

# Stochastic Dynamics during Inflation

NEHOP 25 @ Brussels

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# Primordial perturbations

- **Primordial curvature perturbation  $\zeta$**  provides seed for large scale structure
- $P[\zeta]$  : Probability Distribution Function of  $\zeta$
- **At CMB scales:** (almost) scale-invariant Power Spectrum, fluctuations – Gaussian and adiabatic (perturbation theory)
- **At small scales:** possibility of **large fluctuations**  $\longrightarrow$  PBH formation
- Large fluctuations  $\longrightarrow$  **tail of  $P[\zeta]$**
- Need **non-perturbative methods** to go beyond standard perturbation theory of cosmological perturbations

Non-perturbative methods to calculate  
PDF of primordial curvature perturbation  $\zeta$

1. Classical  $\delta\mathcal{N}$  formalism
2. Stochastic  $\delta\mathcal{N}$  formalism

Non-perturbative methods to calculate  
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**This work: Comparing these formalisms for  
single-field slow-roll potentials**

Non-perturbative methods to calculate  
PDF of primordial curvature perturbation  $\zeta$

1. Classical  $\delta\mathcal{N}$  formalism

2. Stochastic  $\delta\mathcal{N}$  formalism

# Classical $\delta N$ formalism

$$\zeta = \delta N = N(\phi_* + \delta\phi_*, \phi_{\text{end}}) - N(\phi_*, \phi_{\text{end}})$$

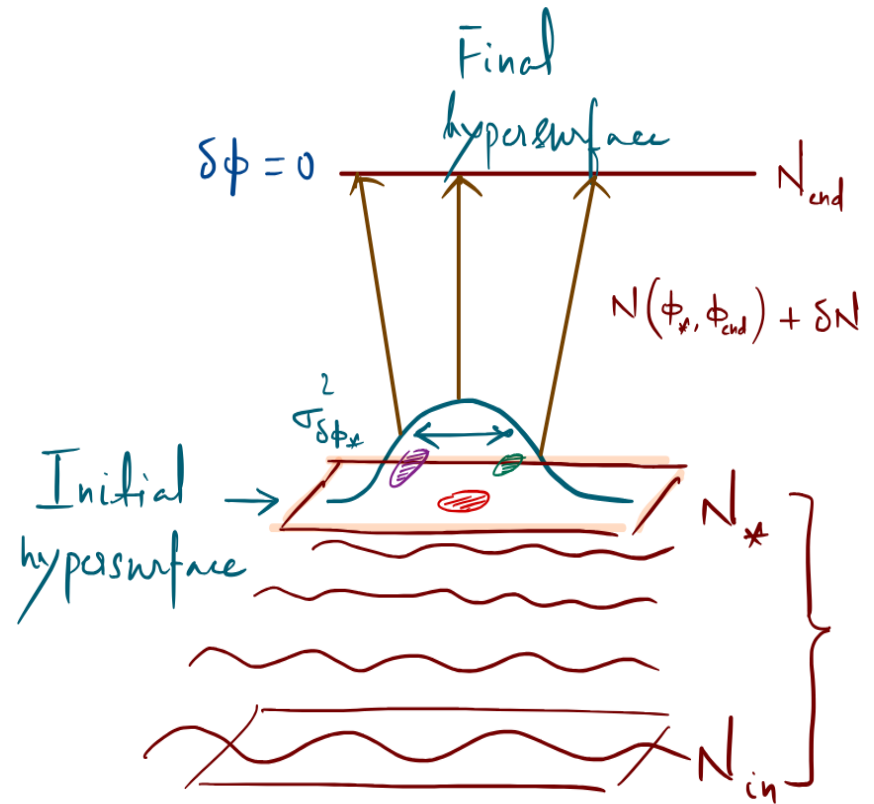
$\delta N$  : perturbed expansion in a family of homogeneous universes

- Non-linear relation between curvature perturbation  $\zeta$  and inflaton fluctuations  $\delta\phi$
- $N$ : Different patches can be effectively assumed as “separate universes”

(see David’s talk)

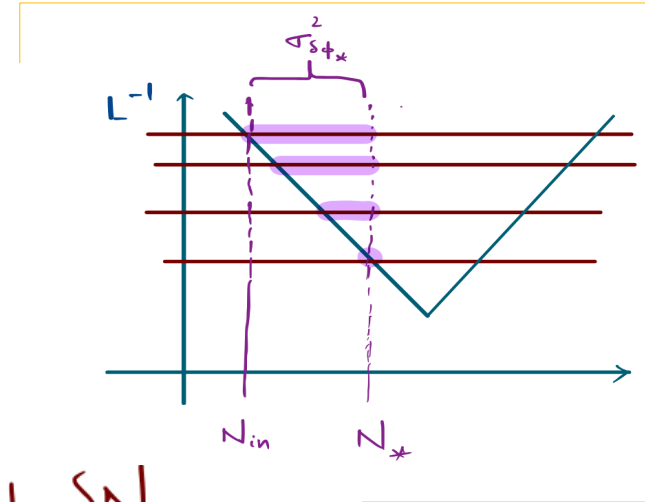
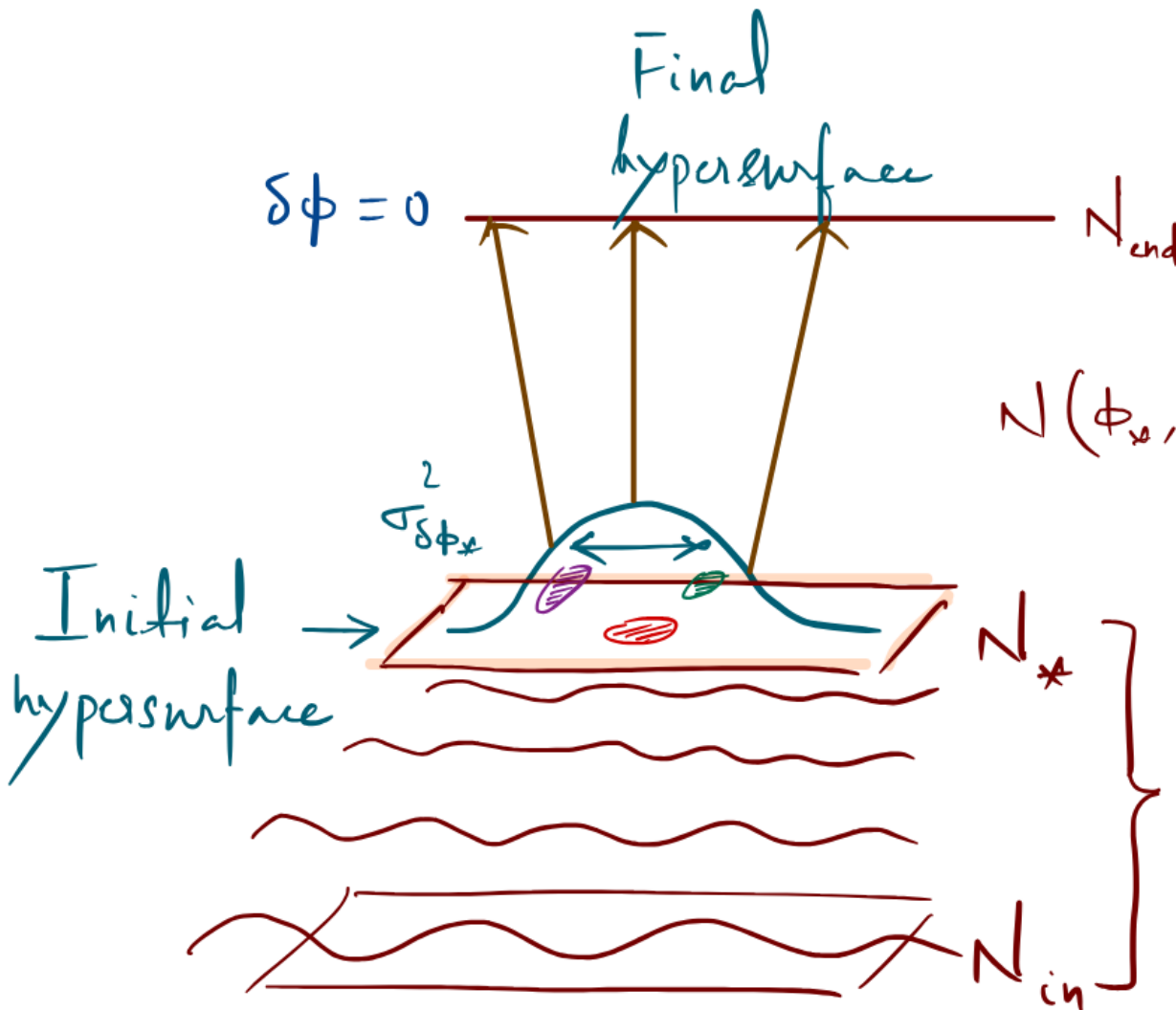
$\Rightarrow$  evolution: background EOM

- Consider field fluctuations  $\delta\phi_*$  on an initial flat hypersurface. These are then evolved using the non-linear equations



# Classical $\delta N$ formalism

$$\zeta = \delta N = N(\phi_* + \delta\phi_*, \phi_{\text{end}}) - N(\phi_*, \phi_{\text{end}})$$



$$\sigma^2_{\delta N} = (2\epsilon_*)^{-1} \sigma^2_{\delta\phi_*}$$

$$\sigma^2_{\delta\phi_*} = \int_{K_{\text{in}}}^{K_*} d\ln k \mathcal{P}_{\delta\phi}$$

# Classical $\delta N$ formalism

$$\zeta = \delta N = N(\phi_* + \delta\phi_*, \phi_{\text{end}}) - N(\phi_*, \phi_{\text{end}})$$

- Change of variables  $\longrightarrow$  Non-Gaussian PDF for the number of e-folds:

$$P[\delta N] = \left| \frac{d\delta N}{d\delta\phi} \right|_*^{-1} P[\delta\phi_*]$$

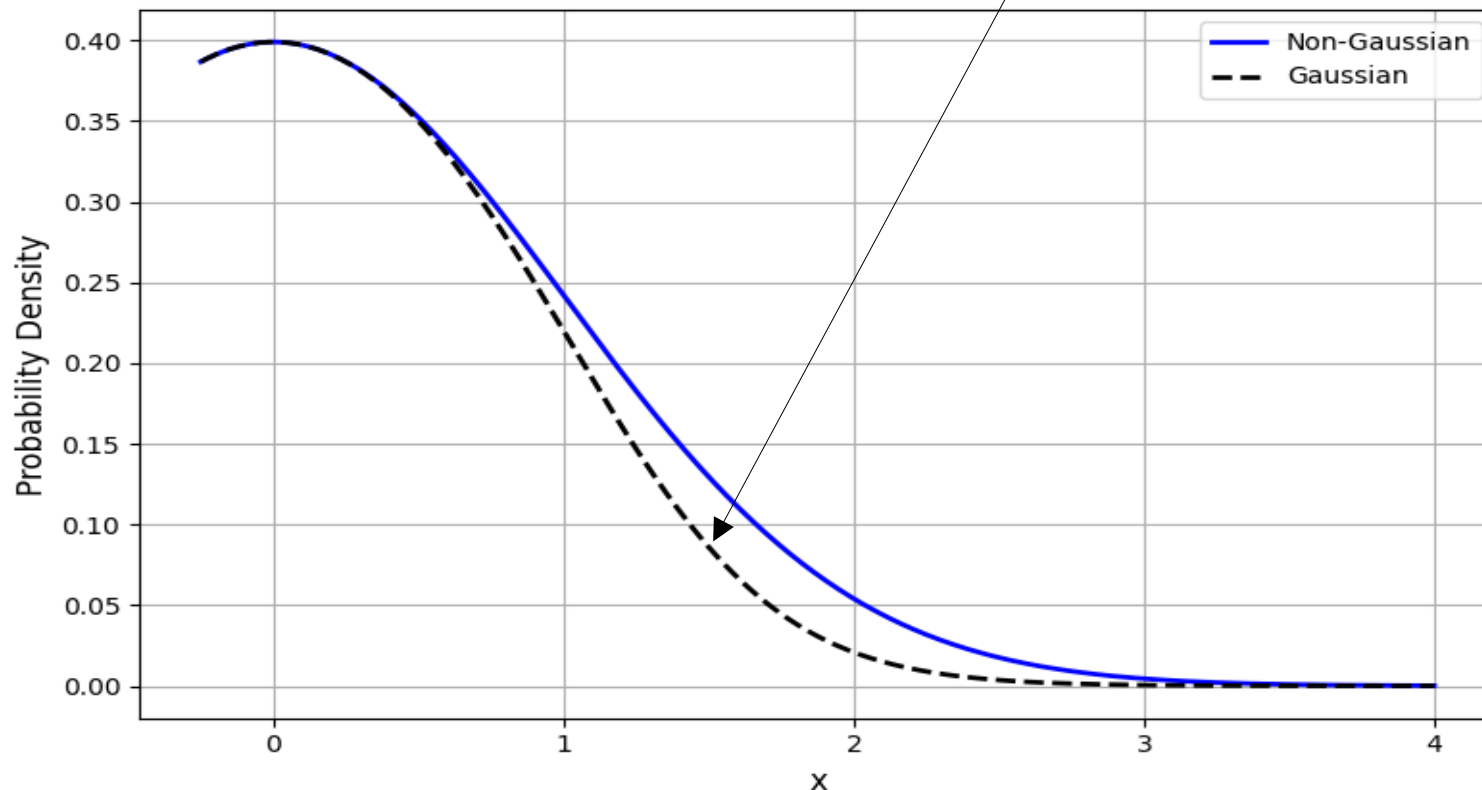
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Field fluctuations are assumed to be Gaussian



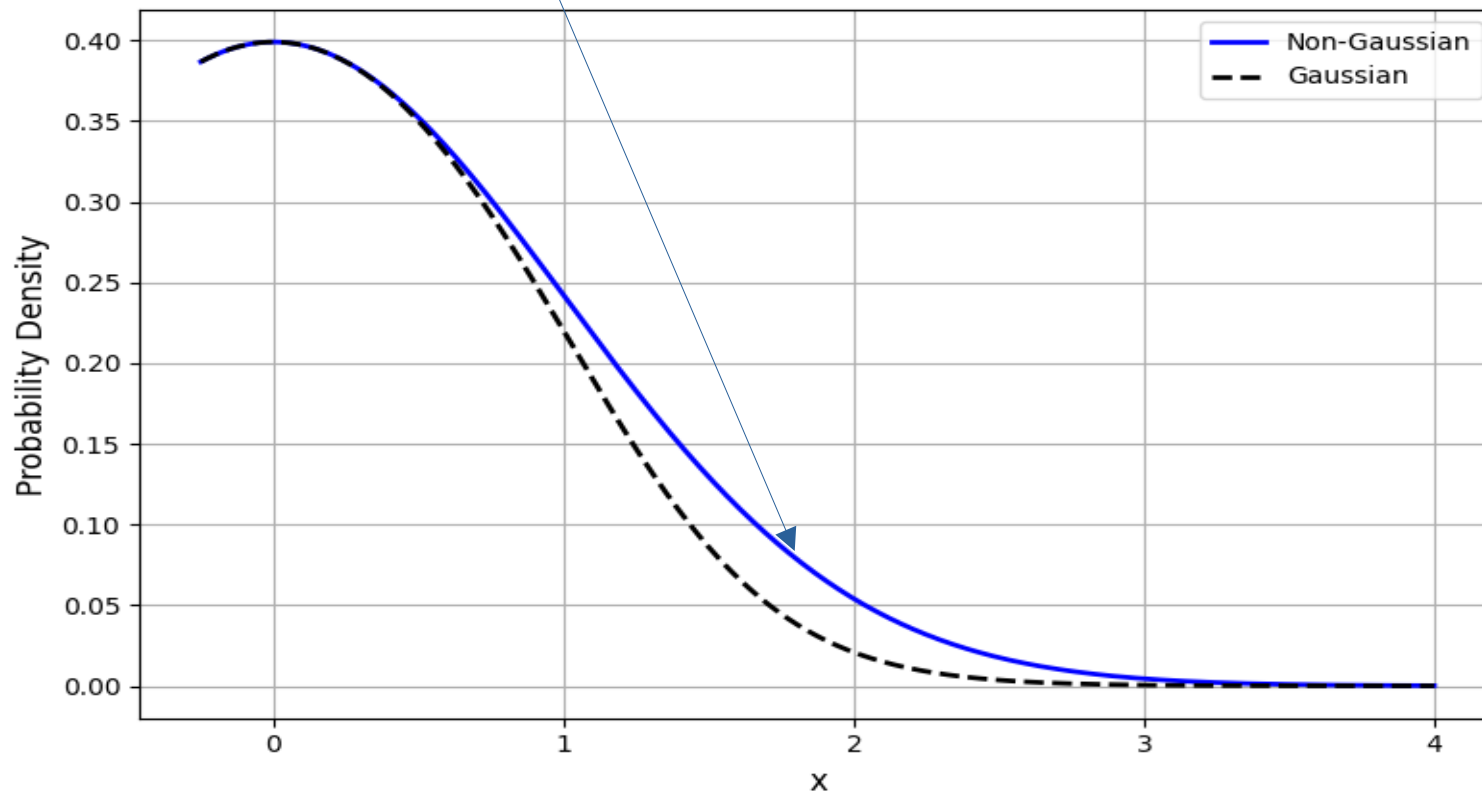
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Non-linear transformation



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## Gaussian PDF

$$P[\delta\phi_*] = \frac{1}{\sqrt{2\pi\sigma_{\delta\phi_*}^2}} \exp\left(-\frac{\delta\phi_*^2}{2\sigma_{\delta\phi_*}^2}\right)$$

with

$$\sigma_{\delta\phi_*}^2 = \int_{k_{\text{in}}}^{k_*} d \log k \mathcal{P}_{\delta\phi}$$

See Alejandro's talk

# Classical $\delta N$ formalism for $m^2 \phi^2$

□ Consider quadratic potential  $V(\phi) = \frac{1}{2}m^2\phi^2$

□ Slow-roll solution:

$$N(\phi_*, \phi_{\text{end}}) = \frac{1}{4m_p^2} (\phi_*^2 - \phi_{\text{end}}^2)$$

□ Using the  $\delta N$  formalism:

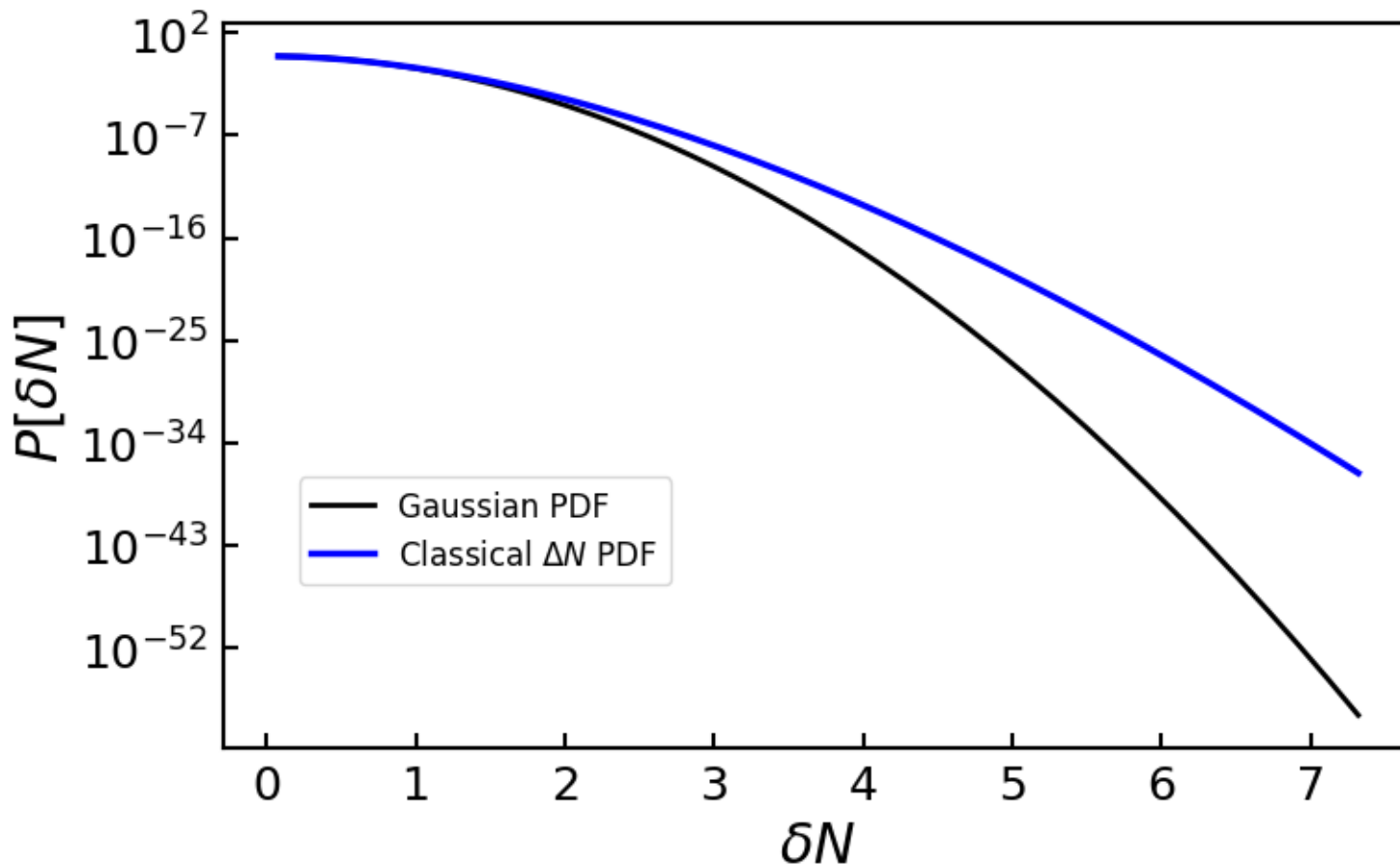
$$\zeta = \delta N = N(\phi_* + \delta\phi_*, \phi_{\text{end}}) - N(\phi_*, \phi_{\text{end}}) \quad \text{and} \quad P[\delta N] = \left| \frac{d\delta N}{d\delta\phi} \right|_*^{-1} P[\delta\phi_*]$$

## Classical $\delta N$ PDF

$$P[\delta N] = \frac{2m_p^2}{\phi_* \sqrt{1 + \frac{4m_p^2}{\phi_*^2} \delta N}} \frac{1}{\sqrt{2\pi\sigma_{\delta\phi_*}^2}} e^{-\frac{\phi_*^2}{2\sigma_{\delta\phi_*}^2} \left( -1 + \sqrt{1 + \frac{4m_p^2}{\phi_*^2} \delta N} \right)^2}$$

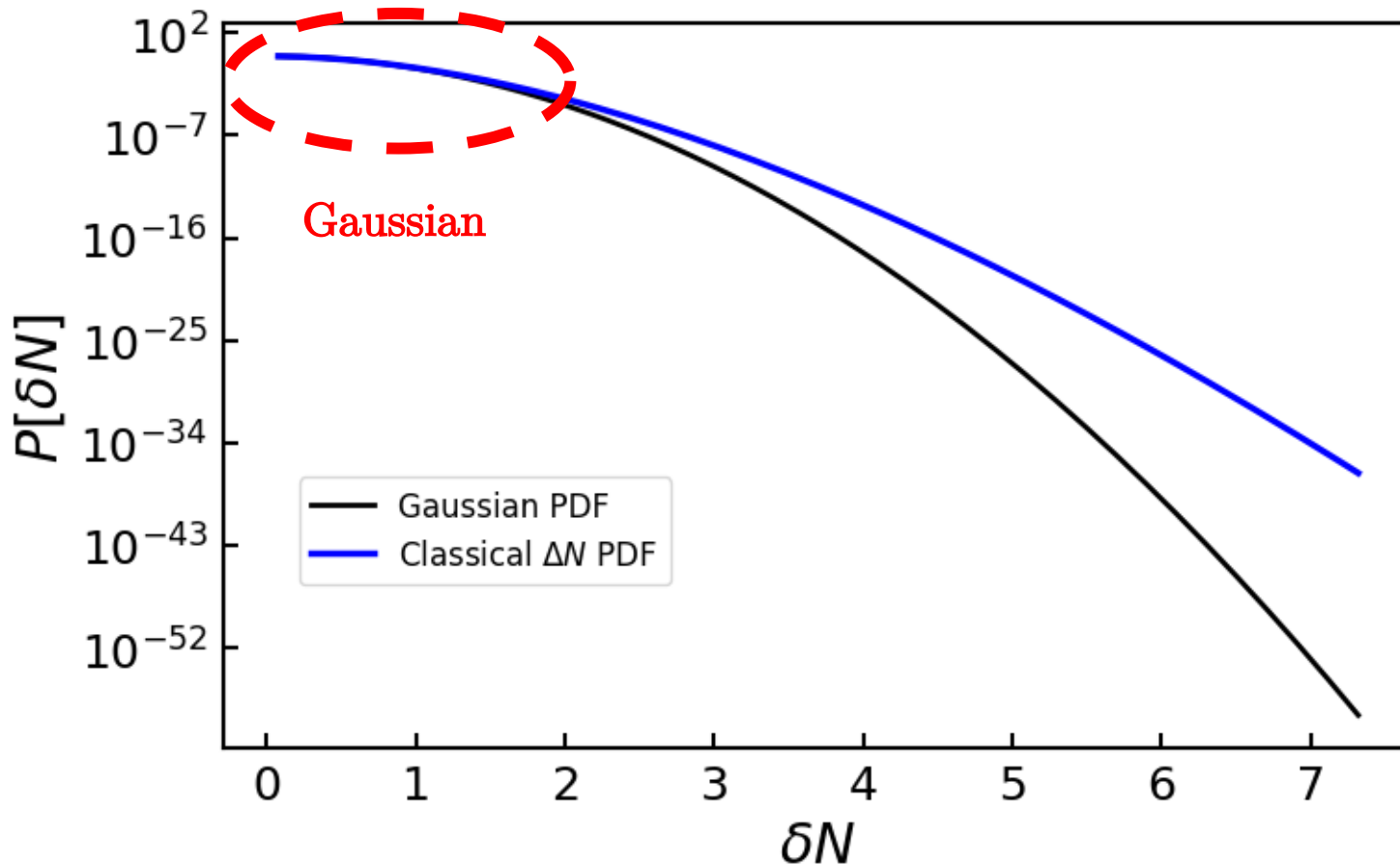
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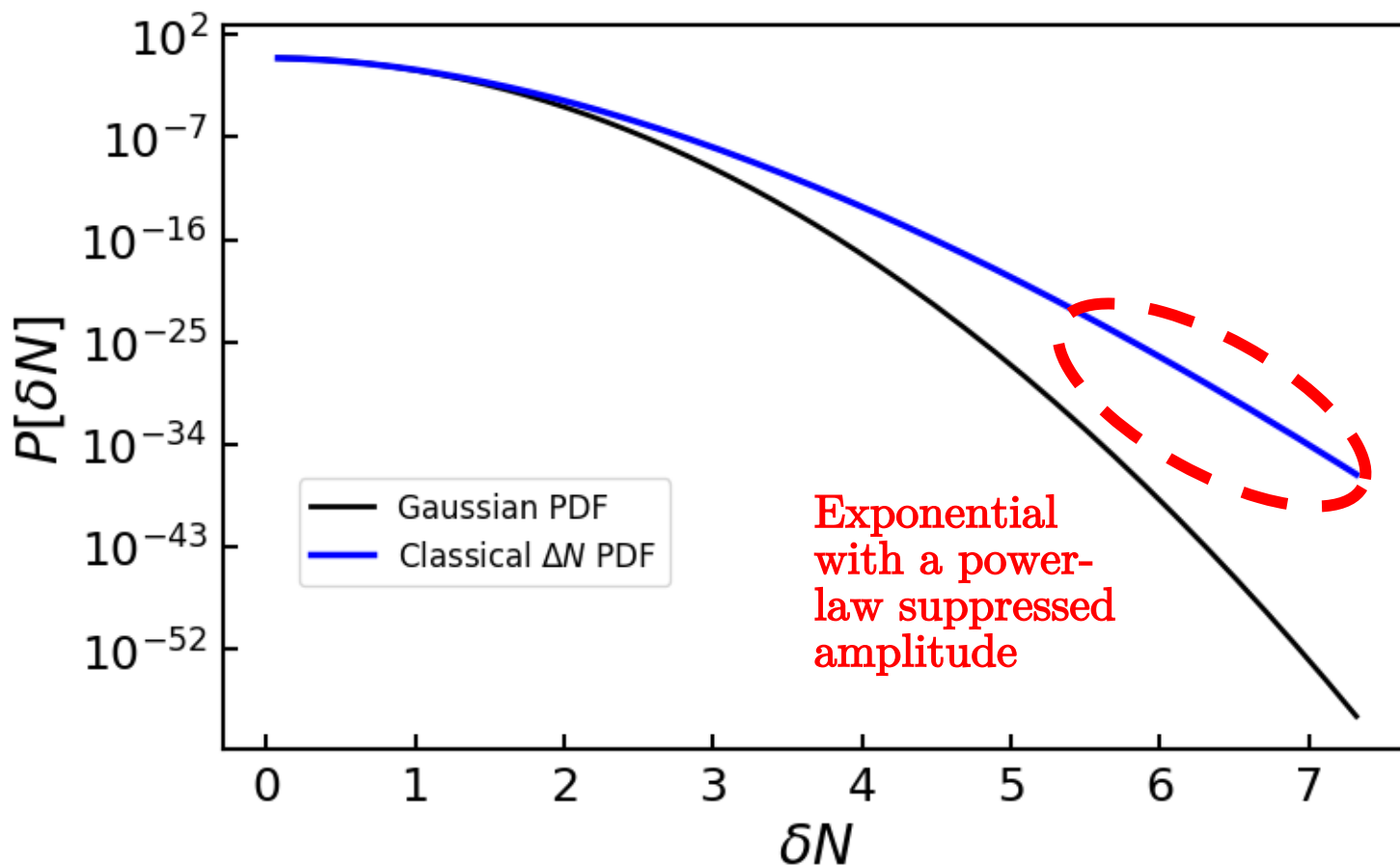
# Classical $\delta N$ formalism for $m^2 \phi^2$

$$\frac{4m_p^2}{\phi_*^2} \delta N \ll 1 : P[\delta N] = \frac{2m_p^2}{\phi_*} \cdot \frac{1}{\sqrt{2\pi\sigma_{\delta\phi_*}^2}} \exp\left(-\frac{4m_p^4}{\phi_*^2 \sigma_{\delta\phi_*}^2} \delta N^2\right)$$



# Classical $\delta N$ formalism for $m^2 \phi^2$

$$\frac{4m_p^2}{\phi_*^2} \delta N \gg 1 : \quad P[\delta N] \approx \frac{m_p}{\sqrt{2\pi\sigma_{\delta\phi_*}^2 \delta N}} \exp\left(-\frac{4m_p^2 \delta N}{\sigma_{\delta\phi_*}^2}\right)$$



# Non-perturbative methods to calculate PDF of primordial curvature perturbation $\zeta$

1. Classical  $\delta N$  formalism

2. Stochastic  $\delta \mathcal{N}$  formalism

# Stochastic Inflation

- Posits that the universe on large scales can be effectively described as a *stochastic system* (see earlier talks in this session)
- Field  $\phi$  is split into “long” and “short” modes:  $\phi(x,t) = \phi_s(x,t) + \Phi(x,t)$
- Integrating out small scales with a Window function (Heaviside func)
- **Langevin equation** for the long modes:

$$\frac{d\Phi}{dN} = \boxed{-\frac{V'(\Phi)}{3H^2(\Phi)}} + \frac{H(\Phi)}{2\pi} \boxed{\xi(N)}$$

Classical term

Stochastic

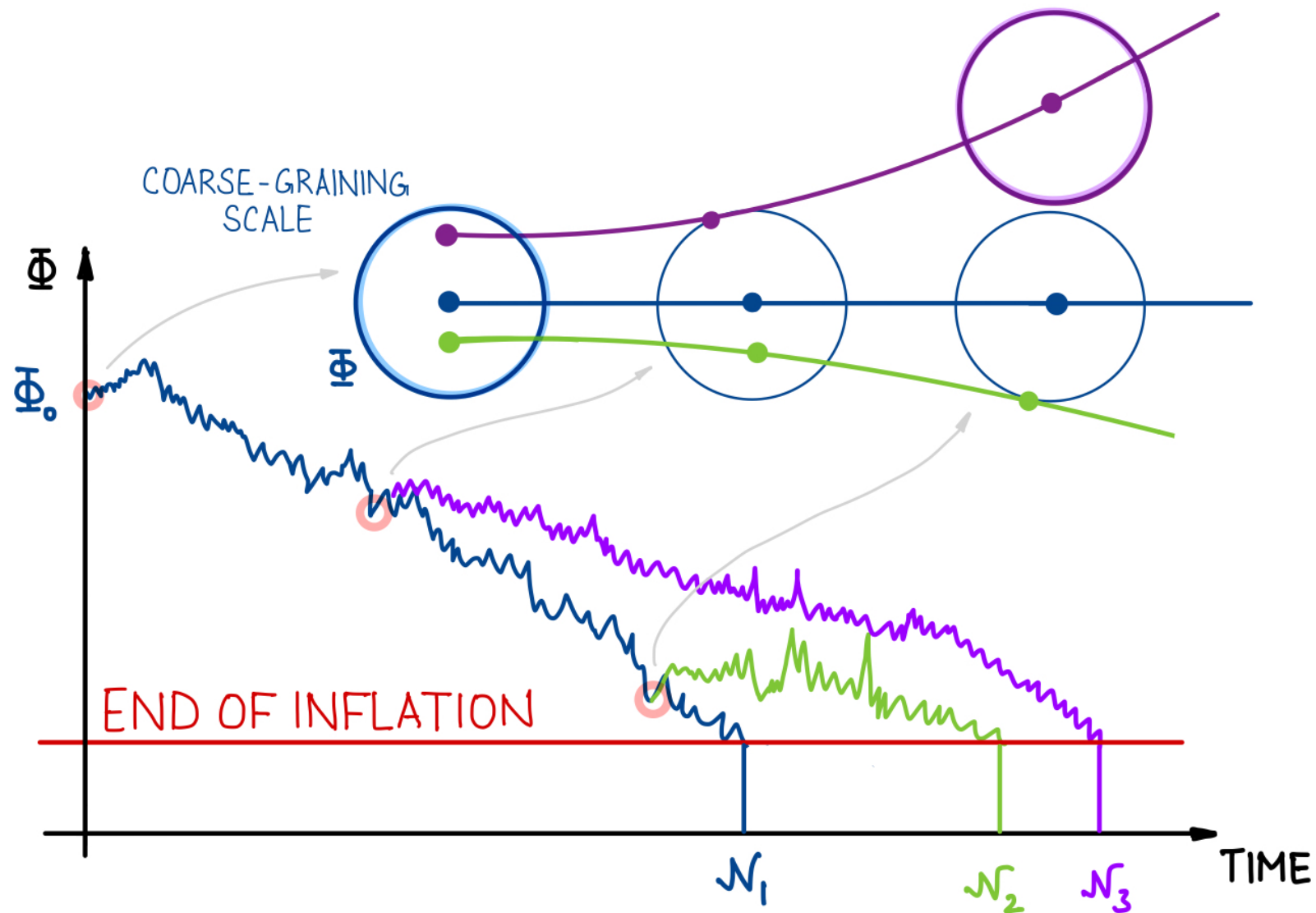
Deterministic

Gaussian White noise

⇒ Dynamics of the long modes is stochastic due to the noise term

# Stochastic Inflation

- First-passage times to the end of inflation:  $\mathcal{N}$

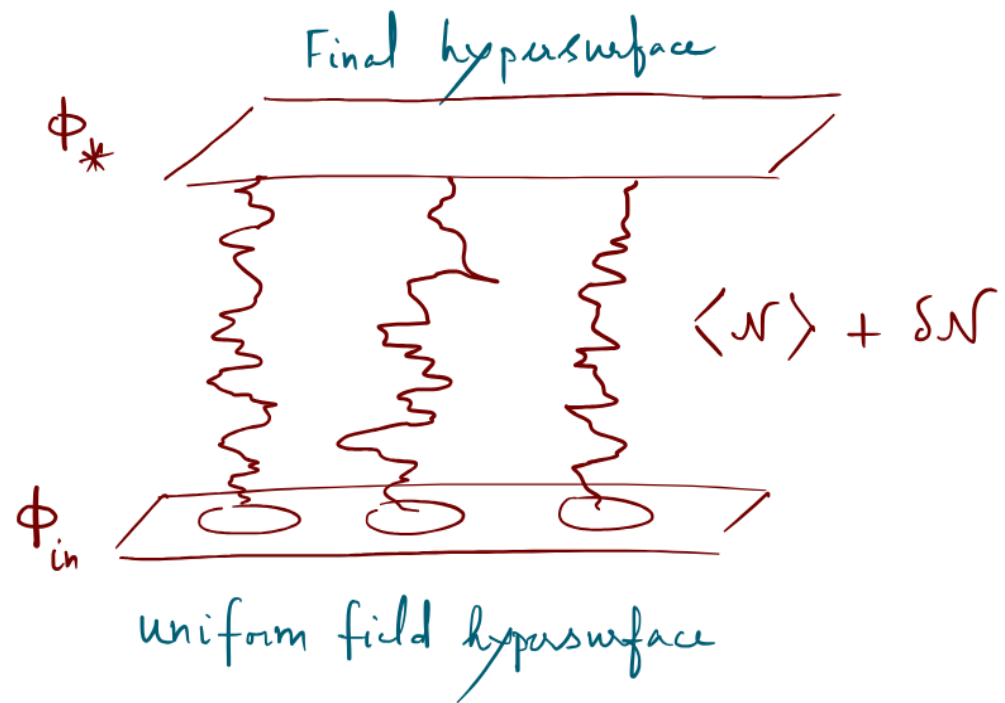


# Stochastic $\delta\mathcal{N}$ formalism

- Statistics of FPTs  $\longrightarrow$  Information about primordial perturbations
- Stochastic  $\delta\mathcal{N}$  formalism: relation between statistics of  $\zeta$  and  $\mathcal{N}$

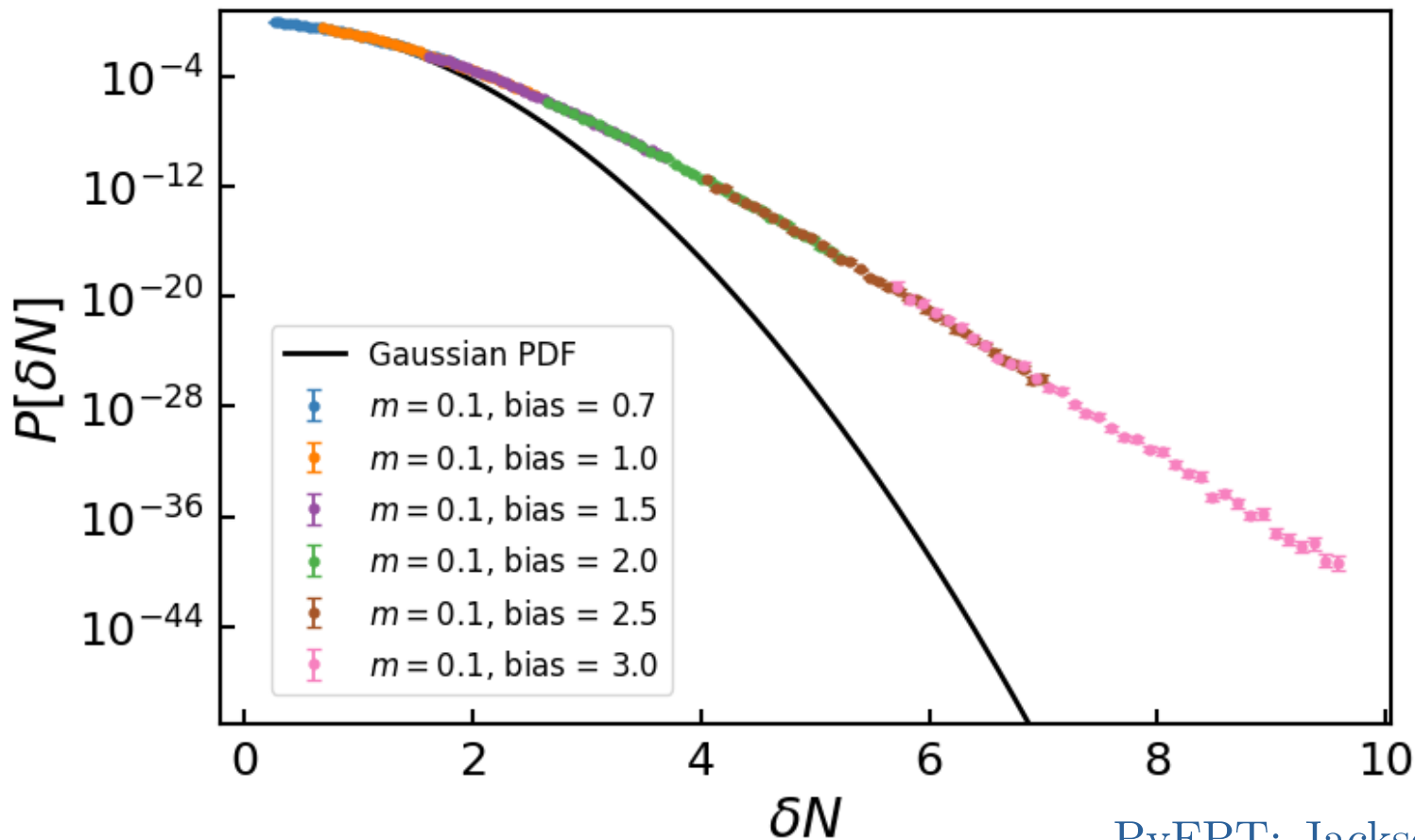
$$\zeta = \delta\mathcal{N} = \mathcal{N} - \langle \mathcal{N} \rangle$$

- Begin from a fixed point in field space (uniform-field hypersurface) and end at  $\phi_{\text{end}}$
- Dynamics is **stochastic** (noise term in the Langevin equation)



# Stochastic dynamics for $m^2 \phi^2$ ( $N = 10$ )

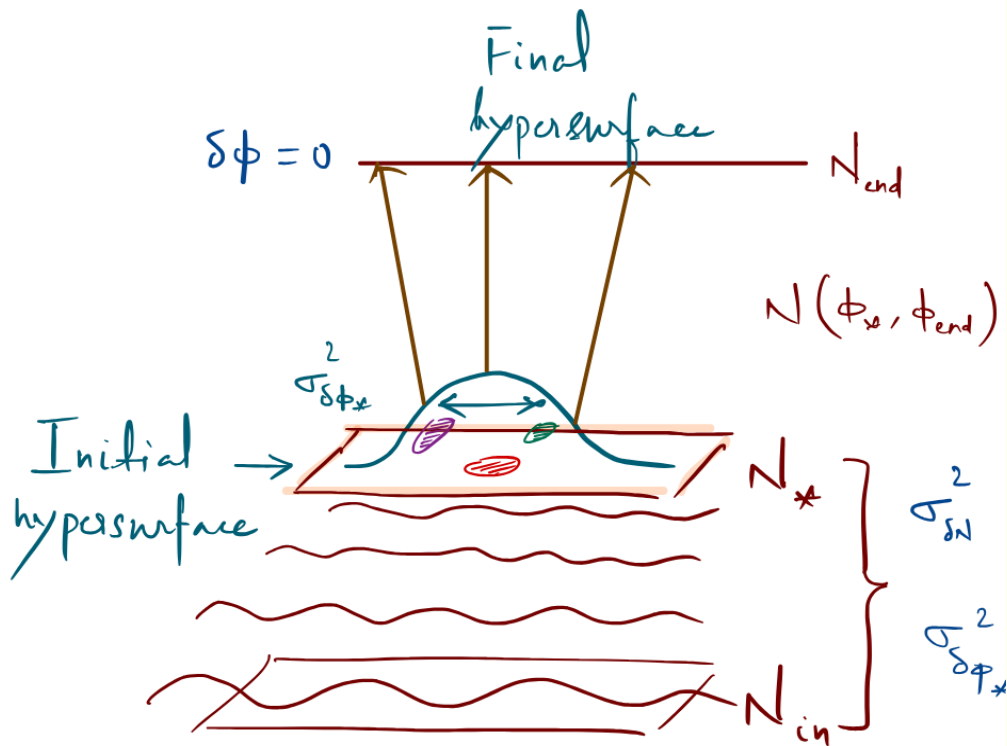
- Reconstructing the statistics of cosmological perturbations by solving many realisations of Langevin equation and collecting the FPTs
- **BUT** direct simulation of the Langevin dynamics cannot effectively reconstruct the tail of the PDF; use **importance sampling**



PyFPT: Jackson et al (2022)

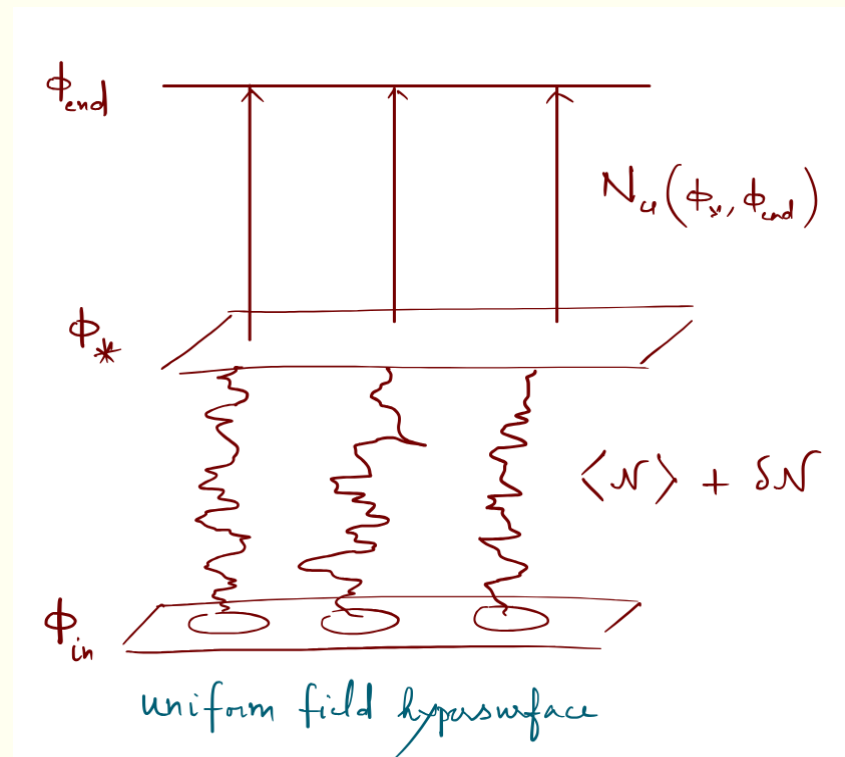
# Classical $\delta N$

- One large kick at  $N_*$
- Evolution is **deterministic** (background Friedmann)



# Stochastic $\delta N$

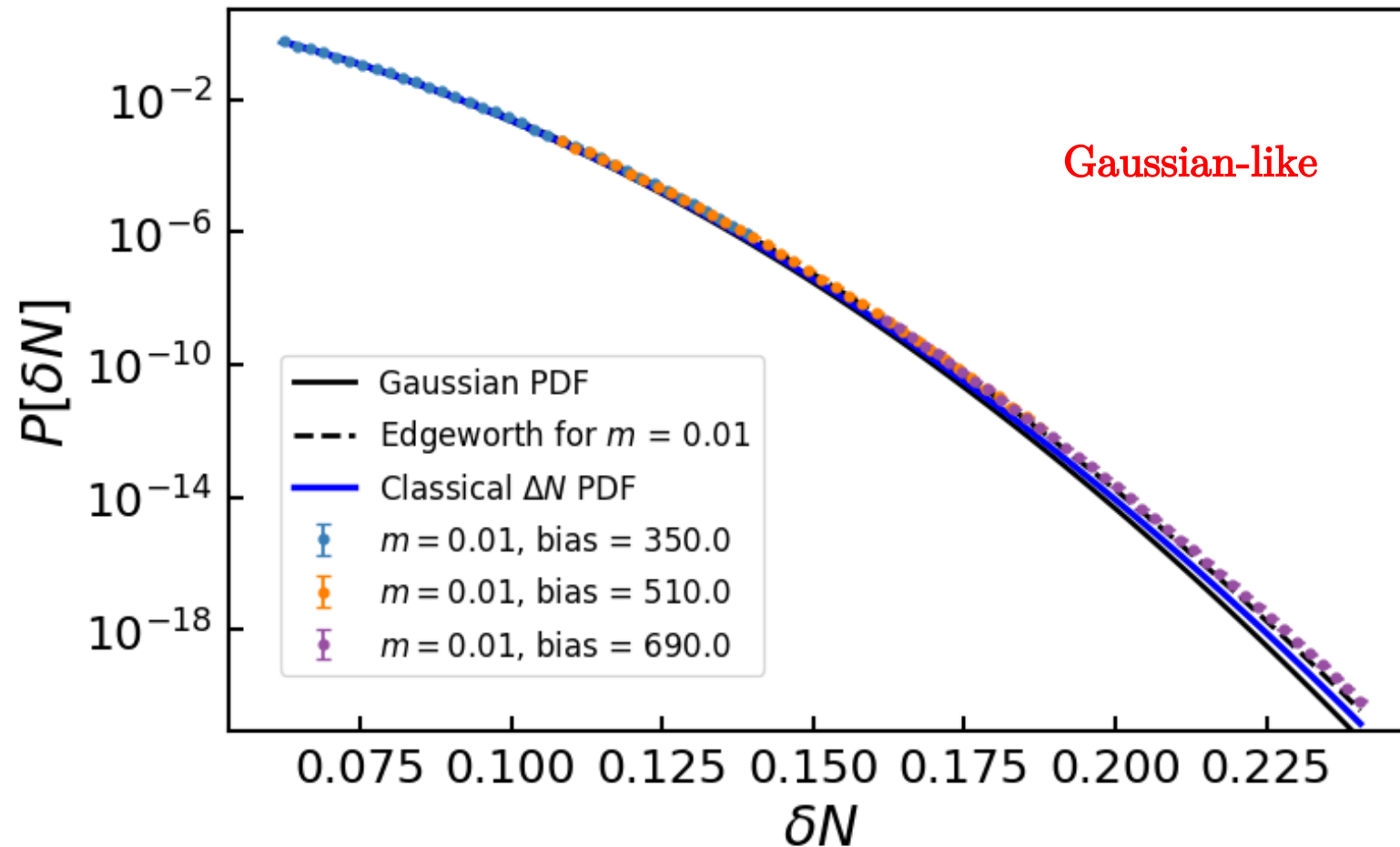
- Many small kicks due to the outflowing short modes
- Evolution is **stochastic** (Langevin dynamics)



**We ask: are these two formalisms equivalent?**

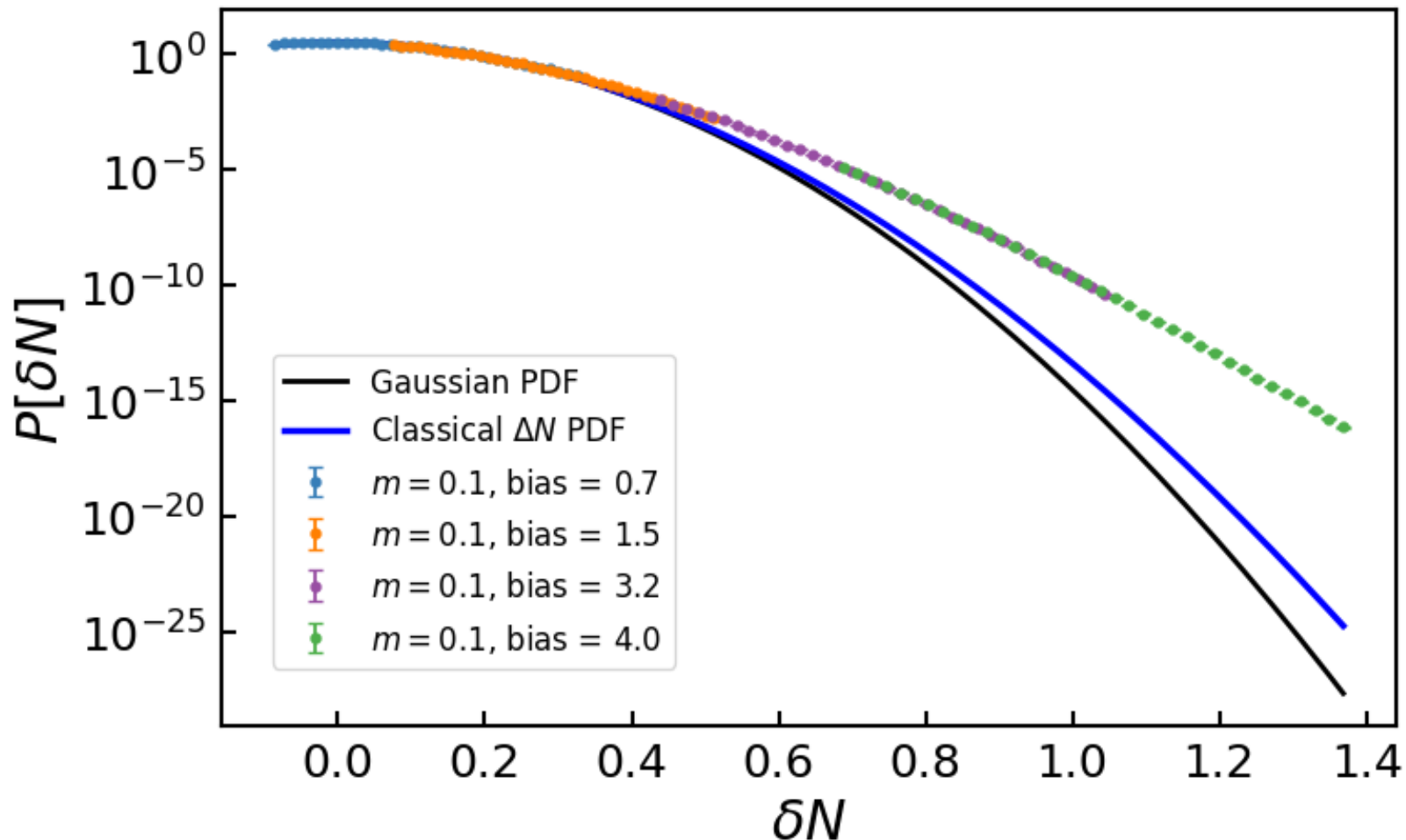
# Classical vs Stochastic for $m^2 \phi^2$ ( $N = 5$ )

- Mass of the inflaton field,  $m = 0.01$  ( $m_p = 1$ ) [  $m$  controls the stochastic effects ]
- $\phi_* = \phi_5$ , *i.e.* field value 5 e-folds before the end of inflation
- $\phi_{\text{ini}} = \phi_{10}$  *i.e.* field value 10 e-folds before the end of inflation
- $\phi_{\text{end}} = \sqrt{2}$



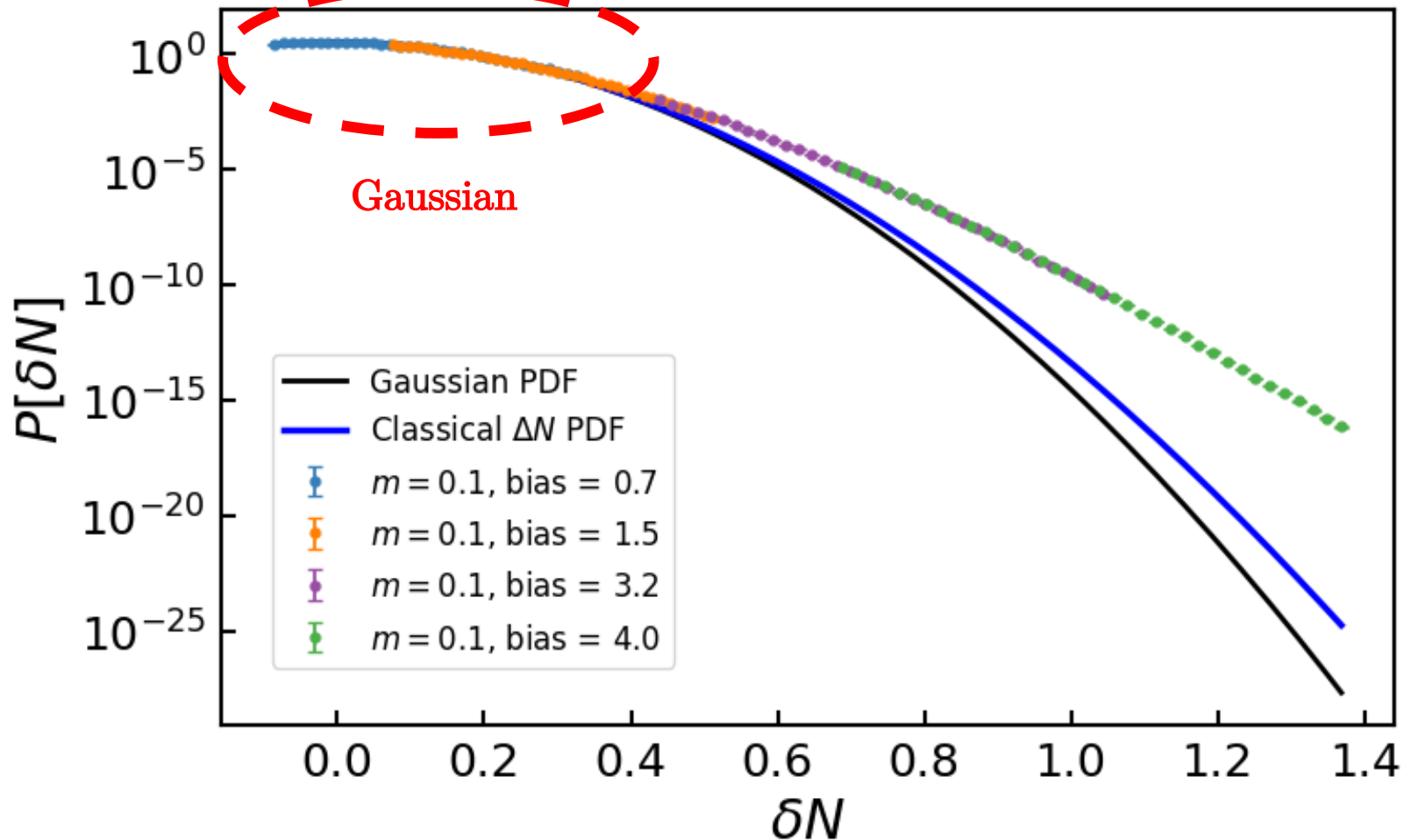
# Classical vs Stochastic for $m^2 \phi^2$ ( $N = 5$ )

- Mass of the inflaton field,  $m = 0.1$
- $\phi_* = \phi_5$ , *i.e.* field value 5 e-folds before the end of inflation
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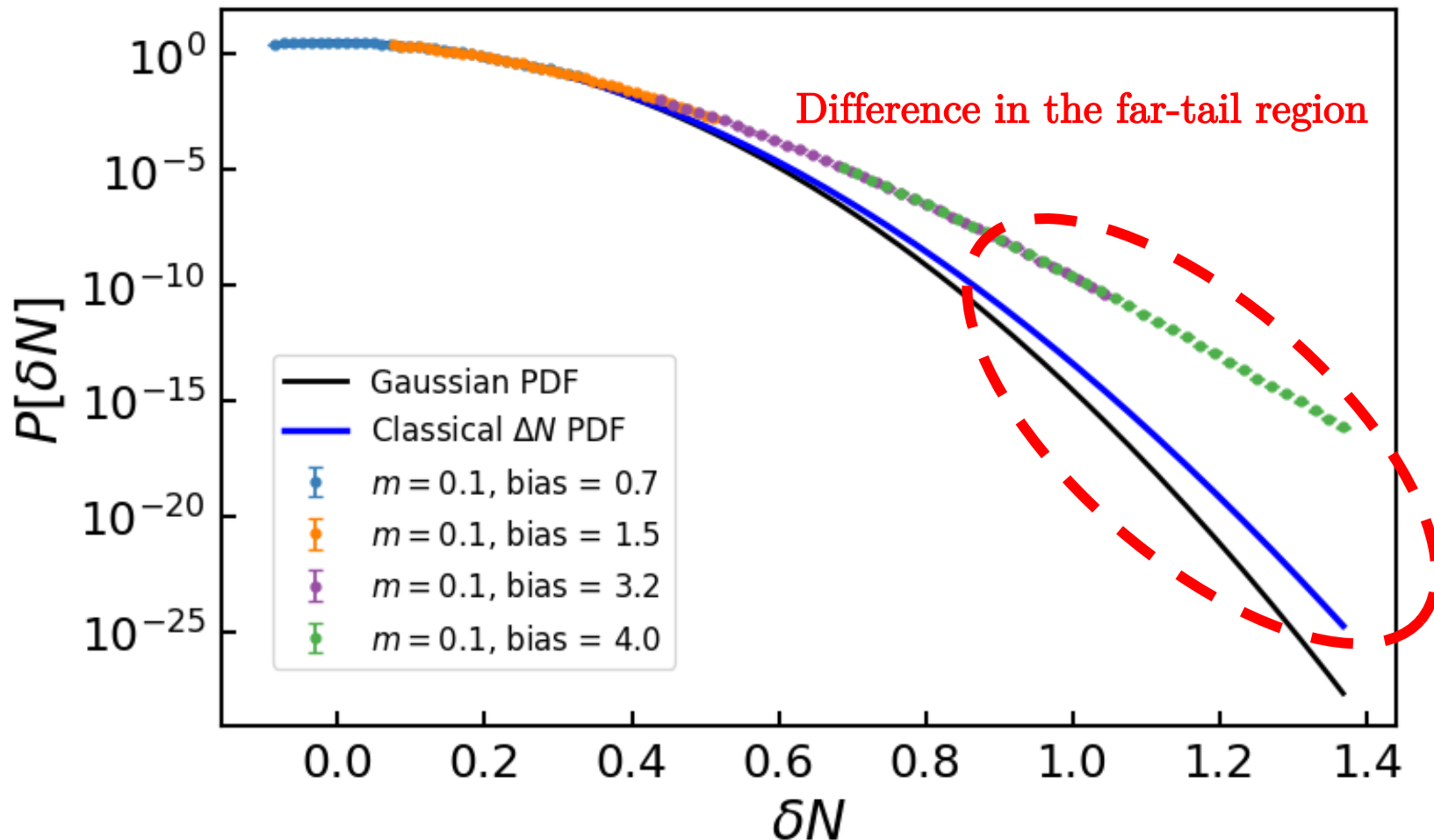
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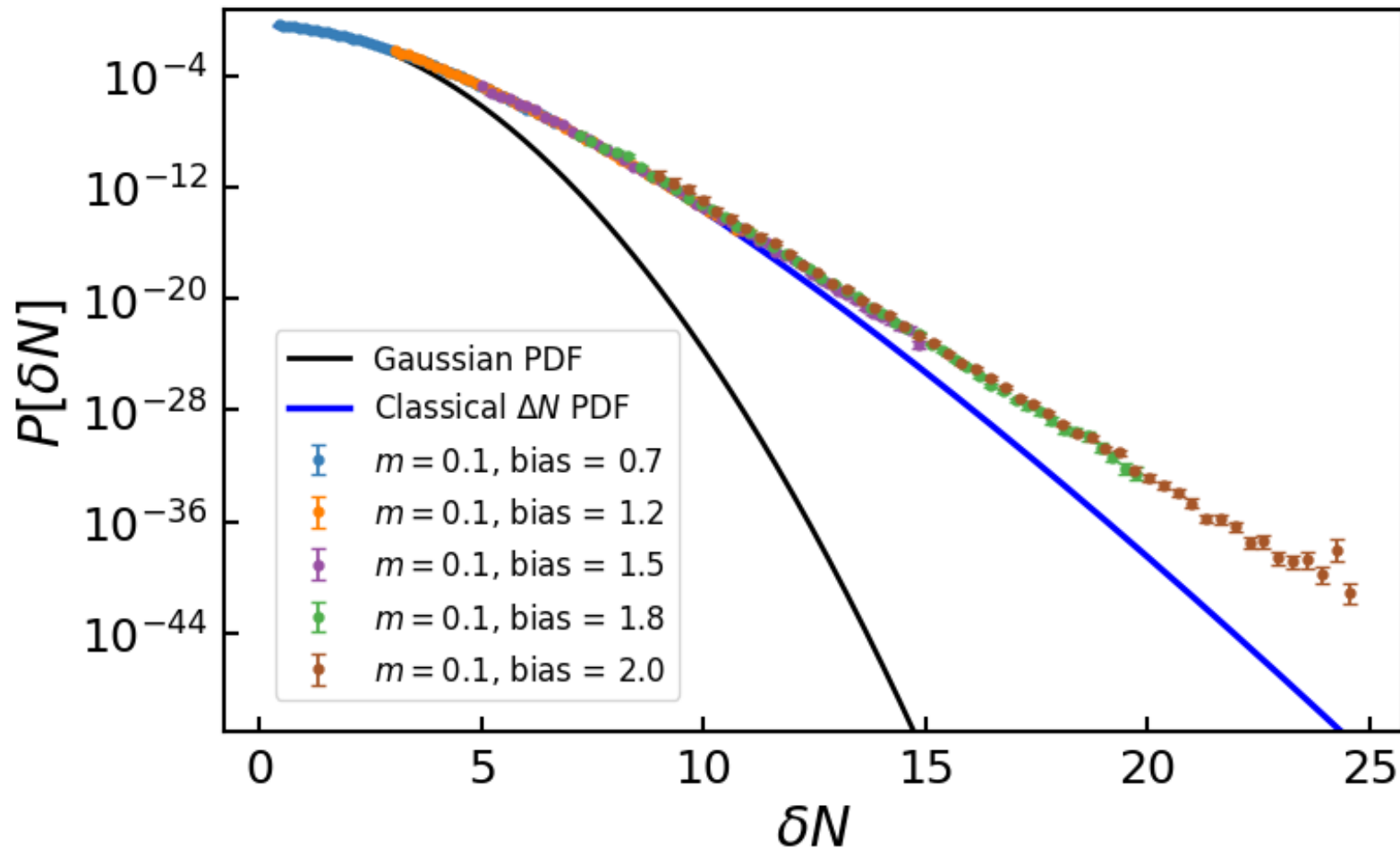
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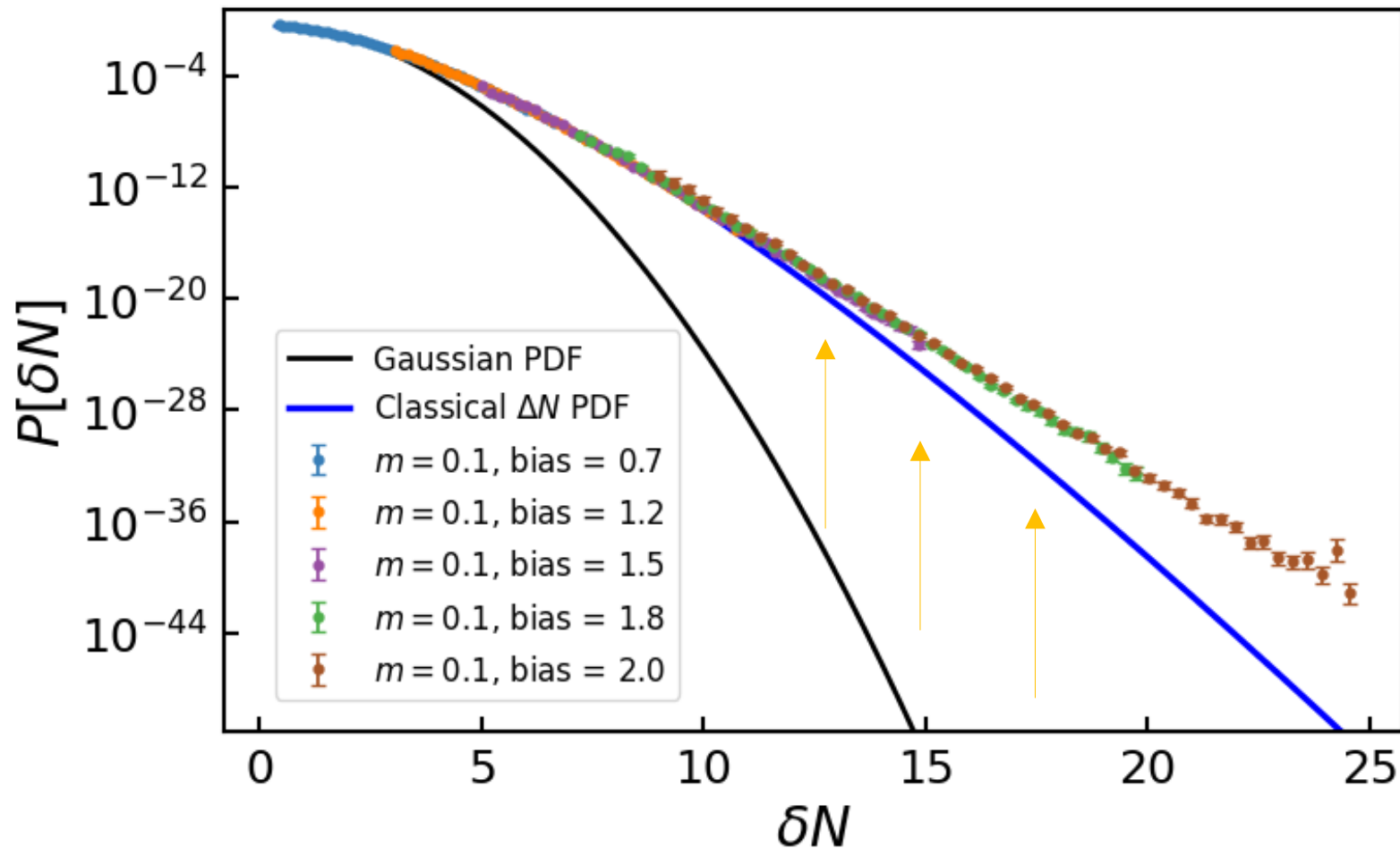
# Classical vs Stochastic for $m^2 \phi^2$ ( $N = 20$ )

- Mass of the inflaton field,  $m = 0.1$
- $\phi_* = \phi_5$ , *i.e.* field value 5 e-folds before the end of inflation
- $\phi_{\text{ini}} = \phi_{25}$  *i.e.* field value 25 e-folds before the end of inflation
- $\phi_{\text{end}} = \sqrt{2}$



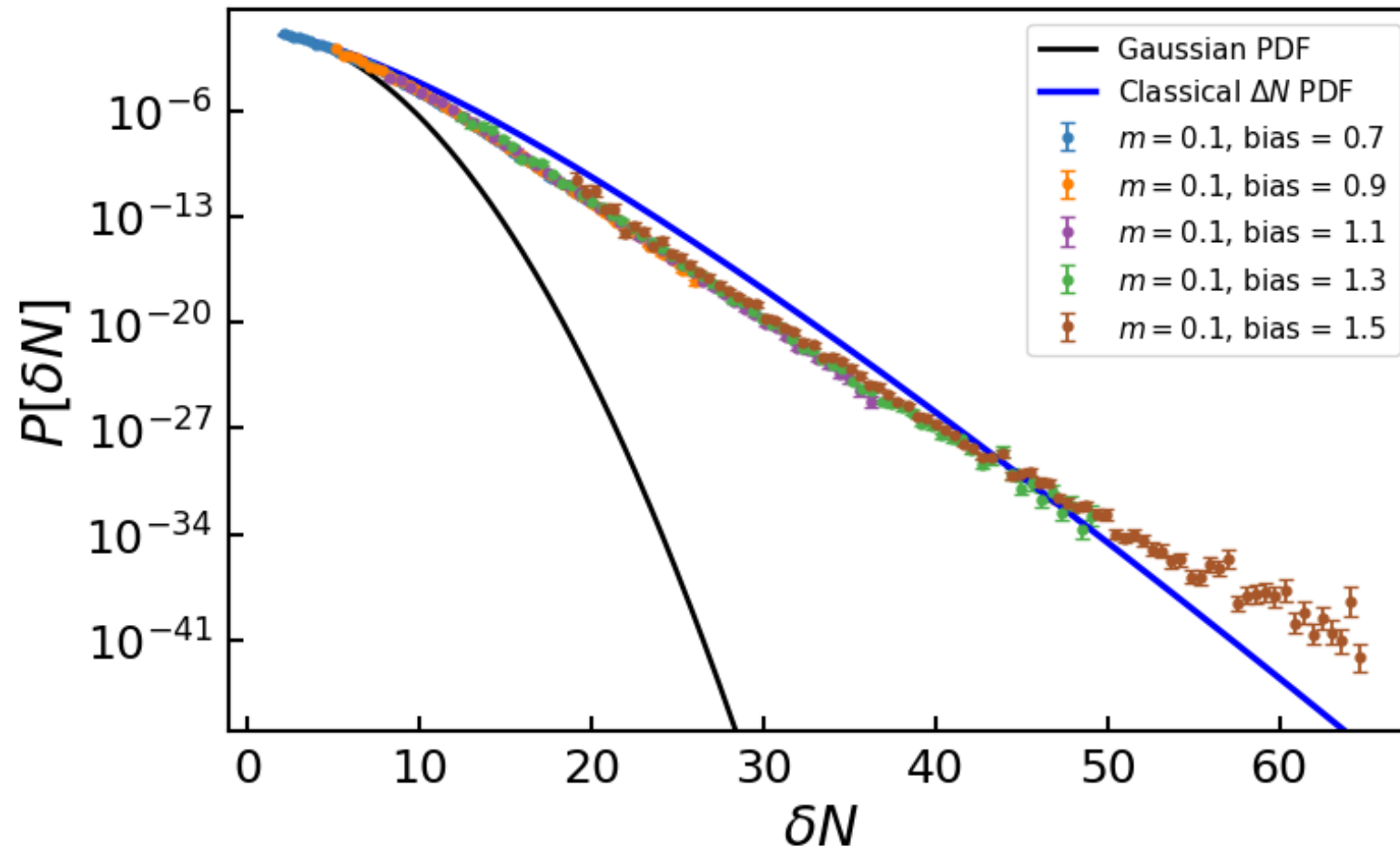
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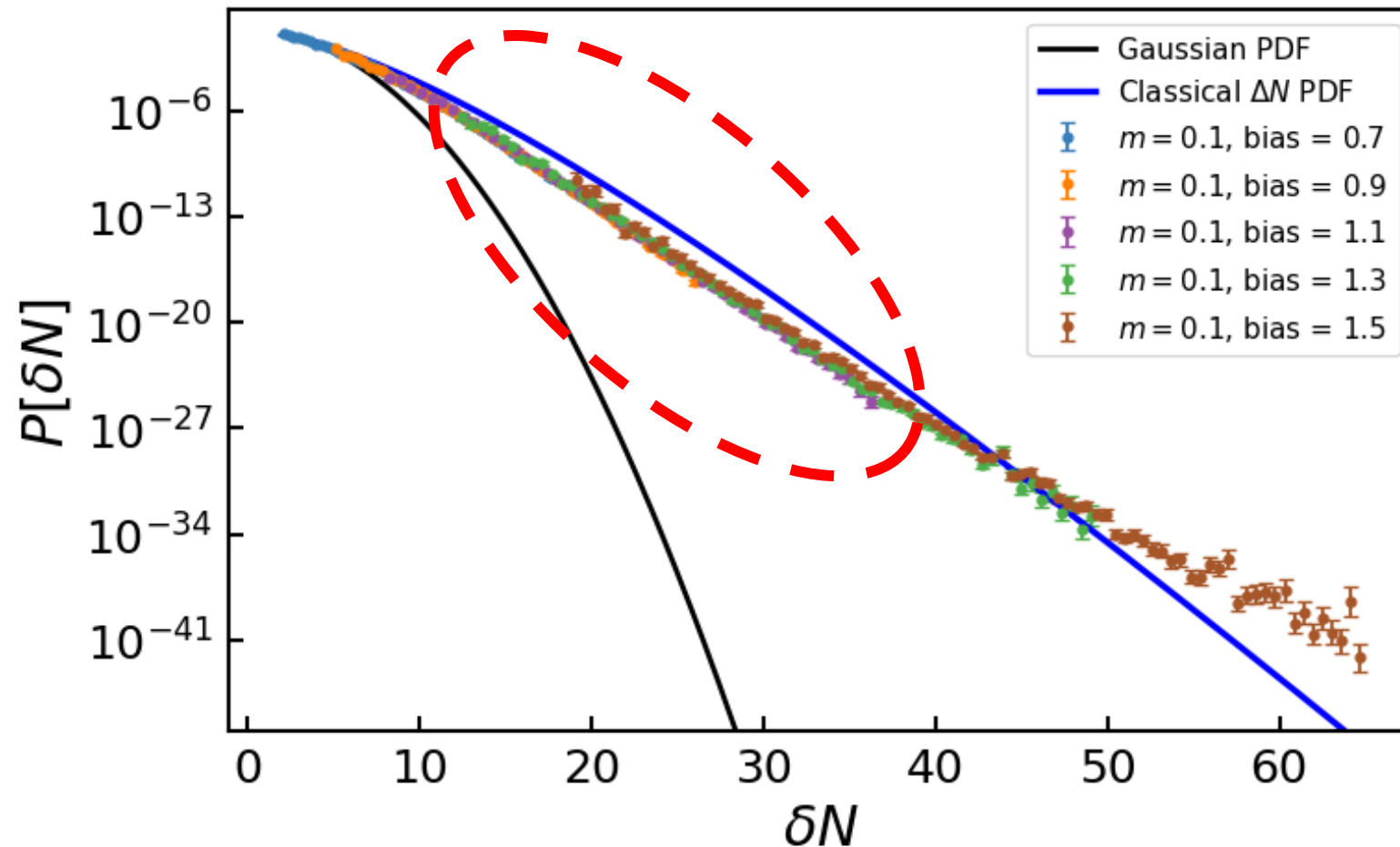
# Classical vs Stochastic for $m^2 \phi^2$ ( $N = 30$ )

- Mass of the inflaton field,  $m = 0.1$
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- $\phi_{\text{ini}} = \phi_{35}$  *i.e.* field value 35 e-folds before the end of inflation
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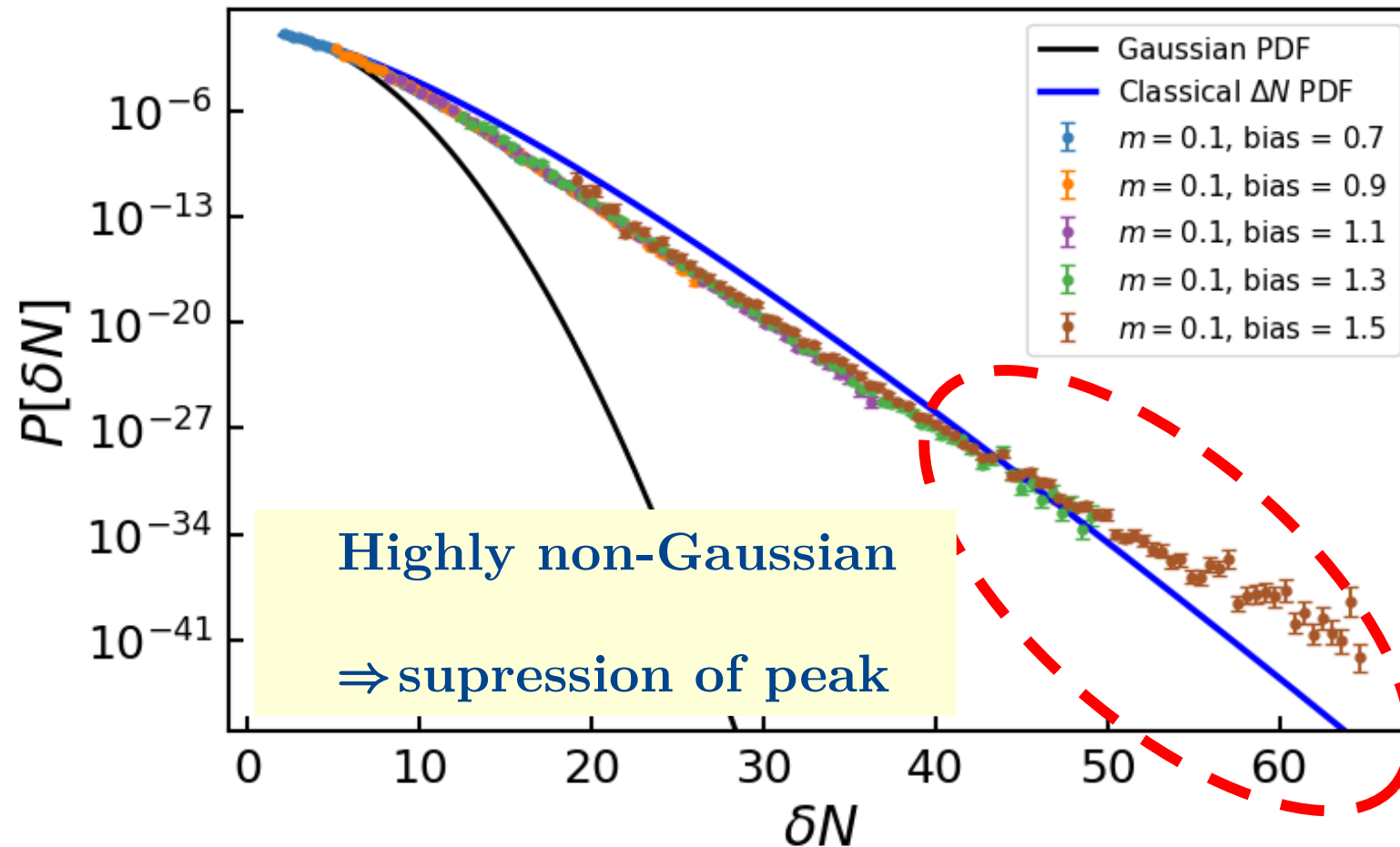
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# Summary

- Large fluctuations are described by the **tail of the PDF**
- Non-perturbative methods to calculate PDF of primordial fluctuations – classical  $\delta N$  formalism and **stochastic  $\delta\mathcal{N}$  formalism**
- **Classical  $\delta N$  formalism**: perturbed initial field hypersurface, evolution is **deterministic**
- **Stochastic  $\delta\mathcal{N}$  formalism**: uniform field initial hypersurface, evolution is **stochastic**
- For  $V(\phi) = \frac{1}{2}m^2\phi^2$ , we computed  $P[\zeta]$  using both the formalisms

## Results:

- $P[\zeta]$  has a **Gaussian peak** (expected from perturbation theory)
- $P[\zeta]$  **tail region**: results are **formalism-dependent**
- SI captures backreaction of the small modes on the long modes

Thank you

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**Back-up slides**

# Importance Sampling: PyFPT

- Sampling method for rare events in the PDF

- Consider a Langevin process:

$$\text{Target (T) : } \frac{dx}{dt} = -D(x, t) + H(x, t)\xi$$

- Introduce a *bias function*  $\Rightarrow$  increase the prob of rare events

$$\text{Sample (S) : } \frac{dx}{dt} = [-D(x, t) + \mathcal{B}(x, t)] + H(x, t)\xi$$

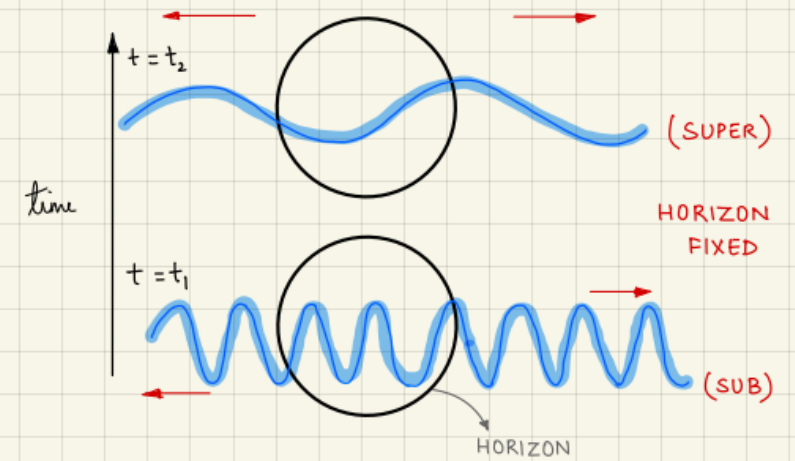
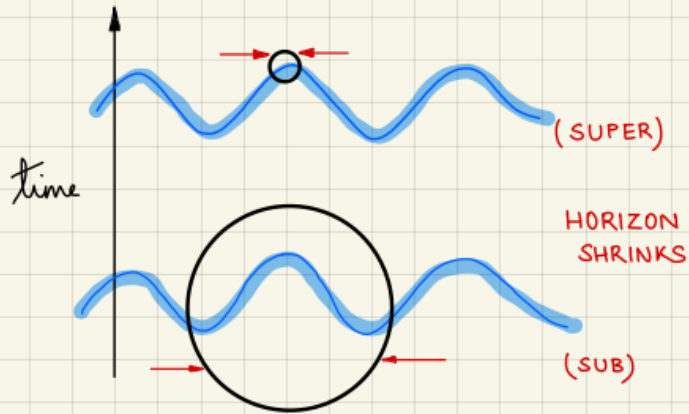
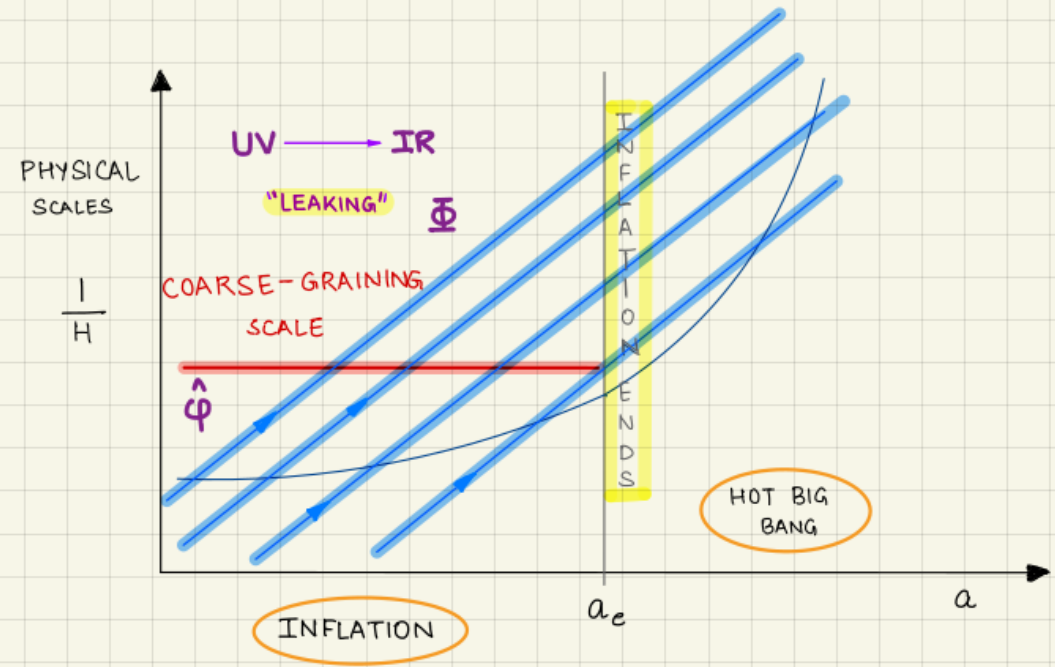
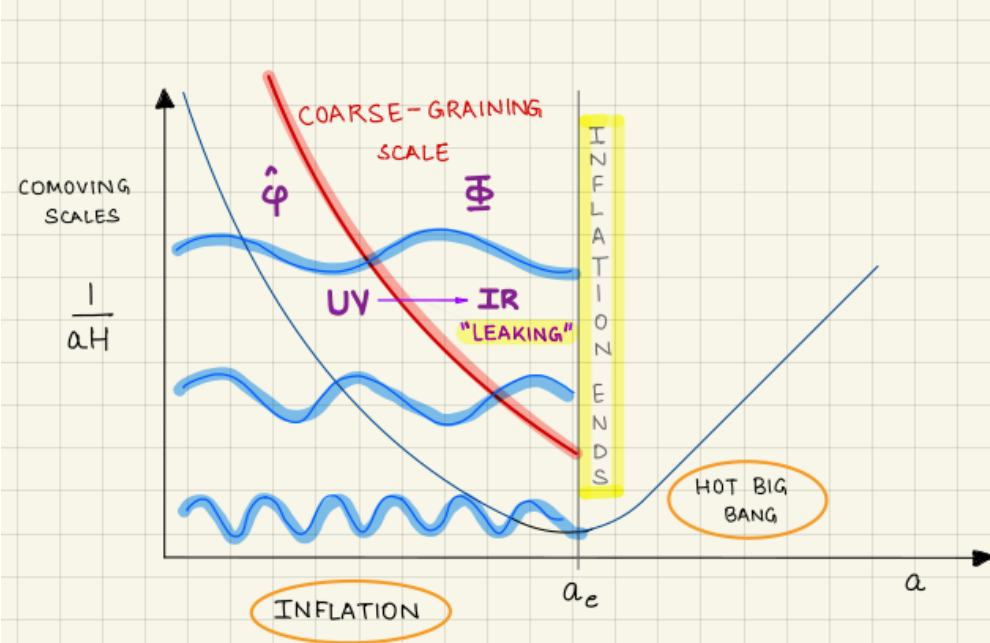
Bias

- Calculate weights: relative probability of a run  $\mathbf{X}$  in both the processes

$$w(\mathbf{X}) = \frac{p_T(\mathbf{X} | x_0)}{p_S(\mathbf{X} | x_0)}$$

- Once a run is complete, the weights and FPTs are recorded to give  $P[\mathcal{M}]$

# Coarse-graining scale



COMOVING LENGTH SCALES

PHYSICAL LENGTH SCALES