UCLouvain
Institut de recherche
en mathématique et physique

CURL
Cosmology, Universe
and Relativity at Louvain

Harvesting primordial black holes from stochastic trees

Chiara Animali

New Horizons in Primordial Black Hole Physics
Brussels, 19 May 2025

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Baptiste Blachier (CURL, UCLouvain)
Vincent Vennin (ENS, Paris)

Large perturbations from inflation

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We are interested in **large** primordial curvature perturbations that could be generated during inflation.

→ Dynamics of eternal inflation.

→ Primordial black holes, scalar induced gravitational waves.

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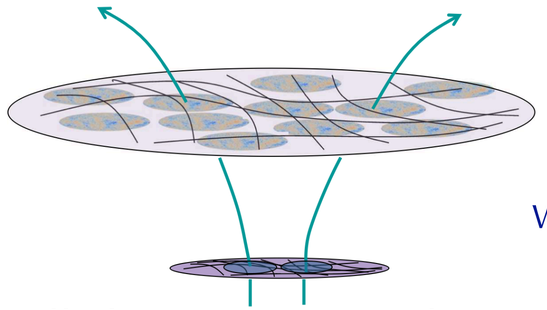

Salopek, Bond [1990]
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Ensemble of independent, locally homogenous and isotropic Hubble sized patches.

Curvature perturbation ζ is the local amount of expansion: $\boxed{\zeta(t,\vec{x}) = N(t,\vec{x}) - \overline{N}(t) \equiv \delta N}$ $\quad N(t,\vec{x}) = \log[a(t,\vec{x})]$

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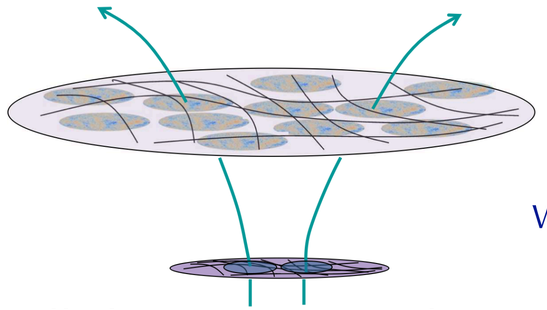

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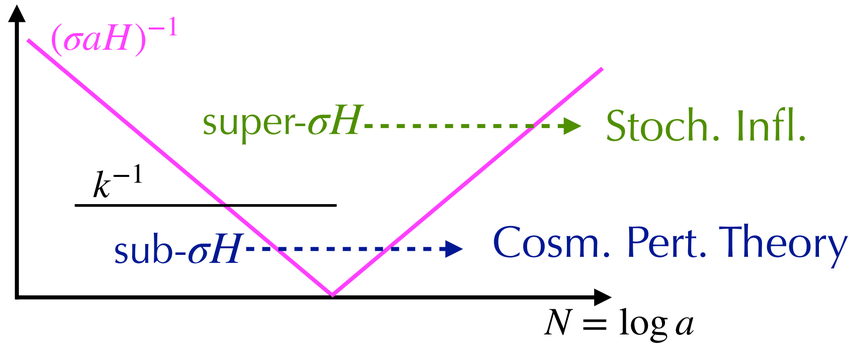
δN formalism

Backreaction → Stochastic inflation formalism

A. Starobinsky [1986]



Stochastic theory for coarse-grained fields:

classical drift

stochastic noise

Alternative way:

implement stochastic inflation on stochastic trees,

modelling inflationary expansion as a branching process

Stochastic trees for inflation

Stochastic trees for inflation

Reference: single-field slow-roll model

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'(\phi)}{3H^2} + \frac{H}{2\pi}\xi(N)$$

Langevin equation

$$\langle \xi(N) \rangle = 0$$
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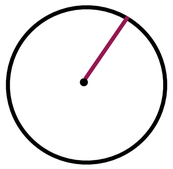
White Gaussian noise

$R_\sigma = (\sigma H)^{-1}$



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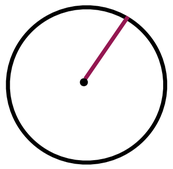

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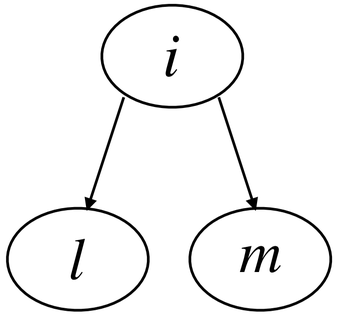

elementary vertex

Children patches have no future causal contact:
separate universe implemented.

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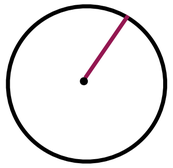

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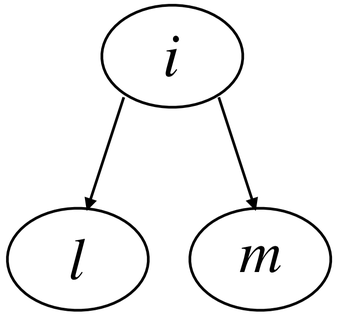

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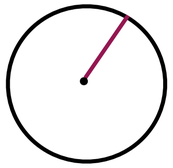

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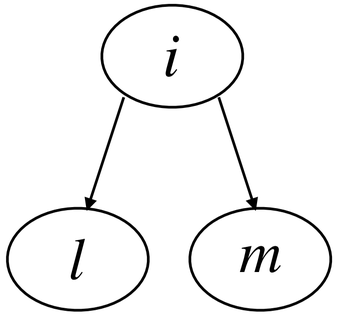

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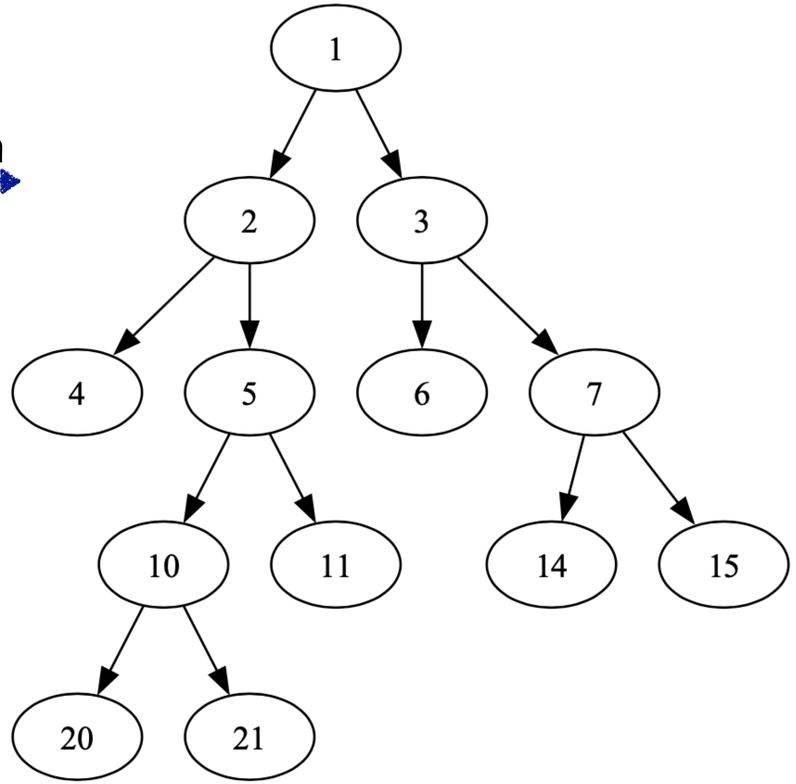

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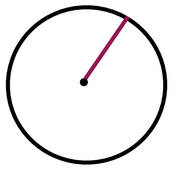

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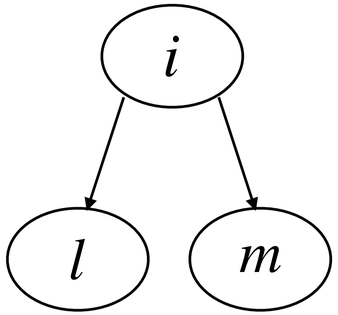

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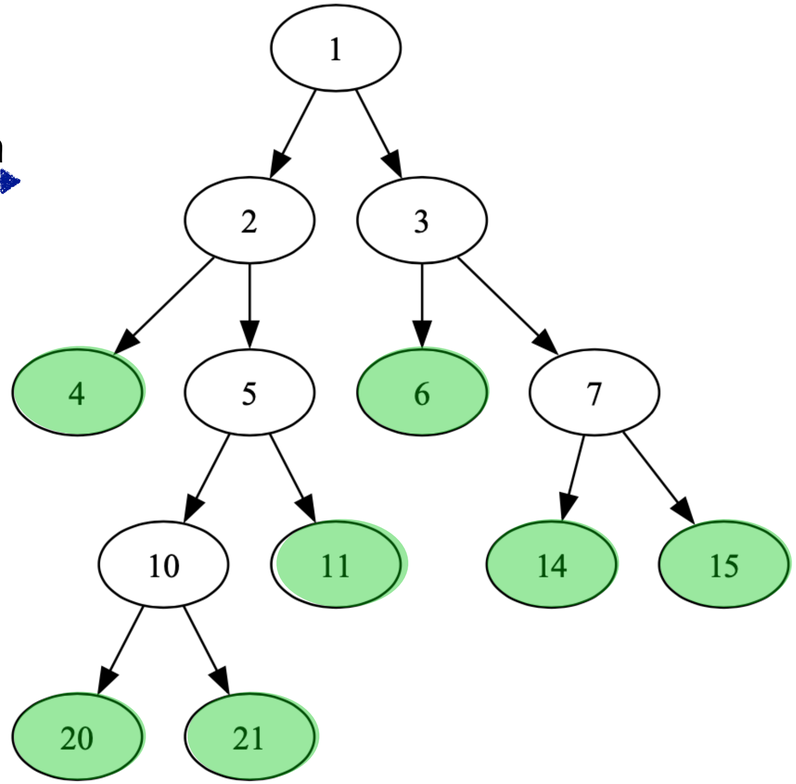

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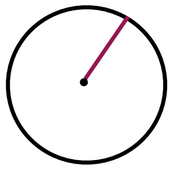

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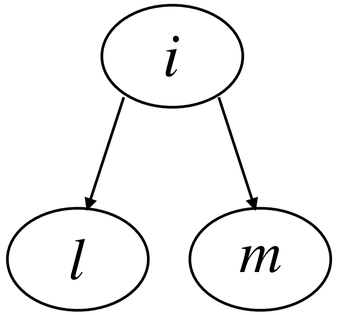

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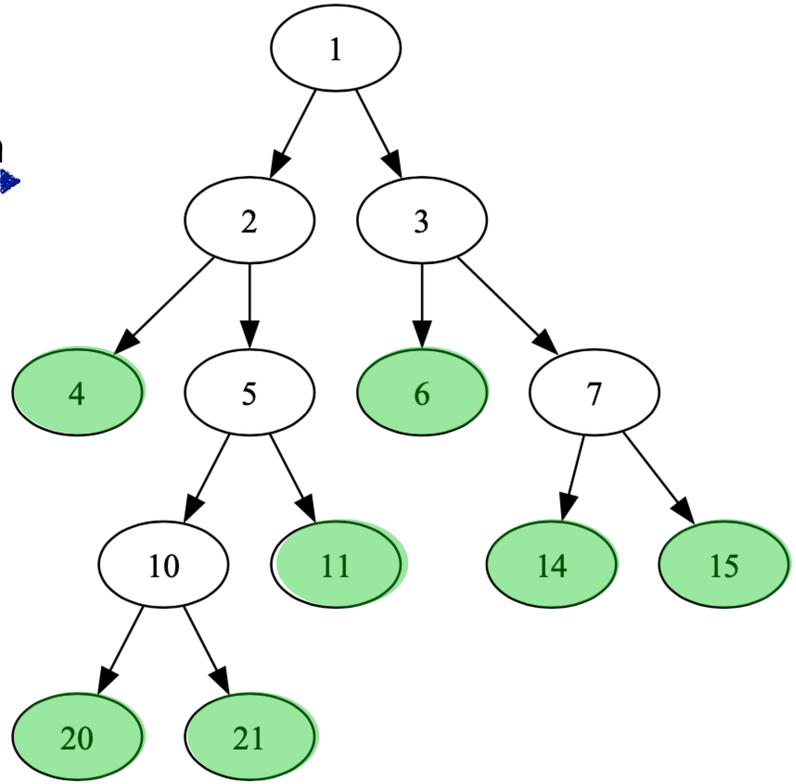

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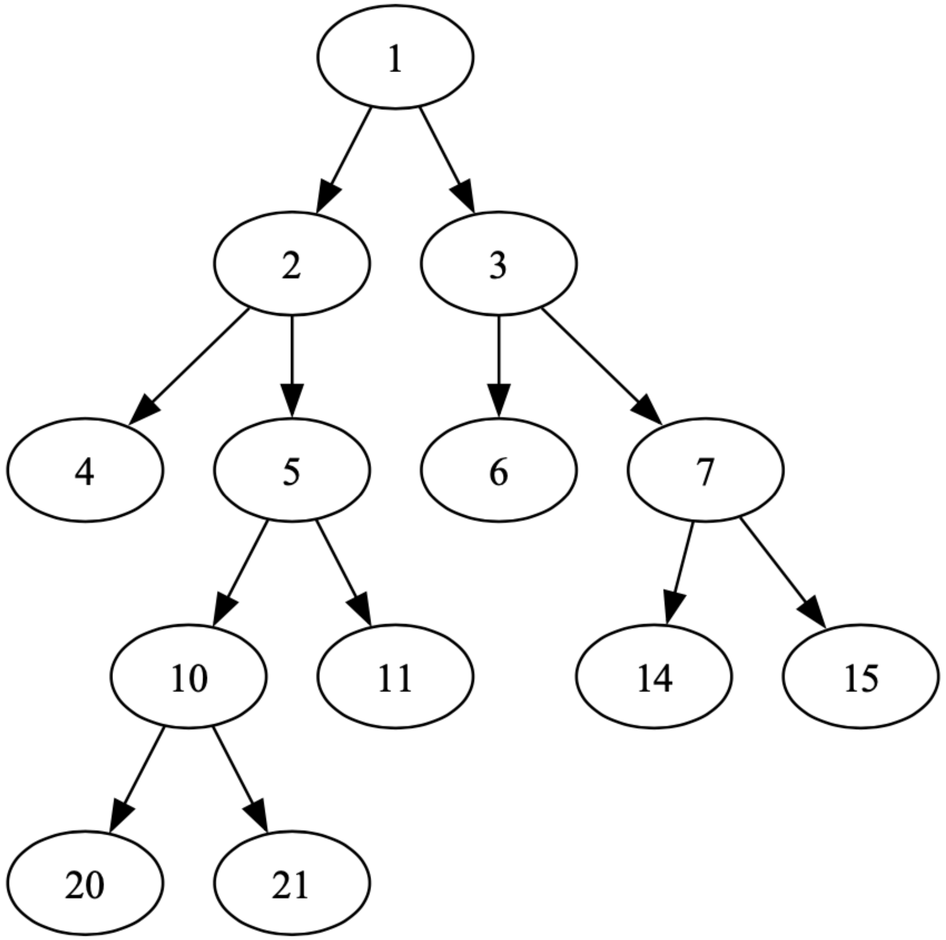
Stochastic trees: curvature perturbation at the end of inflation



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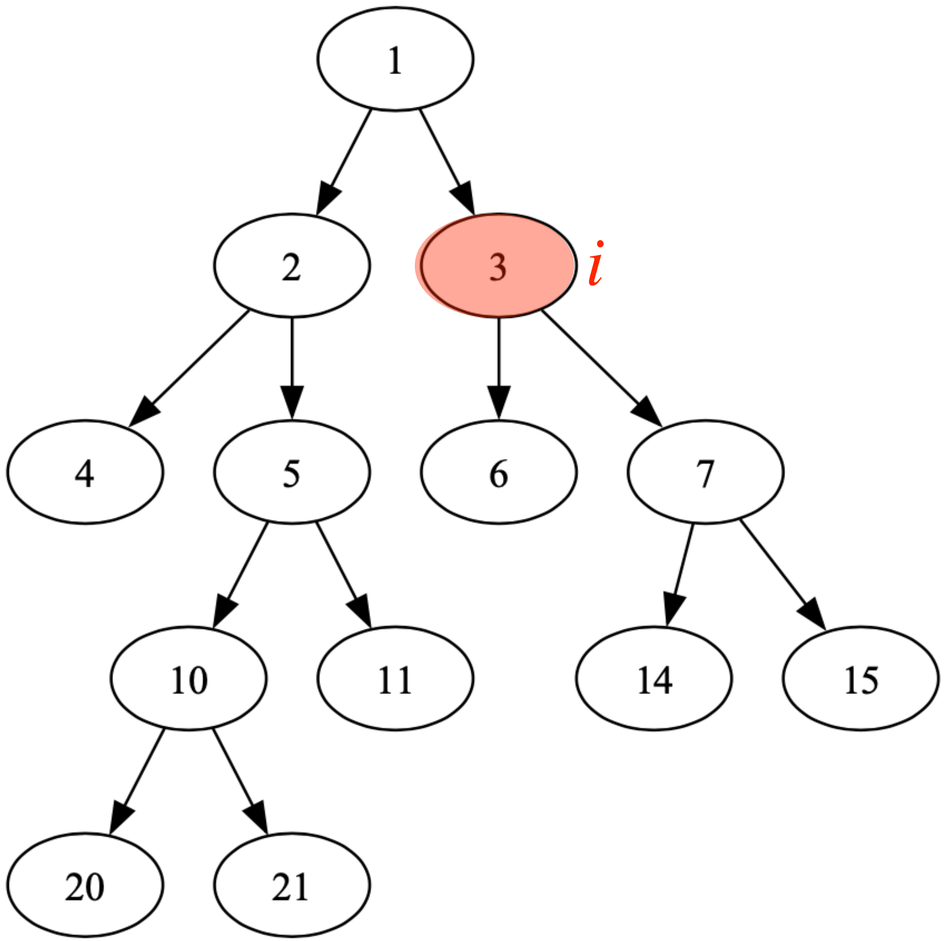

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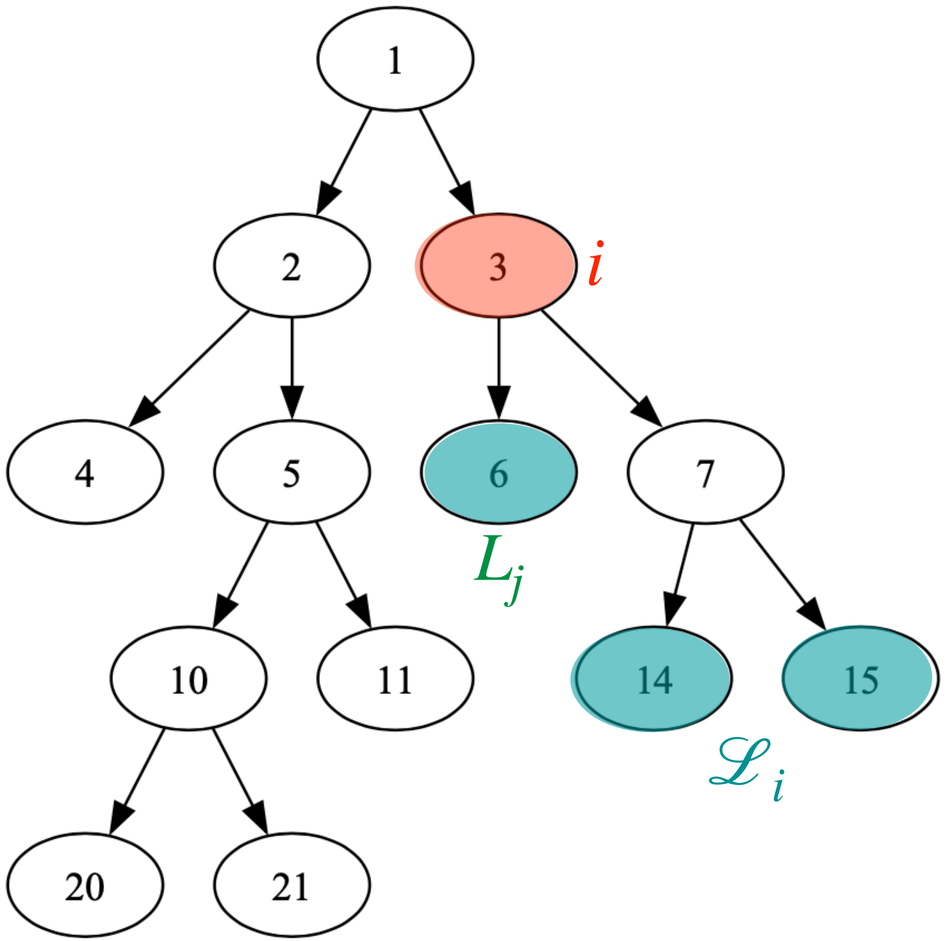

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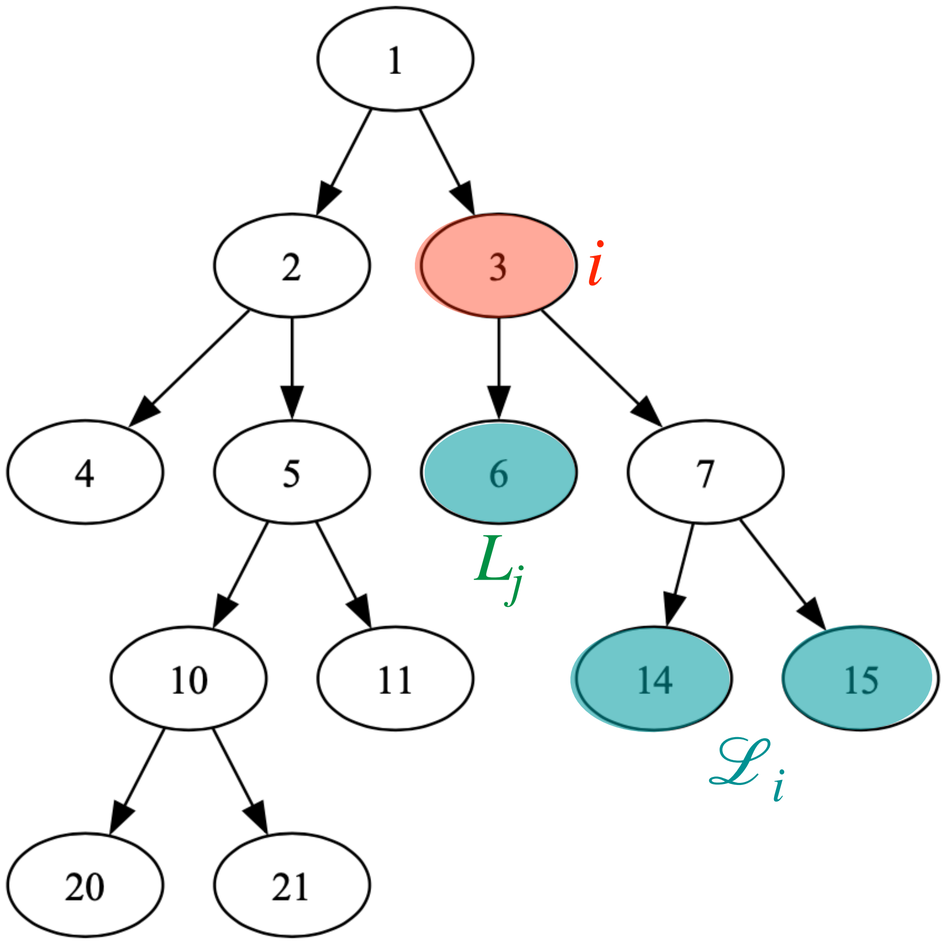

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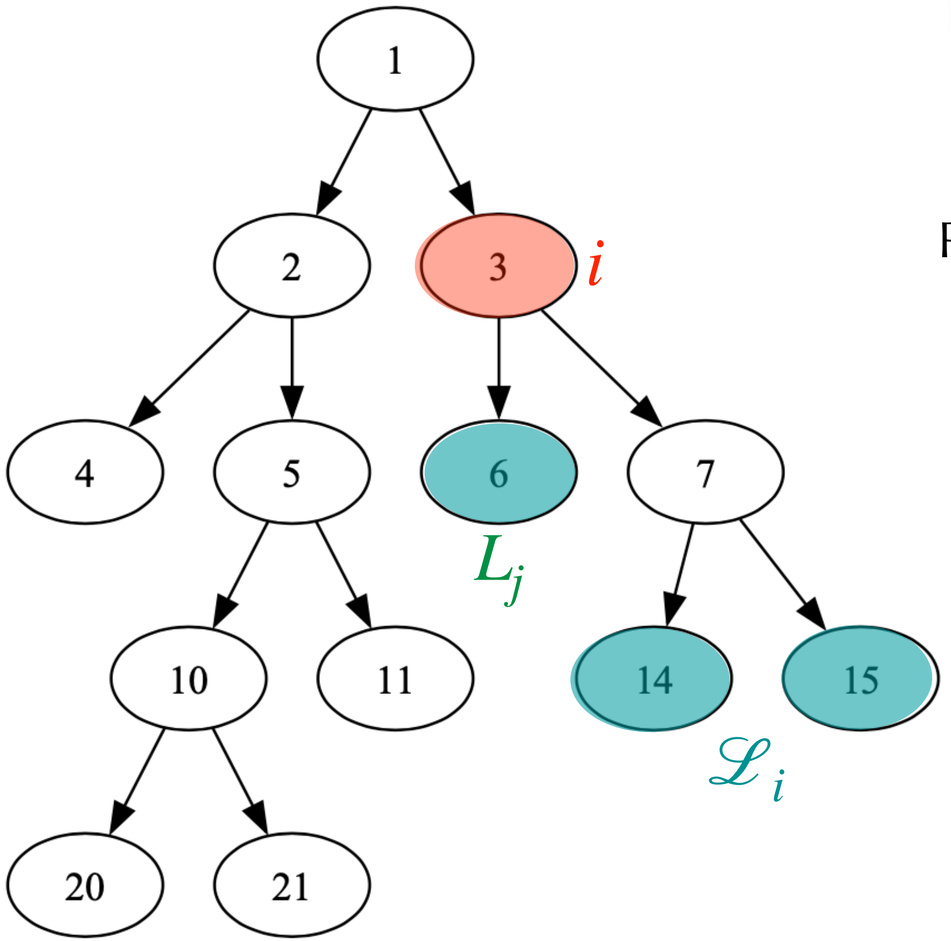

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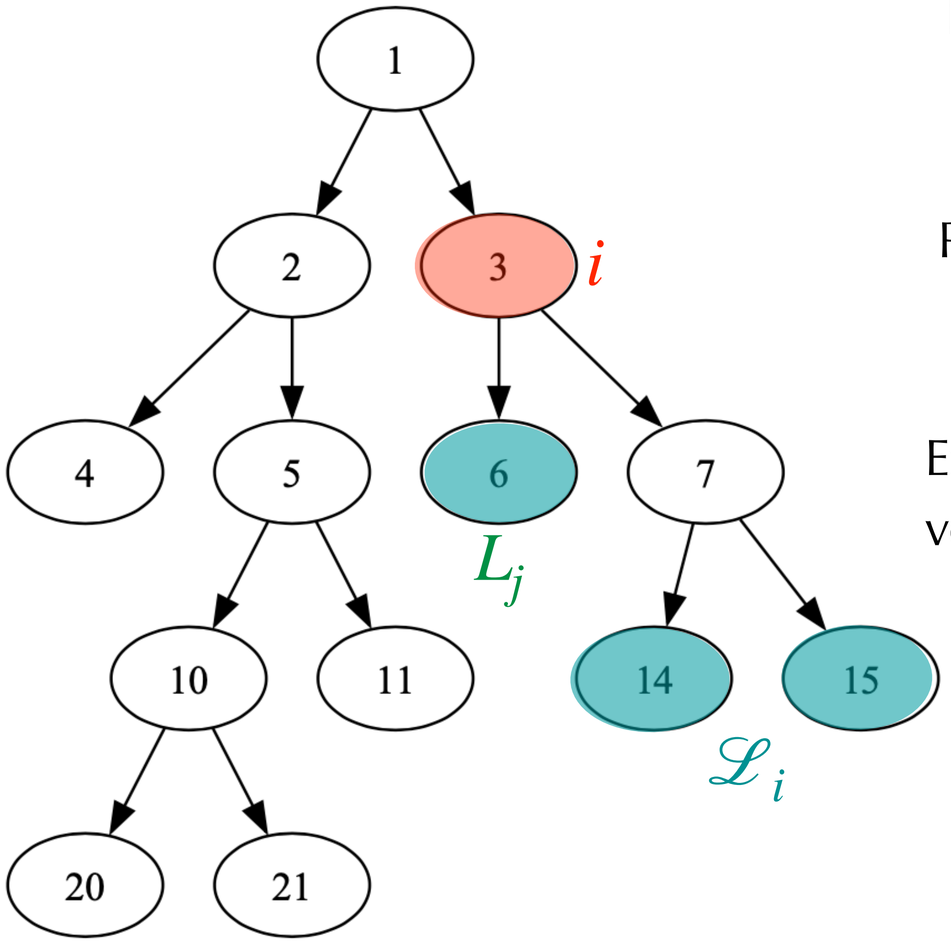

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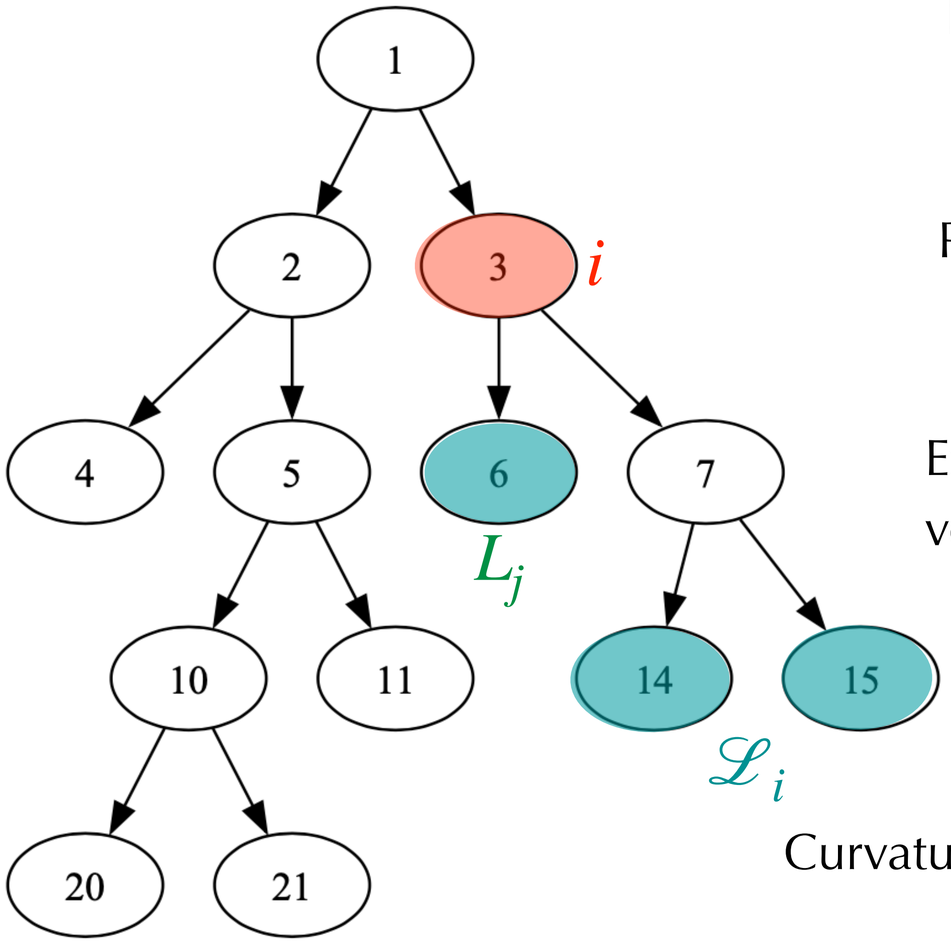

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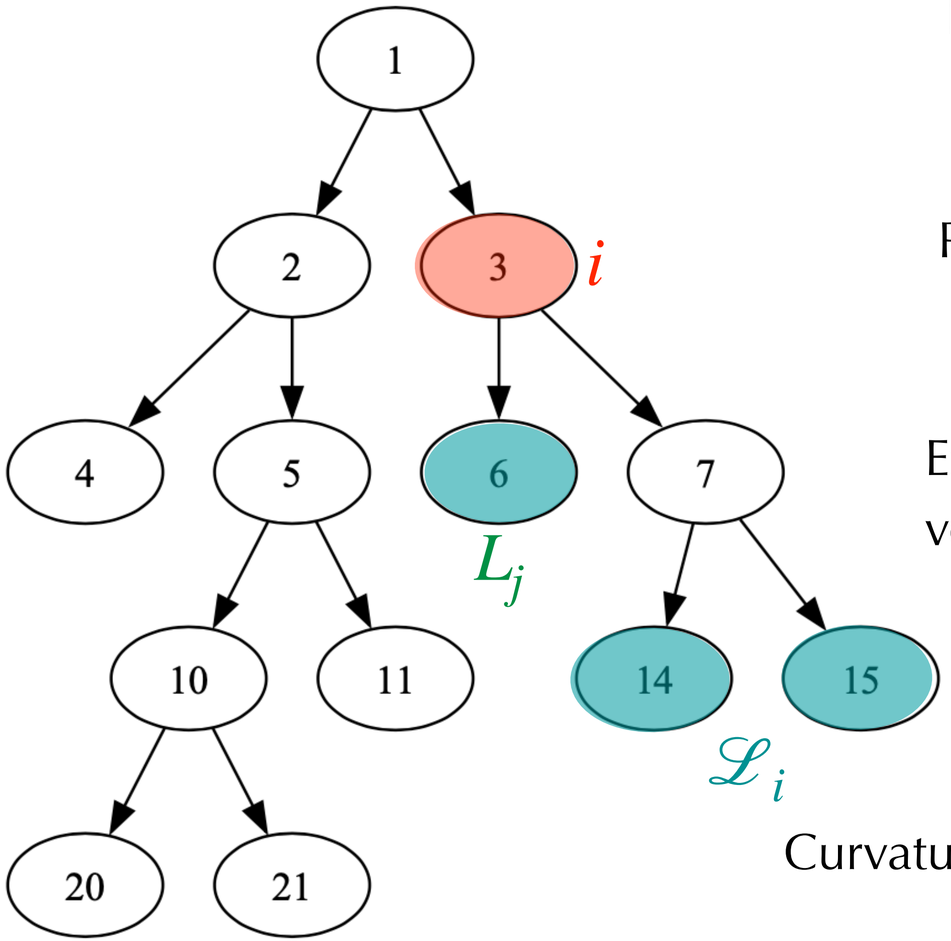

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Curvature perturbation coarse-grained over volumes larger than a leaf:

$$\zeta_i \equiv \zeta_{V_i}(\vec{x}_i) = \frac{1}{V_i} \sum_{j \in \mathscr{L}_i} V_j(\mathcal{N}_{1\to i} + \mathcal{N}_{i\to j} - W_1) = \mathcal{N}_{1\to i} + W_i - W_1$$

How to “harvest” primordial black holes in stochastic trees?

Primordial black holes formation criterion: coarse-shelled curvature perturbation

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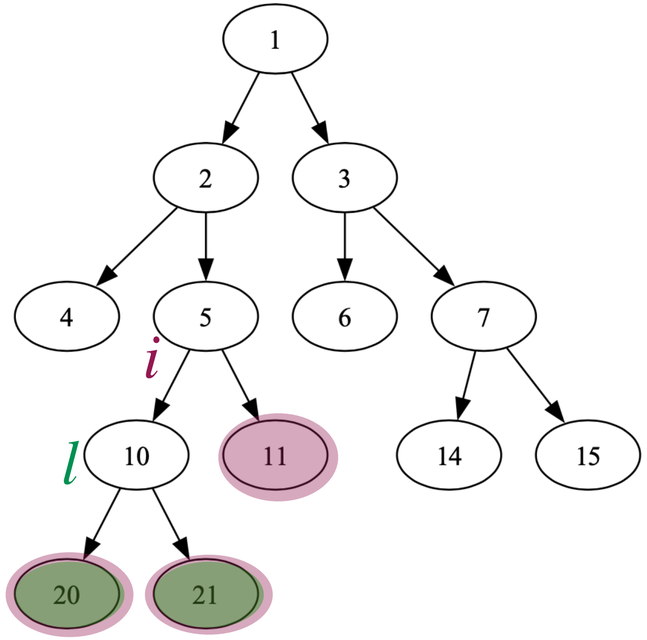
"Coarse-shelled" curvature perturbation $\Delta\zeta(\vec{x}) = \zeta_{R_1}(\vec{x}) - \zeta_{R_2}(\vec{x})$ as a "proxy" for the compaction function.

Tada, Vennin [2021]



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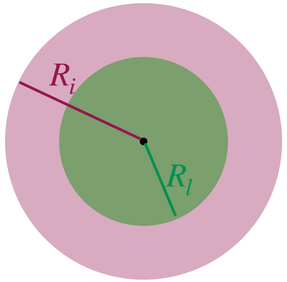

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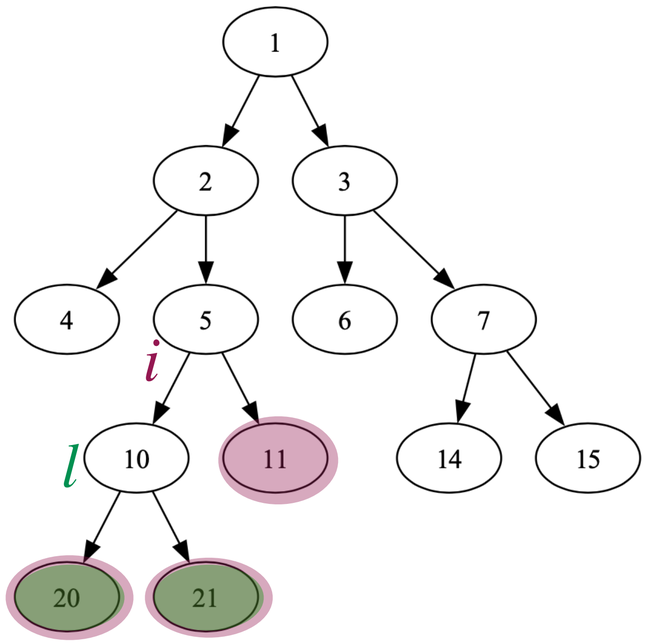

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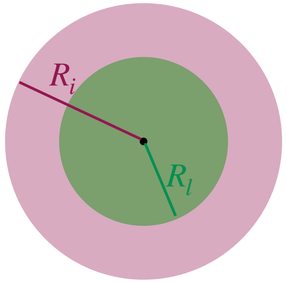

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Nodes for which $\zeta_{li} > \zeta_{li,\mathrm{c}}$ collapse into PBHs: $\zeta_{li,\mathrm{c}} = \frac{1}{2}\log\left(\frac{V_i}{V_l}\right)$ for $w = 1/2$ and $\mathscr{C}_\mathrm{c} = 0.5$.

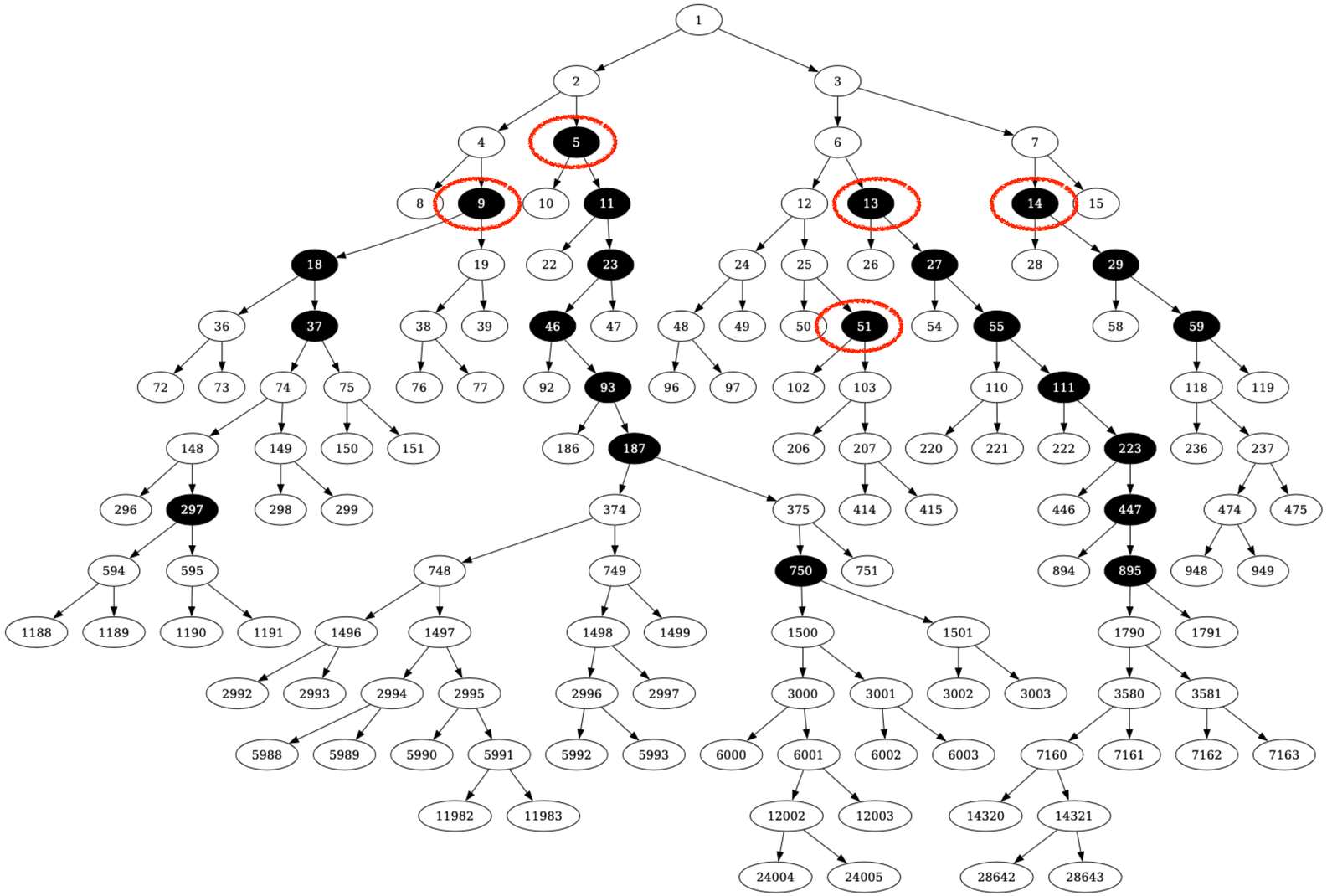
$$\zeta_{li}/\zeta_{li,\mathrm{c}} = \frac{2}{\log(V_i/V_l)}\frac{V_m}{V_i}(W_l - W_m)$$

if $W_l = W_m$ then $\zeta_{li}/\zeta_{li,\mathrm{c}} = 0$

if $V_l \gg V_m$ then $\zeta_{li}/\zeta_{li,\mathrm{c}} \simeq 2(W_l - W_m) \gg 1$

Collapse happens at most asymmetric nodes.

Harvesting primordial black holes



Nested PBH formation along a branch analogous to the cloud-in cloud problem.

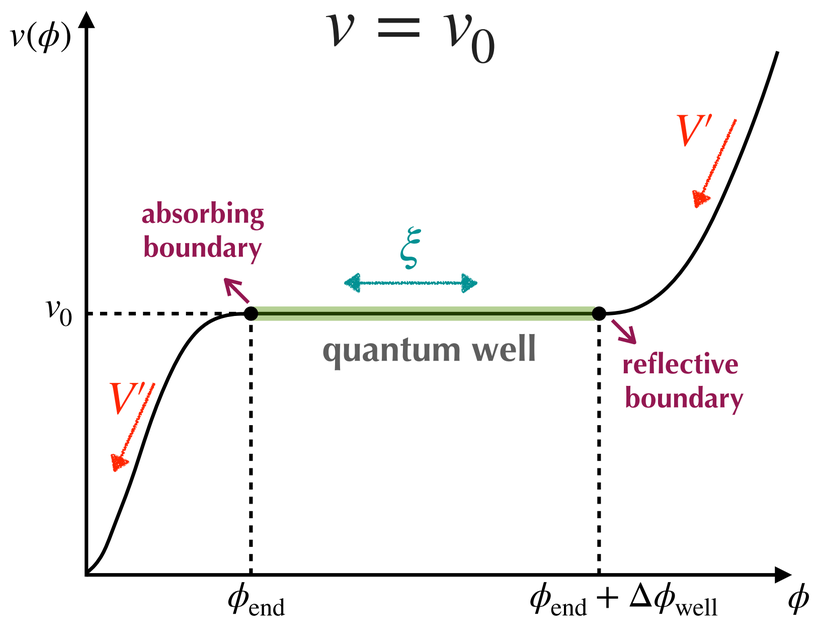
Only the highest (“oldest”) nodes are kept in the PBH inventory. Cloud-in-cloud effects naturally accounted for.

Application: flat-well toy model

Pattison, Vennin, Assadullahi, Wands [2017]

Ezquiaga, Garcia-Bellido, Vennin [2020]

Animali, Vennin [2024]



$$x = (\phi - \phi_{\rm end})/\Delta\phi_{\rm well} \in [0,1]$$

$$\mu^2 = \frac{\Delta\phi_{\rm well}^2}{v_0 M_{\rm Pl}^2}$$

$$\chi_{\mathcal{N}}(t,\phi) = \frac{\cos[\sqrt{it}\,\mu\,(x-1)]}{\cos[\sqrt{it}\,\mu]}$$

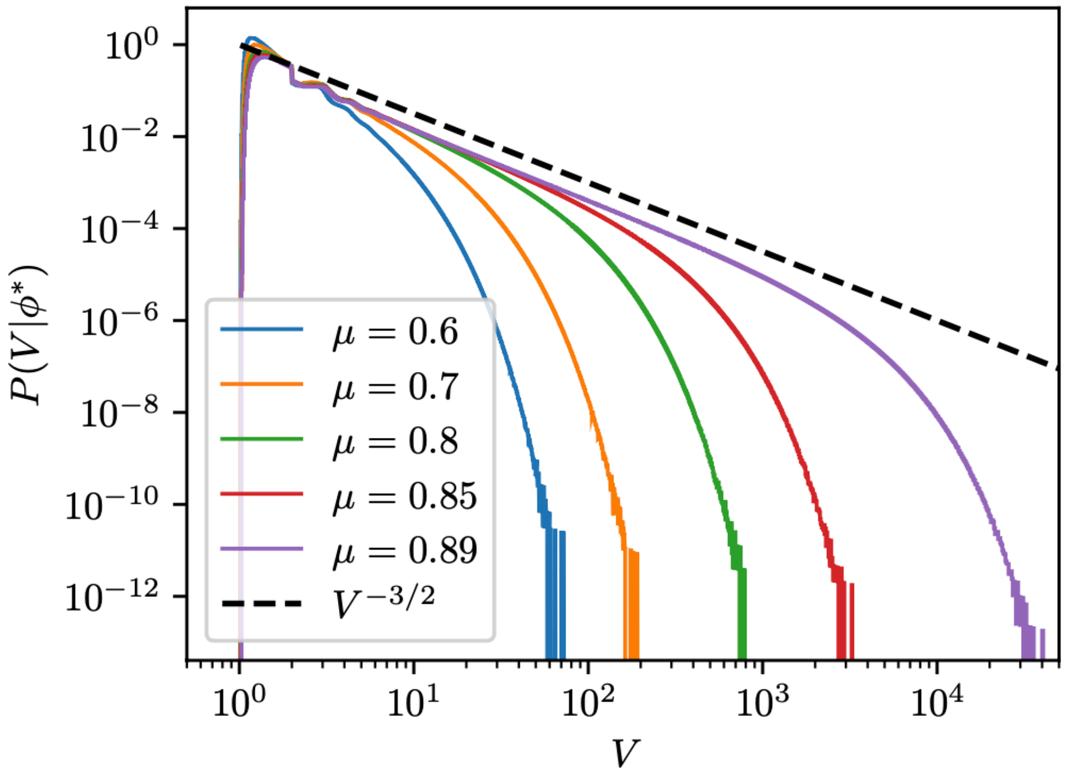
$$P_{{\rm FPT},\phi}(\mathcal{N}) = -\frac{\pi}{2\mu^2}\vartheta_2'\left(\frac{\pi}{2}x, e^{-\frac{\pi^2}{\mu^2}\mathcal{N}}\right)$$

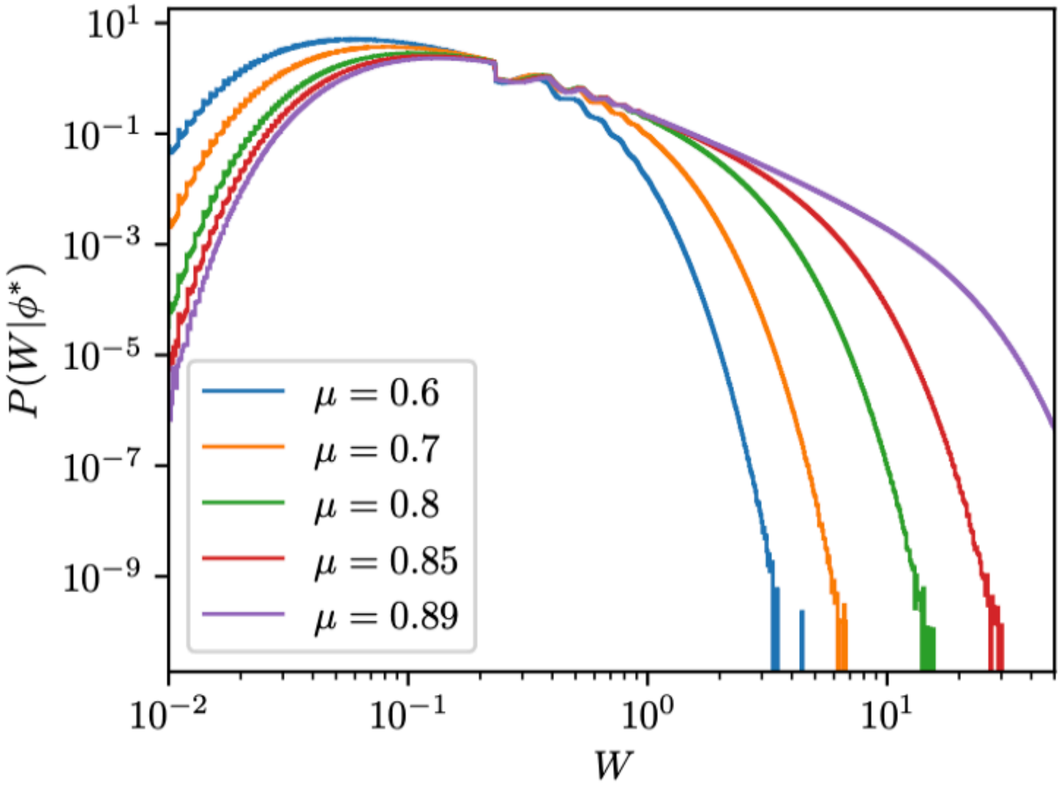
$$\langle V\rangle = \langle e^{3\mathcal{N}}\rangle = \frac{\cos[\sqrt{3}\mu(1-x)]}{\cos(\sqrt{3}\mu)}$$

$$\mu \geq \mu_{\rm c} = \frac{\pi}{2\sqrt{3}}$$ eternal inflation

Probability distributions over the trees

Statistics of the volume V and of the volume-averaged expansion W:





Mean values: $\langle V \rangle = \mathcal{O}(10)\,; \langle W \rangle = \mathcal{O}(1)\,.$

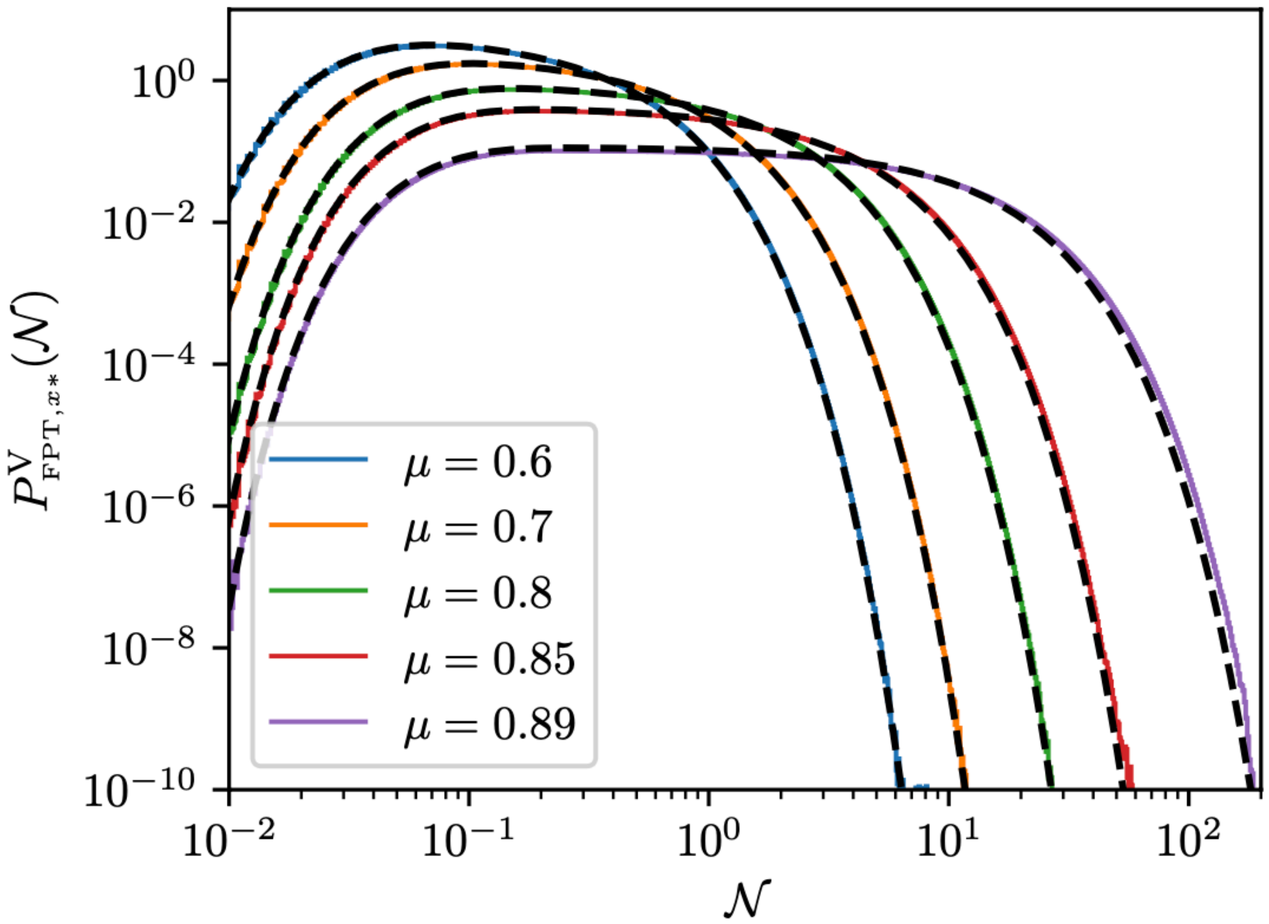
Power-law behaviour followed by exponential tails.

$P(V|\phi_*) \propto e^{z_* V} V^{-3/2}, \quad z_* < 0\,.$ For $\mu \to \mu_c$ $P(V) \propto V^{-3/2}\,.$

S. Winitzki [2008]
S. Dubovsky, L. Senatore and G. Villadoro [2009]

Probability distributions over the leaves

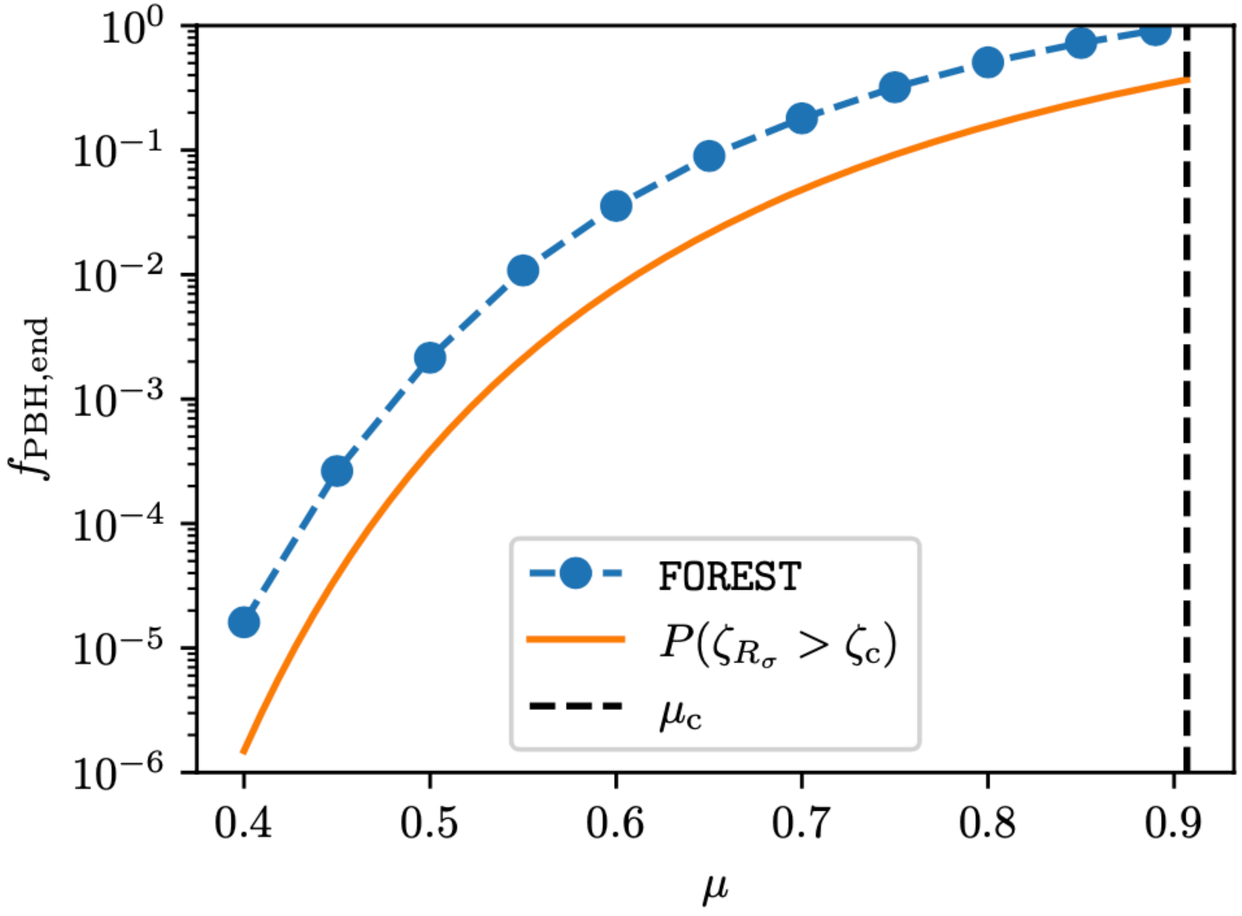
Volume-weighted first-passage-time distribution through the end-of-inflation hypersurface.



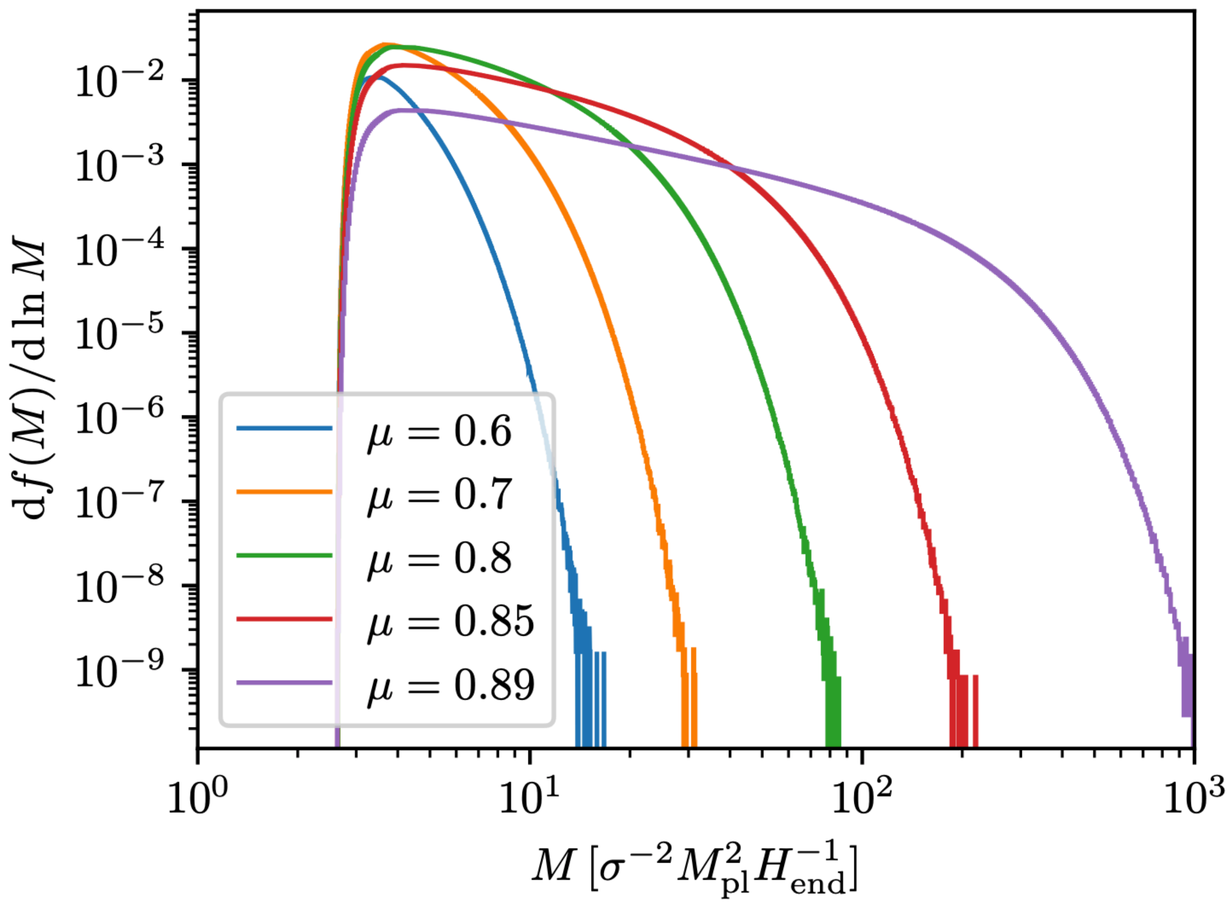
$$P^V_{\mathrm{FPT},x_*}(\mathcal{N}) = \frac{P_{\mathrm{FPT},x_*}(\mathcal{N})\, e^{3\mathcal{N}}}{\int_0^\infty \mathrm{d}\mathcal{N} P_{\mathrm{FPT},x_*}(\mathcal{N})\, e^{3\mathcal{N}}}$$

$$P_{\mathrm{FPT},x_*}(\mathcal{N}) = -\,\pi/(2\mu^2)\vartheta_2'(\pi/2\, x_*, e^{-\pi^2/\mu^2\mathcal{N}})$$

Distribution of primordial black holes



Mass distribution: $M_H(R_i) \simeq M_{\rm Pl}^2 R_i^2 H_{\rm end}$



$\mathrm{d}f/\mathrm{d}\log M \propto M^{-\alpha},\ \alpha \approx 2/3$

Final remarks

Stochastic inflation can be efficiently implemented on stochastic trees, modelling the inflationary expansion as a branching process.

Statistical properties of curvature perturbations and other cosmological fields embedded in the tree structure.

Stochastic trees are ideal tools to “harvest” primordial black holes, directly addressing the cloud-in-cloud problem.

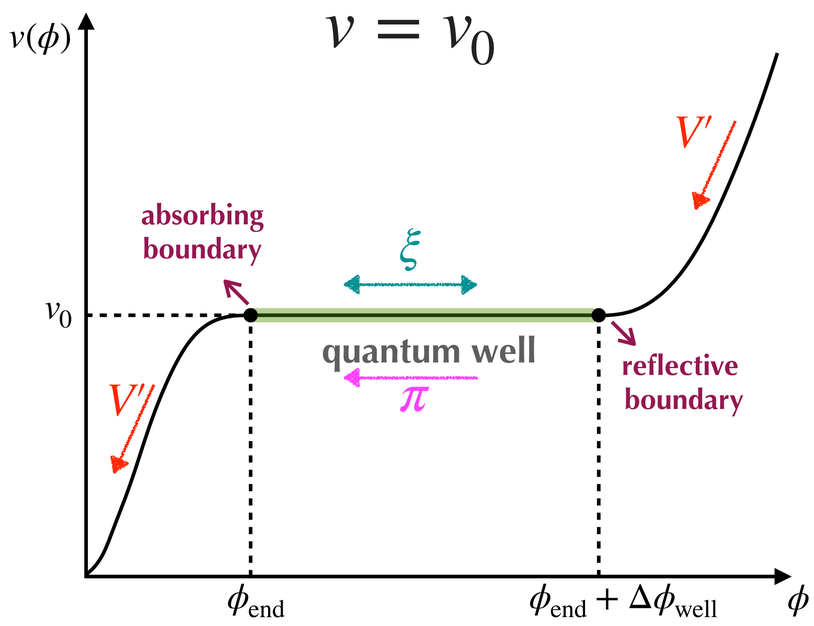
Power-law behaviour followed by exponential tails characterises statistics over the trees and over the leaves, and also the mass function of primordial black holes, in simple toy models.

Much more to explore: ultra-slow-roll scenarios, more realistic models, multiple-point statistics, higher order correlators, clustering…

Thanks!

Appendix slides

Ultra slow-roll model



$$x = (\phi - \phi_{\rm end})/\Delta\phi_{\rm well} \in [0,1] \qquad \mu^2 = \frac{\Delta\phi_{\rm well}^2}{v_0 M_{\rm Pl}^2}$$

Initial field velocity

$$y = \frac{\pi}{\pi_{\rm crit}}, \quad \pi_{\rm crit} = -3\Delta\phi_{\rm well} \qquad \pi = \frac{{\rm d}\phi}{{\rm d}N}$$

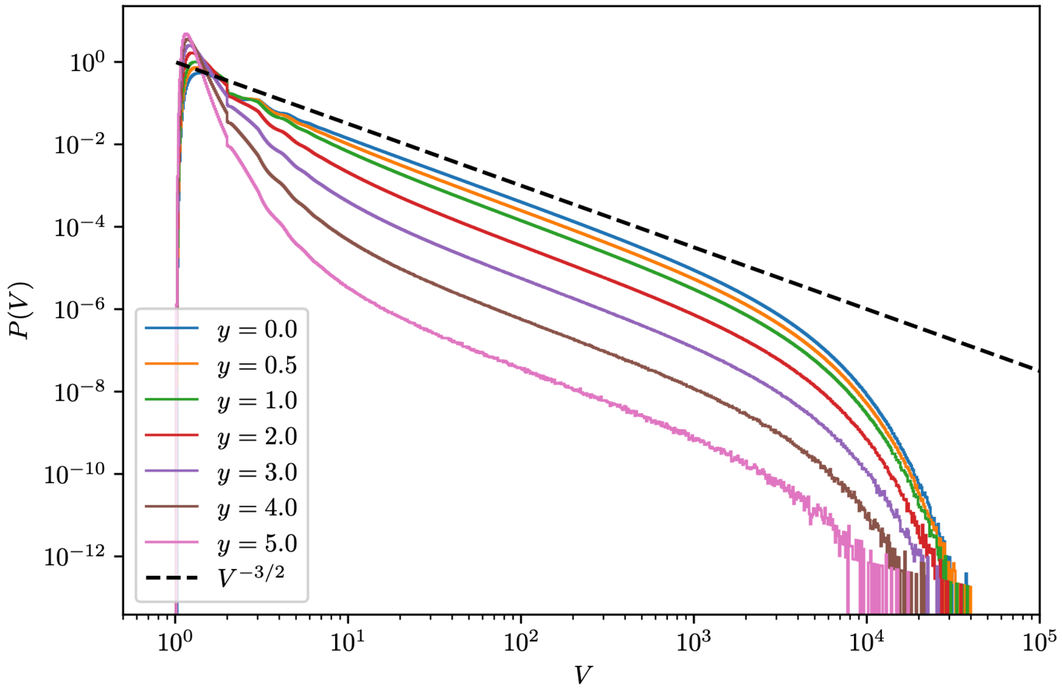
$$\begin{cases} \dfrac{{\rm d}x}{{\rm d}N} = -3y + \dfrac{\sqrt{2}}{\mu}\xi(N) \\[2ex] \dfrac{{\rm d}y}{{\rm d}N} = -3y \end{cases}$$

$y \ll 1$ stochastic limit

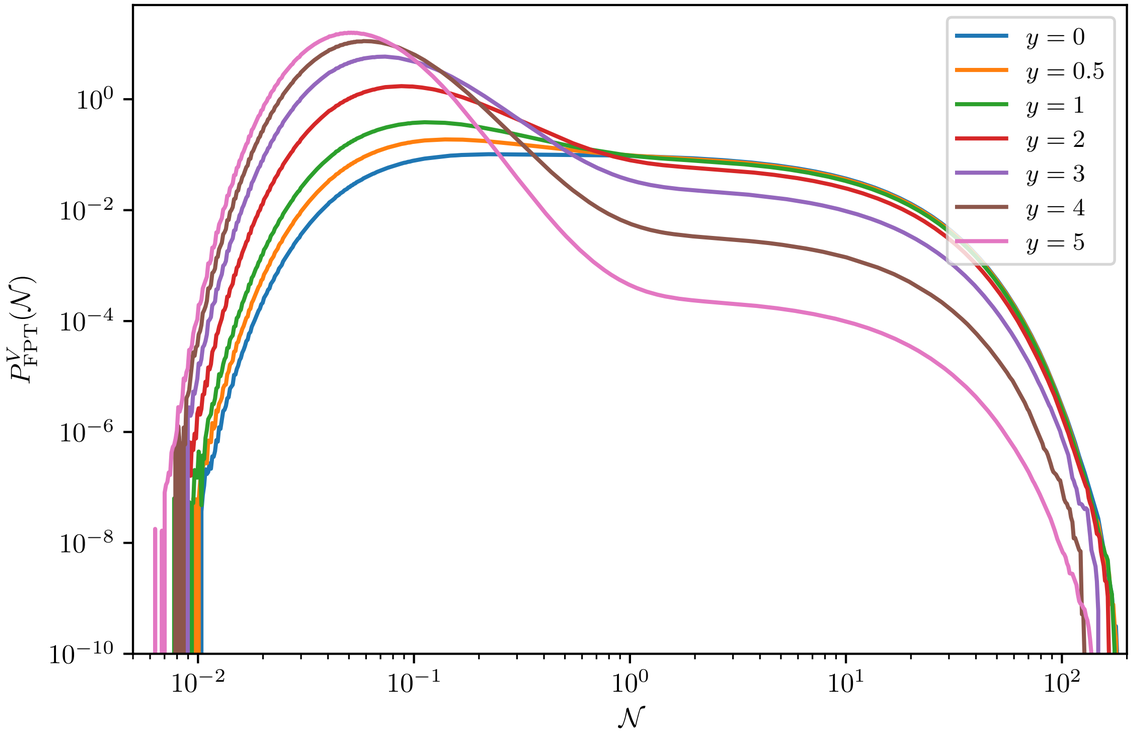
$y \gg 1$ classical limit

Ultra slow-roll model

Volume distribution



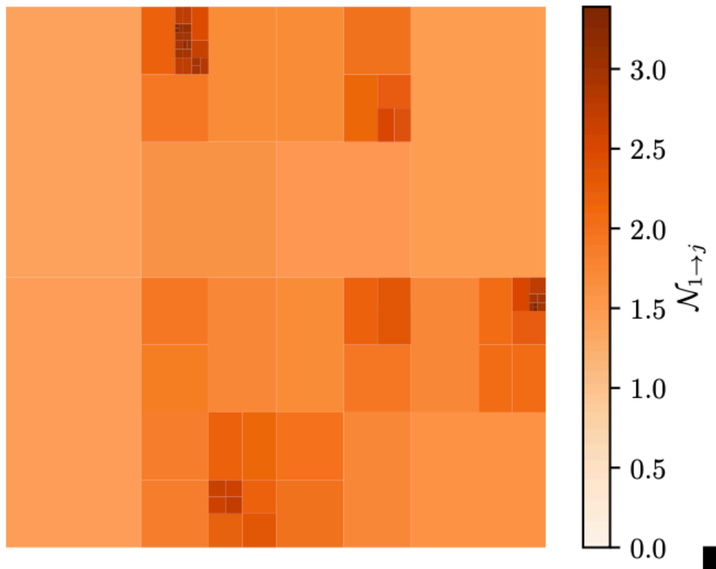
Volume-weighted FPT distribution



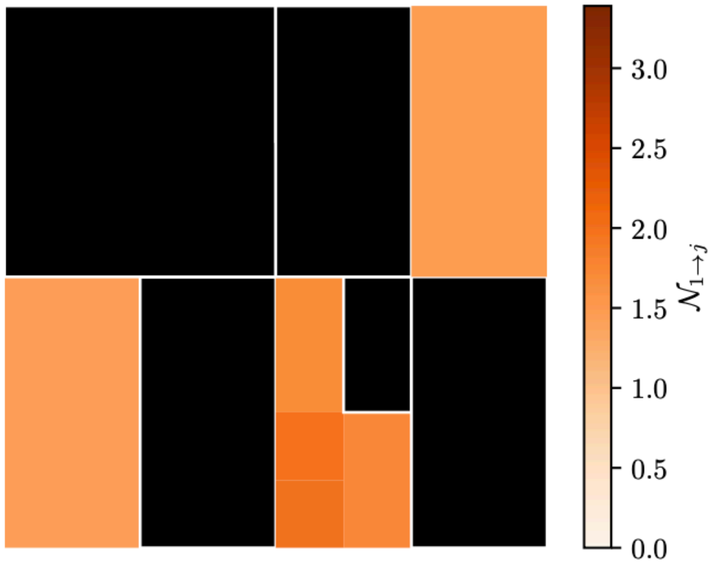
Power-law behaviour $P(V) \propto V^{-3/2}$ followed by exponential tails even for velocity $y > 1$.

Classical regime characterised by non-Gaussian tails where PBHs form.

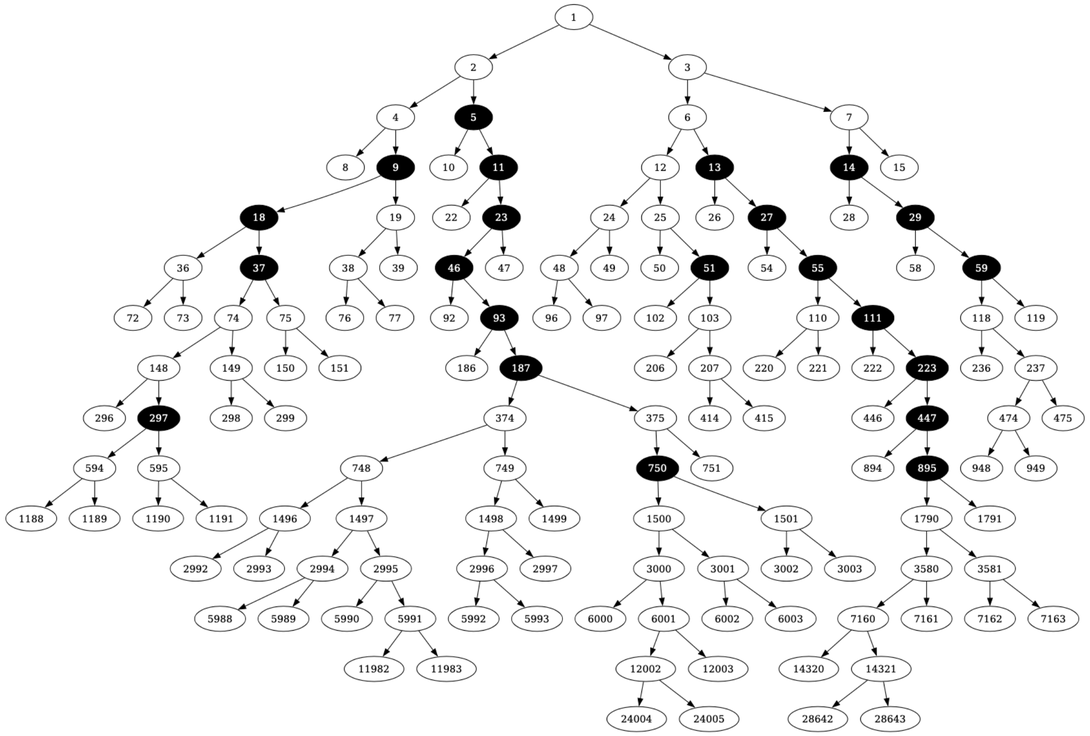
Maps of the curvature perturbation in comoving space



(a) Without PBHs.

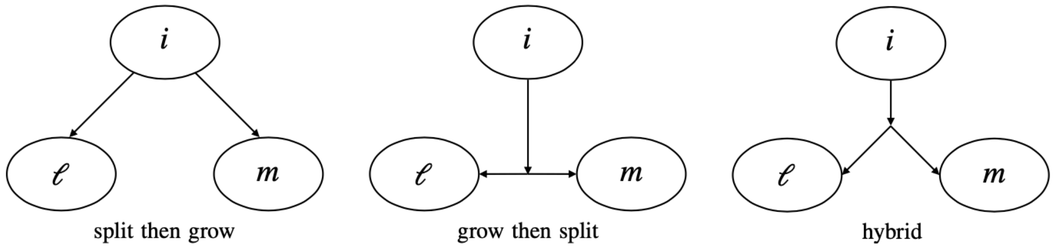


(b) With PBHs.



Discretisation artefacts: branching times

Different branching prescriptions:



Node i grows up to a time $\alpha\Delta N$, then splits, and the child branches are evolved independently for $(1-\alpha)\Delta N$ with $0 \leq \alpha \leq 1$.

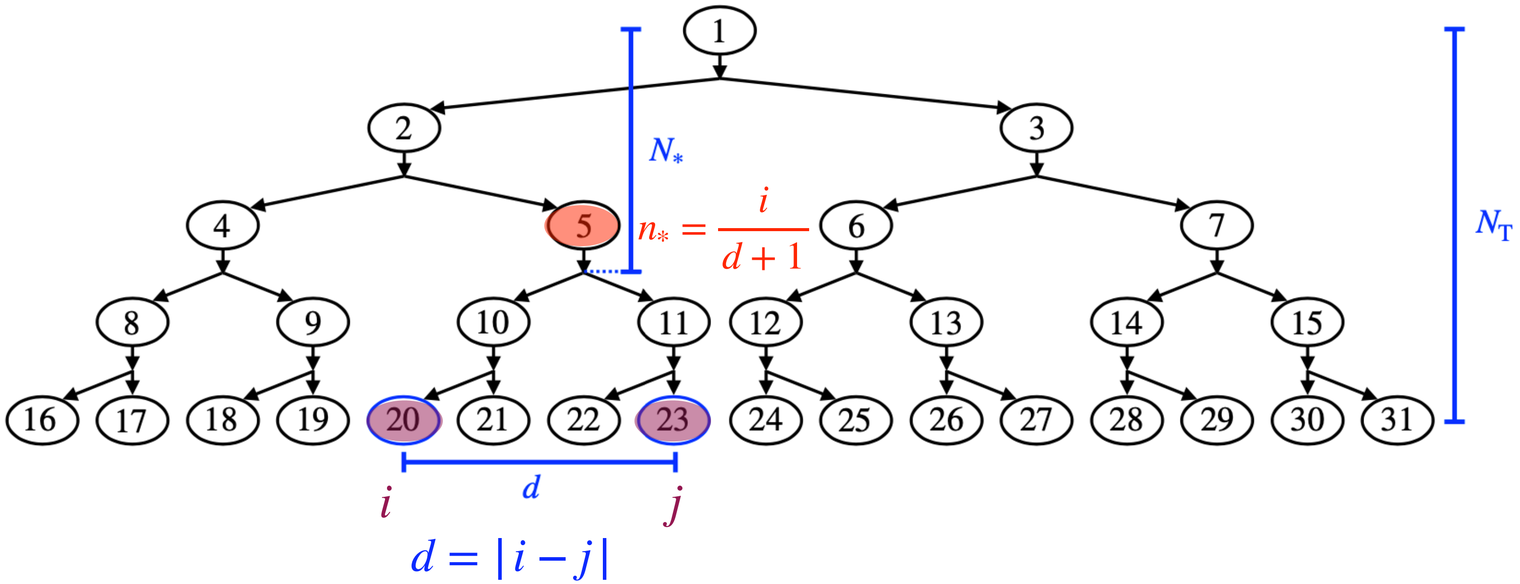
Large scale properties do not depend on the choice of α

$$N_* = N_T - \Delta N_* = N_T - \left[\frac{\log(d+1)}{\log(2)} - \alpha\right]\Delta N$$

<u>Light test scalar field in a fixed binary tree</u>

$$(d+1)V_\sigma = \frac{4}{3}\pi(d_{\rm P}/2)^3 \Rightarrow d+1 = \left(\frac{d_{\rm P}\sigma H}{2}\right)^3$$

$$\Delta N_* = \log(H d_{\rm P}) + \log(2^{-1-\alpha/3}\sigma)$$



$$P(\phi_i, \phi_j) = \int d\phi_* P(\phi_* \,|\, \phi_1, N_*) P(\phi_i \,|\, \phi_*, \Delta N_*) P(\phi_j \,|\, \phi_*, \Delta N_*)$$

depends on α only through ΔN_*
(α dependence reabsorbed in σ)

Discretisation artefacts: branching times

Explicit example: light test field with $V(\phi) = \frac{1}{2}m^2\phi^2$ in de-Sitter universe

Gaussian solution for the stochastic problem:

$$P(\phi\,|\,\phi_{\rm in}, N) = \frac{e^{-\frac{[\phi - \bar{\phi}(N,\phi_{\rm in})]^2}{2s^2(N)}}}{\sqrt{2\pi s^2(N)}}$$

$$\bar{\phi}(N, \phi_{\rm in}) = \phi_{\rm in} e^{-\frac{m^2}{3H^2}N}$$

$$s^2(N) = \frac{3H^4}{8\pi^2 m^2}\left(1 - e^{-\frac{2m^2}{3H^2}N}\right)$$

Starobinsky & Yokoyama [1994]

$$P(\phi_i, \phi_j) = \frac{1}{\sqrt{(2\pi)^2 \det \Sigma}} e^{-\frac{1}{2}(\Delta\phi_i, \Delta\phi_j)\cdot\Sigma^{-1}\cdot\begin{pmatrix}\Delta\phi_i \\ \Delta\phi_j\end{pmatrix}}$$

$$\Delta\phi_i = \phi_i - \bar{\phi}(N_{\rm T}, \phi_1)$$

$$\Sigma_{ii} = \langle \Delta\phi_i^2 \rangle = \Sigma_{jj} = \langle \Delta\phi_j^2 \rangle = s^2(N_T)$$

$$\Sigma_{ij} = \langle \Delta\phi_i \Delta\phi_j \rangle = s^2(N_*) e^{-\frac{2m^2}{3H^2}\Delta N_*}$$

From computation in QFT + renormalisation, late-time limit:

$$\Sigma_{ii} = \frac{3H^4}{8\pi^2 m^2} \qquad \Sigma_{ij} \propto (H d_{\rm p})^{-\frac{2m^2}{3H^2}}$$

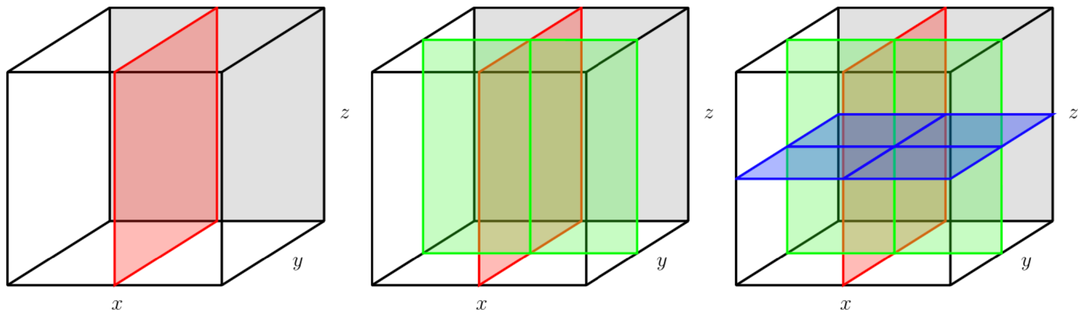
N. A. Chernikov and E. A. Tagirov [1968]

E. A. Tagirov [1973]

T. S. Bunch and P. C. W. Davies [1978]

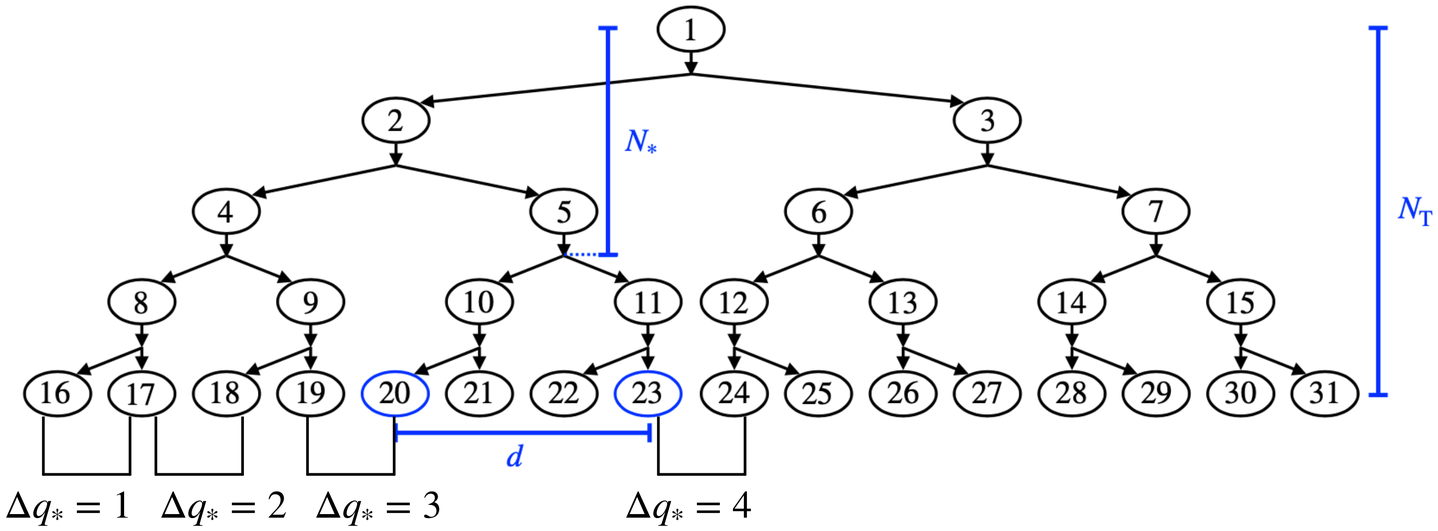
in agreement with the above result

Discretisation artefacts: branching surfaces



(a) ΔN: we divide the x-axis. (b) $2\Delta N$: we divide the y-axis. (c) $3\Delta N$: we divide the z-axis.

Branching surfaces breaks the homogeneity of FLRW spacetime.



Topological distance Δq_* not directly mapped to the geometrical distance d at the end of inflation.

$\Delta N_*(i, j)$ hence $P(\phi_i, \phi_j)$ not just a function of $|i - j|$: breaking of space-translation invariance.

Discretisation artefacts: branching surfaces

Two-point correlation at physical distance $d_{\rm P}$ should be defined by averaging over all pairs of two leaves distant by d on the end-of-inflation hyper surface $\Sigma(d_{\rm P}) = \dfrac{1}{2^{q_{\rm T}} - d} \displaystyle\sum_{i=2^{q_{\rm T}}}^{2^{q_{\rm T}+1}-d-1} \Sigma_{i,i+d}$.

Counting function for the number of pairs: $\beta(d,q) = \begin{cases} 2^q d & \text{if} \quad d \leq 2^{q_{\rm T}-q-1} \\ 2^{q_{\rm T}} - 2^q d & \text{if} \quad 2^{q_{\rm T}-q-1} \leq d \leq 2^{q_{\rm T}-q} \\ 0 & \text{if} \quad d \geq 2^{q_{\rm T}-q} \end{cases}$.

$$\Sigma\left(d_{\rm P}\right) = \frac{1}{2^{q_{\rm T}} - d} \sum_{q=0}^{q_{\rm T}-1} \beta(d,q) \Sigma\left[\left(q_{\rm T} - q\right) \Delta N\right] = \frac{3H^4}{8\pi^2 m^2} \frac{e^{-aq_{\rm T}}}{2^{q_{\rm T}} - d} \left[2^{q_{\rm T}}\left(e^{aq_*} - 1\right) - \frac{2\left(e^a - 1\right) d \left(2^{q_*} e^{aq_*} - 1\right)}{2e^a - 1}\right]$$

$$q_* = \lfloor q_{\rm T} - \ln(d)/\ln(2) \rfloor$$
$$a = 2m^2 \Delta N/(3H^2)$$

At large distances ($1 \ll d \ll D \Rightarrow q_* \gg 1$): $\Sigma\left(d_{\rm P}\right) \simeq \dfrac{3H^4}{8\pi^2 m^2} \left[\dfrac{e^{3a}}{2e^a - 1} - e^{3a - aq_*}\right] \left(\sigma H d_{\rm P}\right)^{-\frac{2m^2}{3H^2}} \underset{\substack{a \ll 1 \\ \text{(slow roll)}}}{\simeq} \dfrac{3H^4}{8\pi^2 m^2} \left[1 - e^{-\frac{2m^2}{3H^2} N_*}\right] \left(\sigma H d_{\rm P}\right)^{-\frac{2m^2}{3H^2}}$

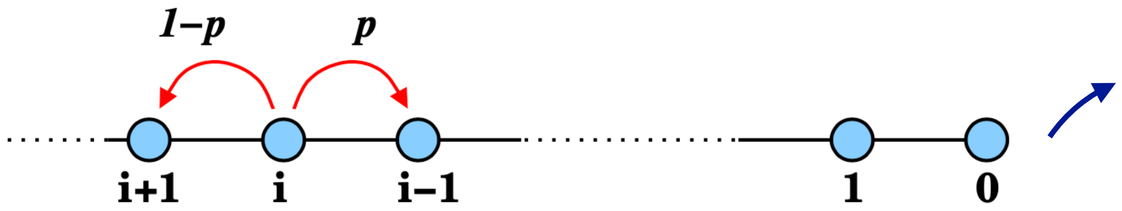
consistent with previous result and with QFT computation

Going beyond

Is it possible to go beyond the large volume approximation?

Creminelli, Dubovsky, Nicholas, Senatore, Zaldarriaga [2008]
Dubovsky, Senatore, Villadoro [2009]

Bacteria model of inflation



End site : bacterium dies

Multi-type Galton-Watson process

Bacteria live on discrete set of positions along a line, replicating into N copies at each time step.

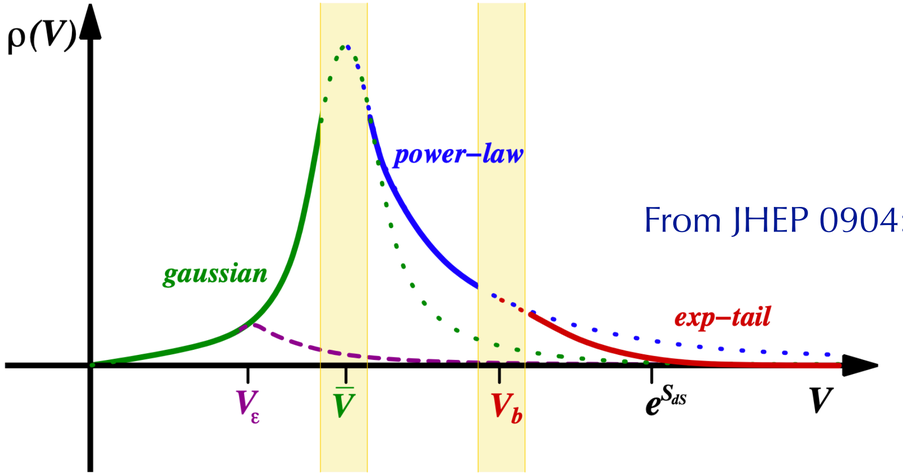
Bacteria → Hubble patches

Sites → Inflaton values

Random hopping → Quantum diffusion

Difference in $(1-p)$ and p → Drift

Number of dead bacteria → Final volume



From JHEP 0904:118,2009

Volume distribution

Volume distribution

Asymptotic behaviour

$$P(V|\phi) \approx \frac{1}{2\sqrt{\pi}}\sigma(\phi)V^{-3/2}\,e^{\xi_* V}$$

Creminelli, Dubovsky, Nicholas, Senatore, Zaldarriaga [2008]
Winitzki [2008]

ξ_* is the rightmost singularity of f in the complex ξ plane.

f always has a singularity at a real, non-positive $\xi = \xi_*$ of the type: $f(\phi;\xi) = f(\phi;\xi_*) + \sigma(\phi)\sqrt{\xi - \xi_*} + \mathcal{O}(\xi - \xi_*)$.

All moments of the distribution are finite for $\xi_* < 0$, $\xi_* = 0$ corresponds to the transition point to eternal inflation.

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Moments of the distribution

Taylor coefficients of $f(\phi;\xi)$ around $\xi = 0$: $\langle V^n \rangle = \int_0^\infty \mathrm{d}V V^n P(V|\phi) = (-1)^n \frac{\partial^n f(\phi;\xi)}{\partial \xi^n}|_{\xi=0}$

Equations for the moments become linear.

$$\langle V \rangle = \int \mathrm{d}V P(V|\phi)V = \langle e^{3\mathcal{N}} \rangle = \int \mathrm{d}\mathcal{N} e^{3\mathcal{N}} P_{\mathrm{FPT},\phi}(\mathcal{N})$$

Not true for higher order moments! $\langle V^2 \rangle = \int \mathrm{d}V P(V|\phi)V^2 \neq \langle e^{6\mathcal{N}} \rangle = \int \mathrm{d}\mathcal{N} e^{6\mathcal{N}} P_{\mathrm{FPT},\phi}(\mathcal{N})$

Bacteria model of inflation: volume distribution

Continuum limit of the bacteria model:

$$\frac{\partial^2}{\partial\phi^2} f(\phi; s_0) - \frac{\partial v}{\partial\phi}\frac{1}{v^2}\frac{\partial f(\phi; s_0)}{\partial\phi} + \frac{3}{M_{\rm Pl}^2 v} f(\phi; s_0)\log(f(\phi; s_0)) = 0$$

$$f(0, s_0) = s_0\,; \qquad \frac{\partial}{\partial\phi} f(\phi; s_0)\,|_{\phi_{\rm b}} = 0$$

$f(\phi; s_0)$ is the Laplace transform of the probability distribution of the volume:

$$f(\phi; s_0) = \int_0^\infty {\rm d}V P(V|\phi) s_0^V \xrightarrow[\xi = -\log(s_0)]{} f(\phi; \xi) = \int_0^\infty {\rm d}V P(V|\phi) e^{-\xi V}$$

Volume distribution $P(V|\phi)$ is the inverse Laplace transform of f:

$$P(V|\phi) = \mathscr{L}^{-1}\{f(\phi; \xi)\}(V) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} {\rm d}\xi e^{\xi V} f(\phi; \xi)$$