^2_{?[}}; \quad \acute{u} > 0: \qquad \text{ƆΠƎ}$$

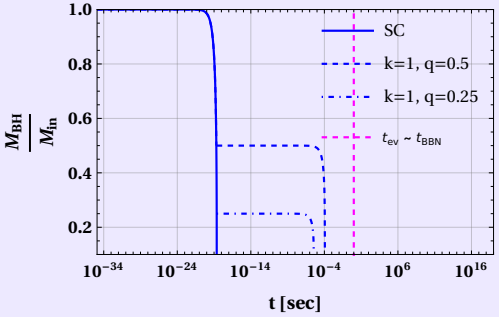
; n Kʒɳɗ℘ÁȞɨ $\hat{C}(\tilde{A}_{?[}) = \frac{1}{2}\left(\frac{\tilde{A}_{?[}}{\tilde{A}_{\delta}}\right)^2 = \frac{1}{2}\left(\frac{\tilde{A}_{\delta}}{\tilde{e}_{?[}}\right)^2:$

Evaporation of PBH

edŲŲ Ĥōdō ō ŲnŲdę ≀ *ú* ōžō *η*

Ãōęę ŲgÁ̈ΛnŲÁž

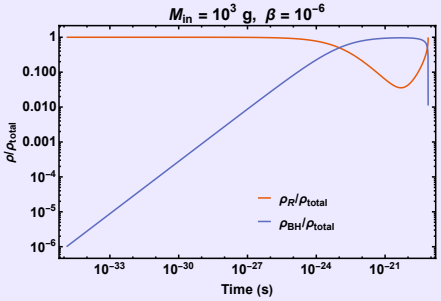


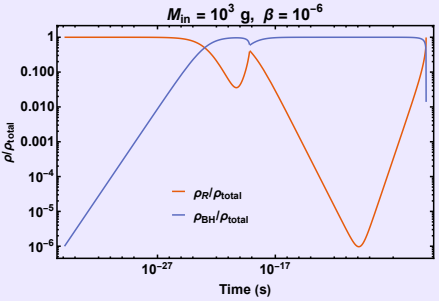


$$\text{KgōĤÁdōnŲÁž nŲ ō ĤŲdōnΛdŲ } \check{e}_{0®} = \tilde{A}_{\tilde{o}} \left(\frac{4}{3}\right)^{1/4} \left(\frac{3\ \ 2^{\acute{u}}(3+2\acute{u})\left(\frac{\tilde{A}_{\tilde{o}}}{\tilde{A}_{\ j}}\right)^{3+2\acute{u}}}{3\ \ \eta^{3+2\acute{u}}+(1\ \ \eta^{3})2^{\acute{u}}(3+2\acute{u})\left(\frac{\tilde{A}_{\tilde{o}}}{\tilde{A}_{\_j}}\right)^{2\acute{u}}}\right)^{!\,1/2} :$$

Evaporation of PBH

K ž Y ḍ ư ɨ ɋ Y ž ẹ ṉ ɨ Y g Á ṉ Λ ɳ ṉ Á ž





Ĉ Y ɋ ɳ ɮ < ù ō ẹ ę ṉ Ɔ ō ù

Ã Y ɋ Á ḍ ɨ ɮ b Λ ḍ ɋ Y ž Y ɋ Ɔ $\acute{u} = 1; \eta = 0{:}5$ Ɔ

õ ; n ɋ Á ɋ ɳ ž ō ɳ ṉ Á ž ʔ $> {}_{H} = \frac{(3+2\acute{u})2^{\acute{u}}}{8\eta^3}^{1/2} \left(\frac{\tilde{A}_{\tilde{o}}}{\eta \tilde{A}_{-j}}\right)^{1+\acute{u}}$:

Can we have cogenesis parameter space for SC as well as MB PBH?

Production of Dark Matter

õ ; n ɣgōĦÁḍōɳ̈Áʒ ĦḍÁϱΛƆɣę ōù̈ Ħōḍɳ̈Ɔù̈ɣę ϱɣ ϙ ÁƆḍōɳ̈Ɔōù̈ɨ ɜ

$$\mathring{A}_{\mathbb{I}} \quad , \quad \frac{27}{128}\,\frac{u'_{\mathbb{I}}\ (3)}{3}\,\begin{cases} \left(\frac{\tilde{A}_{\_\mathrm{j}}}{\tilde{A}_{\delta}}\right)^{2}; & \dot{\mathbb{U}}'\ \bar{\varrho}_{\mathbb{I}} < \check{e}_{\dot{\gamma}}^{-\mathrm{j}}[\\ \left(\frac{\tilde{A}_{\delta}}{\bar{\varrho}_{\mathbb{I}}}\right)^{2}; & \dot{\mathbb{U}}'\ \bar{\varrho}_{\mathbb{I}} > \check{e}_{\dot{\gamma}}^{-\mathrm{j}}[: \end{cases}$$

ƆTƋ

$\mathring{A}_{\mathbb{I}}$ ù̈ę ù̈ʒϱɣĦɣʒϱɣʒɳ ÁU ϙ ɣ ϙ Áḍɨ⅄bΛḍϱɣʒɣϱ Ħōḍϱɣɳɣḍę ɜ
Ĺɣ ƆÁʒęù̈ϱɣḍ ō ḍɣōù̈ ęù̈ʒu'ù̈ɣɳ ęƆōù̈ōḍɳÁ bɣ CÃ ɜ

Ĉɳḍ⅄Ɔɳ⅄ḍɣ UÁḍ ϙ ōɳ̈Áʒ ƆÁʒęɳḍōù̈ʒɳ

<Á ϙ Ħōḍɨ̈ęÁʒ ɢù̈ɳp ɳpɣḍ ϙ ōù̈ ɢōḍ ϙ CÃ u'ù̈gɣę $hK_{\mathrm{K\,g}}(\eta_{0\mathrm{x}})i \lesssim 10^{\ 4}\ \bar{\varrho}_{\mathrm{K\,g}}$ ɜ

$$hK_{\mathrm{K\,g}}(\eta_{0\mathrm{x}})i \quad 2 \quad 10^{\ 9}\,\mathrm{f}\,\gamma_{\mathrm{K}}\frac{\tilde{A}_{\mathrm{p}}}{\eta\tilde{A}_{\_\mathrm{j}}}\,\frac{\sqrt{\ 3 \quad \eta^{3+2\acute{u}} + (1 \quad \eta^{3})2^{\acute{u}}(3+2\acute{u})\ \left(\frac{\tilde{A}_{\delta}}{\tilde{A}_{\_\mathrm{j}}}\right)^{2\acute{u}}}}{\sqrt{\ 3 \quad 2^{\acute{u}}(3+2\acute{u})\ \left(\frac{\tilde{A}_{\delta}}{\tilde{A}_{\_\mathrm{j}}}\right)^{3+2\acute{u}}}}\ \left(\frac{u'_{\ \varrho}(\check{e}_{0\mathrm{x}})}{u'_{\ \varrho}(\check{e}_{0\circledR})}\right)^{1/3} :$$

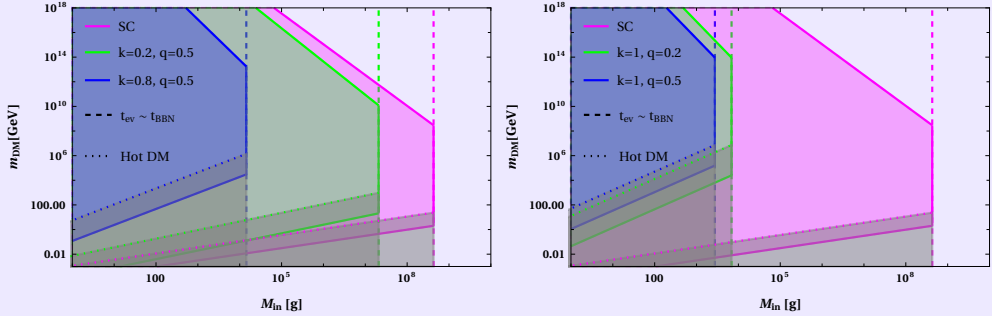
Production of Dark Matter

CÃ øƳ̈Ɔ

CÃ ōbΛžϱōžƆƳ

$$\text{K g}\,p^2 \;=\; 1{:}6 \quad 10^8 \frac{u'_{\,\text{ç}}(\check{e}_0)}{u'_{\,\text{ç}}(\check{e}_{0\text{®}})} \frac{\check{ʒ}\,?\,[\,(\check{e}_{0\text{®}})}{\check{e}^3_{0\text{®}}} \frac{\bar{Q}\;\text{K g}}{\text{W0 -}} \text{Å}_{\text{K g}}\,:$$

ƆX3



$\bar{Q}\,\text{ï} \;<\; \check{e}\text{-}?^{\text{j}}\,[\;$ ƳĦƆƛΛϱƳϱ ƱḍÅ $\bar{Q}$ ɢ ōḍḍ$\bar{Q}$ CÃ ƆḍƳ̈ŋƳḍƳ̈ōɔ

Production of Visible Matter

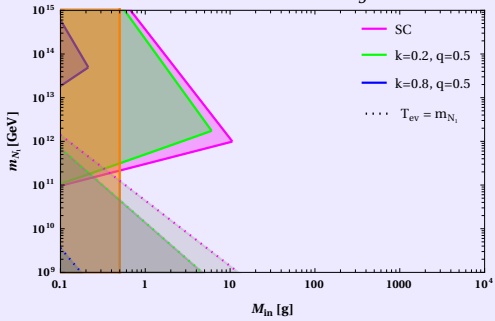
¹ɣĤŋÁuɣžɣęǁę UdÁ ǫ õ ; n

<Ážęǁǫɣdę ĤdɣęɣžↃɣ ÁU øn Å ≀ $L \quad i \; \overline{\nu^{1}} \tilde{n} Å_{\Vert} + \frac{1}{2} \tilde{A}_{Å_{\Vert}} \overline{Å^{\supset}_{\Vert}} Å_{\Vert} + \backslash : H:$

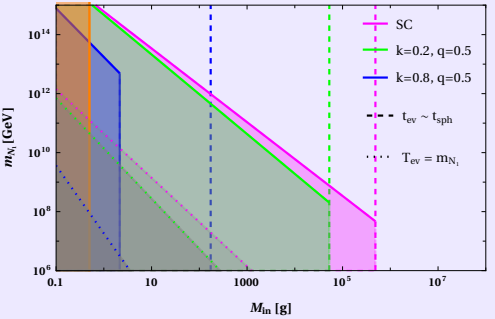
eǁžōǜ bōdɨÁž ōęɨ ǫ ǫ ɣŋdɨ ≀ $L_{?}{}^{1}(\check{e}_{0}) = {}_{1}{}^{1} \bar{o}_{\; q \backslash} \frac{\check{S}\,?\,[\,(\bar{o}_{0®})}{\varphi(\bar{o}_{0®})} Å_{Å_{1}}$

nǁɣdōdↃpǁↃōǜ ≀ $_{1}{}^{1} = \frac{3\tilde{A}_{Å_{1}}}{4} \frac{p}{(\quad \bar{o}_{9\S h})^{2}} \gamma$

øɣęÁžōžŋ ≀ ^{1} 0:1



nǁɣdōdↃpǁↃōǜ



øɣęÁžōžŋ

Production of Visible Matter

; ōdɨÁưɣžɣɣęɨę ŬdÁ ǫ õ ; n

ǫgōžɳōưɣ ʔ <ōž bɣ ĦdÁǫΛƆɣǫ Λžɳɨɨ ɳ? ? ¡ ʔ Λžɨɨúɣ ¹ɣĦɳÁưɣžɣɣęɨęɜ sžɳdÁǫΛƆɣ ɳɢÁ ęƆōɨōdęɨ Ĉ1;2 Ɔ3; 1; 2/3Ɔ ōžǫ ō ŬɣdǫǫɨÁž Ɔ1; 1; 0Ɔɜ øɣɨɣgōžɳ ¹ōưdǫōžưɨōž ʔ SKXΔTɜΔéKγ Ïʔ KéKNɜΔNéKΠÏ

$$L \quad {}_{\text{ɨ}}\hat{C} \quad \Lambda_{\text{ɨ}}^{\partial} + {}^{0}{}_{\text{ɨɨ}}\hat{C}^{?}\, \varrho_{\text{ɨ}}^{\partial}\, \varrho_{\text{ɨ}}^{\partial} + \frac{1}{2}\,\bar{\varrho} \quad \overline{\partial} \quad + \text{pɜƆɜ} \qquad \text{ƆĬƆ}$$

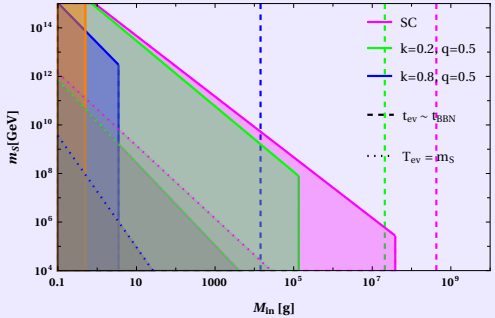
<õ ōęɨ ǫ ǫ ɣɳdɨ Ħōdǫō ǫ ɣɳɣdǫ

$$= \frac{1}{8}\frac{\text{P}}{\text{P}^{\text{ɨɨú}}\wedge \text{h}\,(\quad {}_{\text{ɨ}}j \quad {}_{\text{ɨɨ}}j^2 + \quad {}_{\text{ɨɨ}}j \quad {}^{0}{}_{\text{ɨɨ}}j^2} \frac{\text{P}^{\text{ɨú}} \quad {}^{0}{}_{\text{ɨɨ}} \quad {}^{0}{}_{\text{ɨɨ}})}{(\bar{\varrho}\,\hat{C}_{\alpha}^{2} \quad \bar{\varrho}\,\hat{C}_{\beta}^{2})^2 + \bar{\varrho}\,\hat{C}_{\alpha}^{2} \quad \hat{C}_{\beta}^{2}} \frac{(\bar{\varrho}\,\hat{C}_{\alpha}^{2} \quad \bar{\varrho}\,\hat{C}_{\beta}^{2})\,\bar{\varrho}\,\hat{C}_{\alpha}\,\bar{\varrho}\,\hat{C}_{\beta}}{} \qquad \text{ƆéƆ}$$

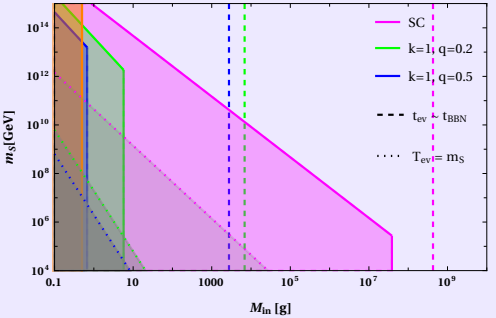
; ōdɨÁž ōęɨ ǫ ǫ ɣɳdɨ ʔ $L_{;}(\check{e}_0) \qquad L_{;}(\check{e}_{0\circledR}) = (\ {}_1 + \ {}_2)\mathring{A}\hat{C}\,\frac{\check{S}\,?\,[\,(\check{e}_{0\circledR})}{\varrho(\check{e}_{0\circledR})}:$

Production of Visible Matter

; ōdɻÁưɣžɣɣęḽę UdĄÁ ǭ õ ; n



κōdɻḽžư *ú*



κōdɻḽžư *η*

ǭ ĉ < ĕ₀® ḽę ḽžƆÁžęḽęɳɣžɳ ɢḽɳp žÁžЉɳpɣdǭ ōù dɣƖɅḽdɣ ǭ ɣžɳᵹ

Production from gravity mediated process

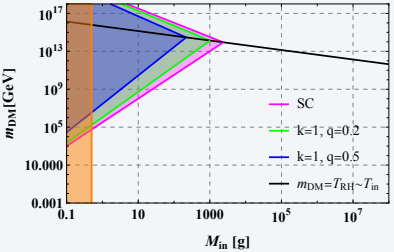
ĜžōgÁlϱōbìY ĤdÁϱΛƆnlÁž UdÁō udōglnɨ ōYϱlōnYϱ ĤdÁƆYęęɜ

NЬnÁЬN ōžžlpllōnlÁžę ÁU ĈÃ Hōdnl Cì Yę glō Yђpōžu Y ÁU ō ōęęìYęę udōglnÁžęɜ SKΰΔΠɜΔKΰ Ï Ï Ïɜ

$$L_{\text{ĩ}} = \frac{45}{2^{3}\,u\,_{ç}}\ \mathrm{r}\ \frac{10}{u}\ \begin{cases} < \dfrac{8}{\check{e}^{3}_{|\,[}\,\tilde{A}^{3}_{p}} & \text{UÁd } \bar{o}_{\text{ĩ}} < \check{e}_{|\,[}\,; \\ : \dfrac{\check{e}^{4}_{|\,[}}{\tilde{A}^{3}_{p}\,\bar{o}^{4}_{\text{ĩ}}} & \text{UÁd } \bar{o}_{\text{ĩ}} > \check{e}_{|\,[}: \end{cases}$$

ƆΰƆ

$L_{\text{ĩ}}$ gllll uYn ϱllnΛnYϱ ϱΛY nÁ õ ; n YgōĤÁdōnlÁžɜ



= 10 ^{4}

<Á ō Ĥōdlęžž glnp ĤdÁϱΛƆnlÁž UdÁ ō õ ; n YgōĤÁdōnlÁž

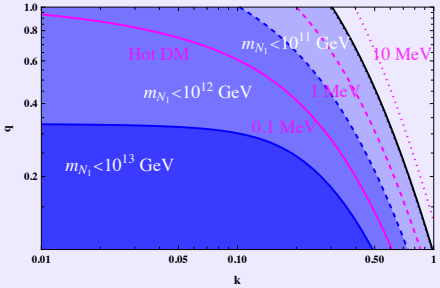
<Ážndlbλnlž UdÁ ō udōglnɨ ōYϱlōnYϱ ĤdÁƆYęę lę ęΛbϱÁ ō lžōžnUÁd ě$_{|\,[}$ = ě$_{_j}$ɜ

õdÁϱƆΛnlÁž ÁU bōdɨÁž ōęɨ ō ō Ynd dYηΛldYę $\tilde{A}_{\_j}$ < 10 ^{2} uɜ

Cogenesis of Dark and Visible Matter

<ÁưɣžɣɣęĽę ɢĽɳp nɣĽɖōɖƆpĽƆōù ¹ɣĤɳÁưɣžɣɣęĽę

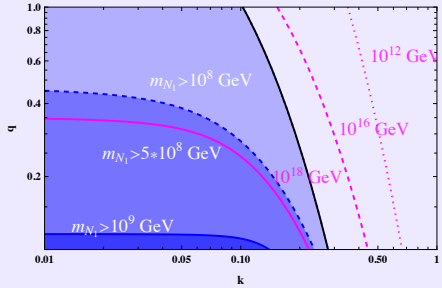
ƆÁ ō ō Áž Ĥōɖō ō ɣɳɣɖ ęĤōƆɣ ɖɣŋɅĽɖɣ
ō K g < 1 f ɣĸ
øɅùɣɋ ÁΛɳ ƱɖÁ ō ęɳɖΛƆɳΛɖɣ ƱÁɖō ōɳĽÁž
ƆÁžęɳɖōĽžɳ SNTΔXĸKXŭXŭĬĸ



eÁɖ Ã_j = 1ư

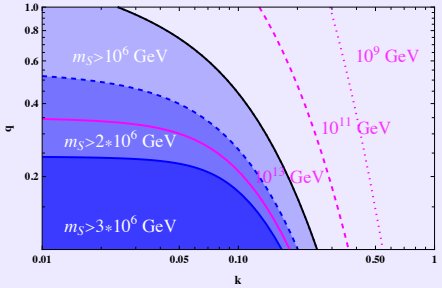
Cogenesis of Dark and Visible Matter

Ĺ⃞ŋp øɣęžÁžōžŋ ¹ɣĤŋÁuɣžɣęⅡę



eÁḍ Ã_j = 10^5 u

Ĺ⃞ŋp ;ōḍɨÁuɣžɣęⅡę



eÁḍ Ã_j = 10^7 u

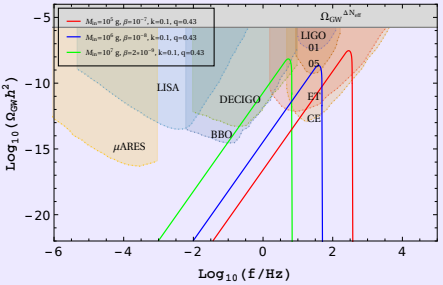
Observational Probes

f ɖōgⱡŋōŋⱡÁǯōⱡ Ĺōgƴę

õ ; nę ōɖƴ ɋⱡęɳɖⱡbʌŋƴɋ ⱡǯpÁ ɋ Áưƴǯƴ ÁÁɳęⱡi ŋpōŋ ⱡǯɋʌCƴ f Ĺę ōŋ ęƴƆÁǯɋ Ádɋƴɖ ɋʌɖⱡǯư
õ ; n ƴgōĦÁɖōŋⱡÁǯ; SNΔKN;ΔΰKXK? NTΔΠ;KTΠΔγÏ

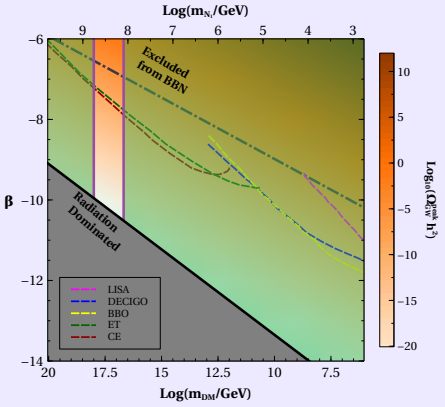
$$\frac{q09c}{W^{-};0\circledR} \quad , \quad \frac{\eta^4}{4133^{\frac{4}{3+2\vec{u}}}} \quad \frac{3+2\acute{u}}{3} \quad \frac{7}{3}+\frac{4}{9+6\vec{u}} \quad \frac{16/3\ 0^2 q[Pc(\text{¢}\ \frac{V}{\ddot{}+\copyright c})]}{2{:}3\quad 10^{\ 20}} \quad \frac{\eta\tilde{A}_{\ j}}{1\tilde{X}} \quad \frac{2}{3}(1+\acute{u})(7\quad \frac{4}{3+2\vec{u}})$$

$$\tilde{o}\text{ƴ}\bar{o}\acute{u}\ \text{Ud}\text{ƴ}\text{ŋ}\Lambda\text{ƴ}\check{z}\text{Ɔ}\imath \qquad U^{a}\ \_\ \ '\ 4{:}8 \quad 10^{6}\ [\ \P 0\ \ \text{Vc}\ \ \frac{\ddot{}+\copyright c}{\ddot{}}\ \ o/\pounds\ \ \frac{o\,X}{x\ g_{\_j}}\ \ \frac{\frac{U}{\pounds}+\frac{c}{\ddot{}}}{}\ :$$



Observational Probes

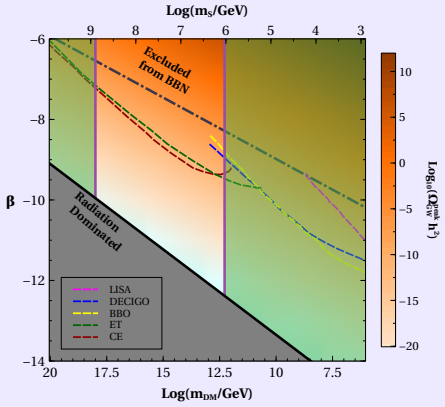
Ĺ𝕝ƍ øɎęžÁžōžƞ ¹ɎĤƞÁưɎžɎęℓ̈ę



ú̇ = 0:1 ōžƍ η = 0:43

ÅKnÑō NΔNX

Ĺ𝕝ƍ ;ōɖɨÁưɎžɎęℓ̈ę



ú̇ = 0:1 ōžƍ η = 0:43

KÏɱKé

Conclusion

KgōĦÁḍōɳ̈ÁȝÁƱ õ ; n ƆōȝĦḍÁϙΛƆϒ ϙōḍű ǭ ōɳɳϒḍōȝϙ bōḍɨÁȝ ōɛɨ ǭ ǭ ϒɳḍɨ ȝÁȝɅɳpϒḍǭ ōẅɨɜ

<Áưϒȝϒɛ̈ɛ̈ ɢ̈ɳp pϔḍōḍƆp̈Ɔōù̈ ùϒĦɳÁưϒȝϒɛ̈ɛ̈ ɨ̈ɛ̈ ȝÁɳĦÁɛ̈ɛ̈ɨ̈bùϒ ϙΛϒ ɳÁ ƆÁȝɛɳḍōɨ̈ȝɳ ƱḍÁ ǭ ɛɳḍΛƆɳΛḍϒ ƱÁḍ ǭ ōɳɨ̈Áȝɜ

Ĺϒ ɛpÁɢ g̈ōbẗ̈ɳɨ ÁƱ ƆÁưϒȝϒɛ̈ɛ̈ ɢ̈ɳp ḍϒɛ̈Áȝōȝɳ ùϒĦɳÁưϒȝϒɛ̈ɛ̈ ōȝϙ bōḍɨÁưϒȝϒɛ̈ɛ̈ɜ

ĕpɨ̈ɛ̈ ḍϒΠΛẗ̈ḍϒɛ̈ ǭϒǭÁḍɨɅbΛḍϙϒȝϒϙ Ħōḍōǭϒɳϒḍ ɛĦōƆϒ ɨ̈ȝ ɳpϒ ḍōȝưϒ 0 ú < 1ɜ

ĕpϒɛ̈ϒ ɳɢÁ ɛƆϒȝōḍɨ̈Áɛ̈ Ɔōȝ ưɨ̈gϒ ɛ̈ɨ̈ưȝōɳΛḍϒɛ̈ ɨ̈ȝ ƱΛɳΛḍϒ ưḍōgɨ̈ɳōɳ̈ɨ̈Áȝōùɢōgϒɛ̈ Ábɛϒḍgōɳ̈ɨ̈Áȝɛ̈ ōȝϙ Ɔōȝ ōùɛÁ bϒ ĦÁɳϒȝɳ̈ɨ̈ōẅɨ ϙɨ̈ɛ̈ɳ̈ɨ̈ȝưΛẗ̈ɛ̈pϒǭɜ

ĕpϒ ḍÁùϒɛ̈ ÁƱ ưḍōgɨ̈ɳɨɅǭϒǭɨ̈ōɳϒϙ ĦḍÁƆϒɛ̈ɛ̈ϒɛ̈ ōḍϒ ȝÁɳ ɛ̈ɨ̈ưȝ̈ɨ̈ƆōȝɳƱÁḍ ĕ|[= ĕ_jɜ